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PSEUDO VALUATION RINGS

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Abstract. In this short paper, we generalize some theorems about pseudo valuation domain to ring and give characterizations of psedo valuation ring.

1. Introduction

Throughout this paper, all rings are commutative with identity and the letter R denotes an integral domain with quotient field K. Hedstrom and Houston ([11]) introduced the concept *pseudo valuation domains(PVD)*. Recall from [11] that an integral domain R, with quotient field K, is called a *pseudo valuation domains (PVD)* in case each prime ideal P of R is *strongly prime*, in the sense that $xy \in P, x \in K, y \in K$ implies that either $x \in P$ or $y \in P$. Recently, Badawi, Anderson and Dobbs([9]) generalized the study of pseudo valuation domains to the context of arbitrary rings. A prime ideal P of a ring A is said to be a *strongly prime* ideal if aP and bA are comparable for all $a, b \in A$([9]). We shall say that a ring A is a *pseudo valuation ring (PVR)* if each prime ideal of A is strongly prime.

In this paper, we show that for nonzero prime ideals $P \subset Q$ of a ring A,if Q is strongly prime then P is also strongly prime (Theorem 3.1). This is a generalized result of Proposition 2.9 and again,this theorem generalizes Theorem 2.5 (Theorem 3.3). Also, we shall prove that if A is a PVR then either A_P is a PVR or A_P

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contains a PVR for each prime ideal P of A (Theorem 3.6) and that every integral overring of a PVR is a PVR (Theorem 3.8). Lastly we show that if the idealization of a ring A and a A-module, $A(M)$, is a PVR then A is also a PVR (Theorem 3.10).

2. Pseudo Valuation Rings

We start by recalling some facts about a pseudo valuation domain.

PROPOSITION 2.1. *([2],[13]) For a prime ideal P of an integral domain R with quotient field K, the followings are equivalent.*

(1) P is strongly prime.

(2) For each fractional ideal I and J of R, if $IJ \subset P$ then $I \subset P$ or $J \subset P$.

(3) P is comparable to each principal fractional ideal of R.

(4) P is comparable to each fractional ideal.

From the above Proposition 2.1, we can easily know the following result and in view of Proposition 2.2,PVR is a generalized concept of PVD.

PROPOSITION 2.2. *Let R be an integral domain with quotient field K and P a prime ideal of R. Then the followings are equivalent:*

(1) aP and bR are comparable for all $a, b \in R$.

(2) $xy \in P, x \in K, y \in K$ implies either $x \in P$ or $y \in P$.

PROPOSITION 2.3. *(a)Let I be an ideal of a ring A and P a strongly prime ideal of A. Then I and P are comparable.*

(b) Any PVR is quasilocal.

Proof. (a) Suppose that I is not contained in P. Then there exists $s \in I - P$. For this s and $a = 1$, either $aP = P \subset sA$ or $sA \subset aP = P$. But $aP = P \subset sA$ since sA is not contained in $P = aP$, and so $P \subset sA \subset I$.

(b) Let $\mathcal{M}$ and $\mathcal{M}'$ be maximal ideals of a PVR A. Then $\mathcal{M}$ is strongly prime. By (a), either $\mathcal{M} \subset \mathcal{M}'$ or $\mathcal{M}' \subset \mathcal{M}$, In any case, $\mathcal{M} = \mathcal{M}'$ since $\mathcal{M}$ and $\mathcal{M}'$ are maximal ideals.

A prime ideal P of a ring A is called *divided* if P is comparable to every principal ideal of A. If every prime ideal of A is divided, then A is called *divided ring*

Now we have the following corollary from the above Proposition 2.3-(a)

COROLLARY 2.4. *If A is a PVR then A is a divided ring.*

THEOREM 2.5. *([9])A quasilocal ring A with maximal ideal $\mathcal{M}$ is a PVR if and only if $\mathcal{M}$ is strongly prime.*

PROPOSITION 2.6. *([9])Any homomorphic image of a PVR is a PVR.*

A ring A is called a *chained ring* if its ideals are linearly ordered by inclusion(equivalently, its principal ideals are linearly ordered by inclusion)

PROPOSITION 2.7. *Any chained ring A is a PVR.*

Proof. Clearly, any chained ring is quasilocal. Now it is enough to show that the unique maximal ideal $\mathcal{M}$ of a chained ring A is strongly prime. Let $a, b \in A$. Then $a\mathcal{M}$ and bA are comparable since A is a chained ring. i.e, $\mathcal{M}$ is strongly prime and so A is a PVR by Theorem 2.5

THEOREM 2.8. *([9]) Let A be a PVR with maximal ideal $\mathcal{M}$ and P a nonmaximal prime ideal of A. Then A_P is a chained ring.*

PROPOSITION 2.9. *([2]) Let P and Q be nonzero prime ideals of an integral domain R with $Q \subset P$. If P is strongly prime then Q is strongly prime*

An ideal $\mathcal{P}$ of a ring A is *prime for its regular elements* if whenever x and y are regular elements of A such that $xy \in \mathcal{P}$, then $x \in \mathcal{P}$ or $y \in \mathcal{P}$. A *regular ideal* of A is one that contains a regular element.

PROPOSITION 2.10. *([7]) Let A be a PVR and let $a, b \in R$. If $a \in Z(A)$ and b is a regular(nonzero divisor) element of A, then $b|a$.*

For a ring A, let $S = \{s \in A : s \text{ is a non-zerodivisor of } A\}$. Then $T = R_S$ is a *total quotient ring* of A. A subring B of T is called an *overring* of A if $A \subset B$.

THEOREM 2.11. *([6]) Let $\mathcal{P}$ be a strongly prime ideal of a ring A containing the zero divisors of A and B be an overring of A. The following statements are equivalent:*

(1) $\mathcal{P}B \cap A = \mathcal{P}$.

(2) B does not contain the reciprocal of any elements of $\mathcal{P}$.

(3) $\mathcal{P}$ is a strongly prime ideal of B.

PROPOSITION 2.12. *Let A be a PVR. Then a regular ideal $\mathcal{P}$ of A is prime if and only if $\mathcal{P}$ is prime for its regular elements.*

Proof. We first prove that each regular ideal $\mathcal{A}$ of A is generated by its set regular elements. Let $\{a_\alpha\}$be the regular elements in $\mathcal{A}$ and let a be any element in $\mathcal{A}$. Then if a is regular then , clearly, a is contained in the ideal which is generated by $\{a_\alpha\}$. Otherwise, $a \in Z(A)$. By Proposition 2.10, $a_\alpha | a$ for regular element a_α of $\mathcal{A}$. Therefore a is contained in the ideal which is generated by $\{a_\alpha\}$ Hence $\mathcal{A}$ is generated by $\{a_\alpha\}$.

Now we are ready to prove Proposition 2.12. If a regular ideal $\mathcal{P}$ is prime, then it certainly prime for its regular elements. For the converse assume that $\mathcal{P}$ is a regular ideal which is not prime. There exist $x, y \in A$ such that $xy \in \mathcal{P}, x \notin \mathcal{P}, y \notin \mathcal{P}$. Since $\mathcal{P}$ is regular ideal,$(\mathcal{P}, x)$ is regular ideal and $(\mathcal{P}, x) = r_{\alpha_1}a_{\alpha_1} + \cdots + r_{\alpha_s}a_{\alpha_s}$ where a_{α_i} is regular element in $(\mathcal{P}, x)$. Similarly, $(\mathcal{P}, y) = t_{\beta_1}b_{\beta_1} + \cdots + t_{\beta_s}b_{\beta_s}$ where β_i is regular element in $\mathcal{P}, y)$. Then we can choose regular elements a_α, b_β such that $a_\alpha \in (\mathcal{P}, x) - \mathcal{P}, b_\beta \in (\mathcal{P}, y) - \mathcal{P}$. Then, clearly, $a_\alpha b_\beta \in \mathcal{P}$ since $xy \in \mathcal{P}$. However, since $\mathcal{P}$ is prime for regular elements $a_\alpha \in \mathcal{P}$ or $b_\beta \in \mathcal{P}$. a contradiction.

3. Main Results

In this section, we generalize Theorem 2.5 and Proposition 2.9 and give some characterizations of Pseudo Valuation Ring. Also, we prove a theorem on the overring of a PVR.

Lastly, we give a relation between PVR and the idealization of a ring and a module.

THEOREM 3.1. *Let P and Q be nonzero prime ideals of a ring A with $Q \subset P$. If P is strongly prime ,then Q is also strongly prime*

Proof. Let $a, b \in A$. We show that aQ and bA are comparable. Since P is strongly prime, aP and bA are comparable. If $aP \subset bA$ then $aQ \subset aP \subset bA$. So we may assume that bA is properly contained in aP. In this case, there exists $p \in P$ such that $b = ap$. If $p \in Q$ then $b = ap \in aQ$ and so $bA \subset aQ$. Thus we may asume $p \notin Q$ and in this case we claim that $Q \subset aP$. Let $q \in Q$. then qA and pP are comparable since P is strongly prime. If $pP \subset qA$ then $pP \subset qA \subset Q$.However Q is a prime ideal of A and hence either $p \in Q$ or $P \subset Q$, a contradiction. Therefore we have $qA \subset pP$ for all $q \in Q$. i.e, $Q \subset pP$. Hence $aQ \subset apP = bP \subset bA$.

In view of Proposition 2.2, we have Proposition 2.9 as the following corollary

COROLLARY 3.2. *Let P and Q be nonzero prime ideals of an integral domain R with $Q \subset P$. If P is strongly prime then Q is strongly prime.*

THEOREM 3.3. *The following statements are equivalent:*
(1) A ring A is a PVR.
(2) Every maximal ideal of a ring A is strongly prime.
(3) A ring A is quasilocal with the maximal ideal $\mathcal{M}$ and $\mathcal{M}$ is strongly prime.

Proof. (1) $\Rightarrow$ (2) It is clear by definition of PVR.
(2) $\Rightarrow$ (3) It follows from Proposition 2.3-(a).
(3) $\Rightarrow$ (1) Any prime ideal P of A is contained in the maximal ideal $\mathcal{M}$(hence prime ideal) of A and since $\mathcal{M}$ is strongly prime by our assumption, P is strongly prime by Theorem 3.1. Thus A is a PVR.

Now we have Theorem 2.5 as the following corollary

COROLLARY 3.4. *A quasi local ring with maximal ideal $\mathcal{M}$ is a PVR if and only if $\mathcal{M}$ is strongly prime.*

PROPOSITION 3.5. *The prime ideals below a strongly prime ideal P of a ring A are linearly ordered.*

Proof. Let Q be strongly prime and let $P_i \subset Q$ for $i = 1, 2$. Then P_1 and P_2 are strongly prime ideals by Theorem 3.1 . By Proposition 2.3 -(a), P_1 and P_2 are comparable.

THEOREM 3.6. *Let A be a PVR. Then for every prime ideal P of A, either A_P is a PVR or A_P contains a PVR.*

Proof. We know that any PVR is quasilocal by Proposition 2.3-(b). Let A be a PVR with the unique maximal ideal $\mathcal{M}$ and let P be any prime ideal of A. Let $P = \mathcal{M}$. In this case, any element $s \in A - P = A - \mathcal{M}$ is regular since $(A, \mathcal{M})$ is quasilocal. Hence the canonical homomorphism $\phi : A \to A_P$ defined by $\phi(a) = \frac{a}{1}$ is injective and so $A \cong \phi(A) \subseteq A_P$. By Proposition 2.6, $\phi(A)$ is a PVR. Hence A_P contains a PVR.

Now let $P \neq \mathcal{M}$. i.e, let P be a nonmaximal prime ideal of a ring A. Then by Theorem 2.8, A_P is a chained ring and hence PVR by Proposition 2.7.

PROPOSITION 3.7. *The total quotient ring T of a PVR A is a PVR*

Proof. If $A = T$ then it is clear. Assume that $A \neq T$. Then, by [7]-Proposition 6 T is a chained ring and so T is a PVR by Proposition 2.7.

THEOREM 3.8. *If B is an integral overring of a PVR A,then B is a PVR.*

Proof. Since A is a PVR, it is quasilocal by Proposition 2.3-(b). Let its unique maximal ideal be $\mathcal{M}$. Then $\mathcal{M}$ is a stongly prime ideal of A and $Z(A)$, the set of zero divisors in A, is contained in $\mathcal{M}$. We know that B does not contain the reciprocal of any elements of $\mathcal{M}$. In fact, Since B is integral over A, if $1/m \in B$ for some $m \in \mathcal{M}$ then there exist $a_i \in A$ such that $(1/m)^n + a_1(1/m)^{n-1} + \cdots + a_n = 0$. Hence $1 \in \mathcal{M}$, a contradiction. Therefore $\mathcal{M}$ is a strongly prime ideal in B by Theorem 2.11. Furtheremore $\mathcal{M}$ is a maximal ideal in B. For, suppose that $\mathcal{M} \subseteq \mathcal{B} \subset B$ for some ideal $\mathcal{B}$ in B. then $\mathcal{M} = \mathcal{M} \cap A \subseteq \mathcal{B} \cap A \subseteq A$. Since $\mathcal{M}$ is a maximal ideal in A and $\mathcal{B} \neq B$, $\mathcal{M} = \mathcal{B} \cap A$. If $\mathcal{B}$ is prime in B then $\mathcal{M} \cap A = \mathcal{B} \cap A$. Hence $\mathcal{M} = \mathcal{B}$ by Incomparability Theorem. If $\mathcal{B}$ is not prime in B then there exist a prime ideal $\mathcal{Q}$ in B such that $\mathcal{M} \subseteq \mathcal{B} \subseteq \mathcal{Q}$ in B. Thus $\mathcal{M} = \mathcal{M} \cap A \subseteq \mathcal{Q} \cap A$. Since $\mathcal{Q} \cap A \neq A$ and $\mathcal{M}$ is a maximal ideal in A, $\mathcal{M} = \mathcal{Q} \cap A$. i.e, $\mathcal{M} \cap A = \mathcal{Q} \cap A$. Again, by Incomparability Theorem ,$\mathcal{M} = \mathcal{Q}$

and so $\mathcal{M} = \mathcal{B}$. Since $\mathcal{M}$ is strongly prime in B and strongly prime ideal is comparable to every ideal (Proposition 2.3-(a)), we know that B is quasilocal. Thus B is a PVR by theorem 3.3.

From the above theorem we have the following corollary.

COROLLARY 3.9. *Let A be a PVR with maximal ideal $\mathcal{M}$. then every overring B of A such that $B \subset A'$(the integral closure of A in total quotient ring) is a PVR with maximal ideal $\mathcal{M}$.*

Let A be a commutative ring with unity and M an A-module. Consider $A(M) = \{(r,m) | r \in A, m \in M\}$ and let (r,m) and (s,n) be two elements of $A(M)$. Define:

1. $(r,m) = (s,n)$ if $r = s$ and $m = n$
2. $(r,m) + (s,n) = (r+s, m+n)$
3. $(r,m)(s,n) = (rs, rn + sm)$

Under these definition, $A(M)$ becomes a commutative ring with unity and $A(M)$ is called the *idealization* of a ring A and an A-module M. We can find some basic results about $A(M)$ ([10]).

In A(M) we can easily show that $(r,m) \in A(M)$ is a unit of $A(M)$ if and only if r is a unit of A.

THEOREM 3.10. *Under the above notations, if $A(M)$ is a PVR then A is also a PVR*

Proof. Let $a, b \in A$ and c any nonunit in A. Suppose that $a \nmid b$ in A. Clearly, $(a,0) \nmid (b,0)$ in $A(M)$ and $(c,0)$ is a nonunit in $A(M)$. Since $A(M)$ is a PVR $(b,0)|(a,0)(c,0) = (ac,0)$ by Theorem 5 in [9](or Proosition 1.1 in [10]). Hence $b|ac$ and again,A is a PVR by theorem 5 in [9].

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