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SEMI–INVARIANT SUBMANIFOLDS OF CODIMENSION 3 IN A COMPLEX HYPERBOLIC SPACE

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Abstract. In this paper we prove the following : Let M be a semi-invariant submanifold with almost contact metric structure (ϕ, ξ, g) of codimension 3 in a complex hyperbolic space $H_{n+1}\mathbb{C}$. Suppose that the third fundamental form n satisfies $dn = 2\theta\omega$ for a certain scalar $\theta(\leq \frac{c}{2})$, where $\omega(X,Y) = g(X, \phi Y)$ for any vectors X and Y on M. Then M has constant eigenvalues correponding the shape operator A in the direction of the distinguished normal and the structure vector ξ is an eigenvector of A if and only if M is locally congruent to one of the type A_0, A_1, A_2 or B in $H_n\mathbb{C}$.

0. Introduction

A submanifold M is called a *CR submanifold* of a Kaehlerian manifold $\tilde{M}$ with complex structure J if there exists a differentiable distribution $T : p \to T_p \subset M_p$ on M such that T is J-invariant and the complementary orthogonal distribution $T^{\perp}$ is totally real, where M_p denotes the tangent space to M at each point p in M([1],[2], [22]). In particular, M is said to be a *semi-invariant submanifold* provided that dim $T^{\perp} = 1$. The unit

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normal vector field in $JT^{\perp}$ is called the *distinguished normal* to the semi-invariant submanifold ([20]). A semi-invariant submanifold admits an induced almost contact metric structure, and many results are known by using this structure ([4], [18], [19], etc.).

A typical example of a semi-invariant submanifold is real hypersurface. When the ambient manifold $\tilde{M}$ is a complex hyperbolic space $H_n\mathbb{C}$, real hypersurfaces were investigated by many geometers in connection with the shape operator and the induced almost contact metric structure ([3], [6], [7],[12], [14], [15] and [16] etc.). One of them, Berndt ([3]) showed that all real hypersurfaces with constant principle curvatures of a complex hyperbolic space $H_n\mathbb{C}$ are realized as the tubes of constant radius over certain submanifolds when the structure vector ξ is principal. Nowadays in $H_n\mathbb{C}$ they are said to be of type A_0, A_1, A_2, and B. He proved the following:

THEOREM B ([3]). *Let M be a real hypersurface of $H_n\mathbb{C}$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following:*

(A_0) *a self-tube, that is, a horosphere,*

(A_1) *a geodesic hypersphere or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*

(A_2) *a tube over a totally geodesic $H_k\mathbb{C}(1 \leq k \leq n-2)$,*

(B) *a tube over a totally real hyperbolic space H_nR.*

On the other hand, submanifolds of codimension 3 addmitting an almost contact metric structure in a complex space form have been studied in ([8], [9],[10],[11],[13],
[21]) by using properties of the third fundamental form of M. From this point of view, Ki, Song and Takagi ([11]) proved the following:

THEOREM K-S-T. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanfold of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$ such that the third fundamental tensor satisfies $dn = 2\theta\omega$ for a certain scalar $\theta(< \frac{c}{2})$, where $\omega(X,Y) = g(X,\phi Y)$ for any vectors X and Y on M. Then M has constant eigenvalues corresponding the shape operator A in the direction of distinguished normal and the structure vector ξ is an eigenvector* *of A if and only if M is locally congruent to a homogeneous real hypersurfaces of $P_n\mathbb{C}$.*

The main purpose of the present paper is to extend Theorem B under certain conditions on a semi-invariant submanifold of codimension 3 in a complex hyperbolic space $H_{n+1}\mathbb{C}$.

1. Preliminaries

Let $\tilde{M}$ be a real $2(n+1)$-dimensional Kaehlerian manifold equipped with parallel almost complex structure J and a Riemannian metric tensor G, which J-Hermitian and covered by a system of coordinate neighborhoods $\{W; y^A\}$.

Let M be a real (2n-1)-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; x^h\}$ and immersed isometrically in $\tilde{M}$ by the immersion $i : M \to \tilde{M}$.

Throughout the present paper the following convention on the range of indices are used, unless otherwise stated :

$$A, B, \cdots = 1, 2, \cdots, 2n+2 \ ; \ i, j, \cdots = 1, 2, \cdots, 2n-1.$$

The summation convention will be used with respect to those system of indices. When the argument is local, M need not to be distinguished from $i(M)$.Thus, for simplicity, a point p in M may be identified with $i(p)$ and a tangent vector X at p may also be identified with the tangent vector $i_*(X)$ at $i(p)$ via the differential $i*$ of i. We represent the immersion i locally by $y^A = y^A(x^h)$ and $B_j = (B_j{}^A)$ are also (2n-1)-linearly independent local tangent vectors of M, where $B_j{}^A = \partial_j y^A$ and $\partial_j = \partial/\partial x^j$. Three mutually orthogonal unit normals C, D and E may then be chosen. The induced Riemannian metric tensor g with components g_{ji} on M is given by $g_{ji} = G(B_j, B_i)$ because the immersion i is isometric.

Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to the induced Riemannian metric, equations of the Gauss for M of $\tilde{M}$ is obtained :

$$\nabla_j B_i = A_{ji}C + K_{ji}D + L_{ji}E, \tag{1.1}$$

where A_{ji}, K_{ji} and L_{ji} are components of the second fundamental forms in the direction of normals C, D and E respectively.

Equations of the Weingarten are also given by

$$\begin{aligned} \nabla_j C &= -A_j{}^h B_h + l_j D + m_j E, \\ \nabla_j D &= -K_j{}^h B_h - l_j C + n_j E, \\ \nabla_j E &= -L_j{}^h B_h - m_j C - n_j D, \end{aligned} \tag{1.2}$$

where $A = (A_j{}^h), A_{(2)} = (K_j{}^h)$ and $A_{(3)} = (L_j{}^h)$, which are related by $A_{ji} = A_j{}^r g_{ir}, K_{ji} = K_j{}^r g_{ir}$ and $L_{ji} = L_j{}^r g_{ir}$ respectively, and l_j, m_j and n_j being components of the third fundamental forms.

In the sequel, we denote the normal components of $\nabla_j C$ by $\nabla_j{}^\perp C$. The normal vector field C is said to be *parallel* in the normal bundle if we have $\nabla_j{}^\perp C = 0$, that is, l_j and m_j vanish identically.

On the other hand, a submanifold M is callel a *CR submanifold* of a Kaehlerian manifold $\tilde{M}$ if there exists a differentiable distribution $T : p \to T_p \subset M_p$ on M satisfying the following conditions, where M_p denotes the tangent space to M at each point p in M :

(1) T is invariant, that is, $JT_p = T_p$ for each p in M, (2) the complementary orthogonal distribution $T^\perp : p \to T_p{}^\perp \subset M_p$ is totally real, that is, $JT_p{}^\perp \subset M_p{}^\perp$ for each p in M, where $M_p{}^\perp$ denotes the normal space to M at $p \in M$([1], [2], [22]). In particular M is said to be a *semi-invariant submanifold* provided that *dim* $T^\perp = 1$. In this case the unit normal vector field in $JT^\perp$ is called a *distinguished normal* to the semi-invariant submanifold and denoted this by C([4], [20]). More precisely, we choose an orthonormal basis $e_1, \cdots, e_{n-1}, e_n$ of M_p in such a way that $e_1, \cdots, e_{n-1} \in T$. Then we see that

$$G(Je_n, e_i) = -G(e_n, Je_i) = -G(e_n, \sum_{k=1}^{n-1} F_{ik} e_k) = 0$$

for $i = 1, \cdots, n-1$. Also we have $G(Je_n, e_n) = 0$ because J is skew-symmetric. Therefore Je_n is orthogonal to M_p. We put $C = -Je_n$. Then we can write

(1.3)
$$JB_i = \phi_i{}^h B_h + \xi_i C, \quad JC = -\xi^h B_h, \quad JD = -E, \quad JE = D$$

in each coordinate neighborhood, where we have put $\phi_{ji} = G(JB_j, B_i), \xi_i = G(JB_i, C), \xi^h$ being associated component of ξ_h. By the property of the almost Hermitian structure J, it is clear that ϕ_{ji} is skew-symmetric. A tensor fied of type (1,1) with components $\phi_i{}^h$ will be denoted by ϕ. By properties of the almost complex structure J it follows that

$$\phi_i{}^r \phi_r{}^h = -\delta_i{}^h + \xi_i \xi^h, \quad \xi^r \phi_r{}^h = 0, \quad \xi_r \phi_i{}^r = 0,$$
$$\xi_r \xi^r = 1, \quad g_{rs} \phi_j{}^r \phi_i{}^s = g_{ji} - \xi_j \xi_i.$$

Since J is parallel, by differentiating the first equation of (1.3) covariantly along M and using (1.1), (1.2) and (1.3), and by comparing the tangential and normal parts, we find (see [21])

$$\nabla_j \phi_i{}^h = -A_{ji} \xi^h + A_j{}^h \xi_i, \tag{1.4}$$

$$\nabla_j \xi_i = -A_{jr} \phi_i{}^r, \tag{1.5}$$

$$K_{ji} = -L_{jr} \phi_i{}^r - m_j \xi_i, \tag{1.6}$$

$$L_{ji} = K_{jr} \phi_i{}^r + l_j \xi_i. \tag{1.7}$$

The last two relations give

$$K_{jt} \xi^t = -m_j, \quad L_{jt} \xi^t = l_j, \tag{1.8}$$

$$(1.9) \qquad m_t\xi^t = -k, \quad l_t\xi^t = l$$

where $k = T_rA_{(2)}, l = T_rA_{(3)}$.

Here we may assume that $l = 0$. In fact, for a normal vector v of M we denote by A_v the second fundamental tensor of M in the direction of v. Then we have $A_{-v} = -A_v$. Hence there is a unit normal vector D' of M in the plane spanned by two vectors D and E such that $T_rA_{D'} = 0$, which proves our assertion. Therefore we have by (1.9)

$$(1.10) \qquad l_t\xi^t = 0.$$

Transforming (1.7) by $\phi_k{}^j$ and using (1.6), we obtain

$$-K_{ik} - m_i\xi_k = K_{st}\phi_i{}^s\phi_k{}^t + \xi_i\phi_{kt}l^t,$$

which implies

$$m_k\xi_i - m_i\xi_k = \xi_i\phi_{kt}l^t - \xi_k\phi_{it}l^t,$$

or, using (1.9)

$$(1.11) \qquad \phi_{it}l^t = m_i + k\xi_i.$$

Similarly we have

$$(1.12) \qquad \phi_{ir}m^r = -l_i$$

because of (1.10).

Transforming (1.6) and (1.7) by $L_k{}^i$ and using (1.6), (1.7) and (1.8), we have respectively

$$(1.13) \qquad K_{jr}L_i{}^r + K_{ir}L_j{}^r = -(l_jm_i + l_im_j),$$

$$L_{ji}^2 - K_{ji}^2 = l_j l_i - m_j m_i. \tag{1.14}$$

The ambient Kaehlerian manifold $\tilde{M}$ is assumed to be of constant holomorphic sectional curvature c, which is called a *complex space form* and denoted by $M_{n+1}(c)$. Then equations of the Gauss and Codazzi are given by

$$\begin{aligned} R_{kjih} =& \frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}) \\ &+ A_{kh}A_{ji} - A_{jh}A_{ki} + K_{kh}K_{ji} - K_{jh}K_{ki} \\ &+ L_{kh}L_{ji} - L_{jh}L_{ki}, \end{aligned} \tag{1.15}$$

$$\begin{aligned} &\nabla_k A_{ji} - \nabla_j A_{ki} - l_k K_{ji} + l_j K_{ki} - m_k L_{ji} + m_j L_{ki} \\ &\quad = \frac{c}{4}(\xi_k \phi_{ji} - \xi_j \phi_{ki} - 2\xi_i \phi_{kj}), \end{aligned} \tag{1.16}$$

$$\nabla_k K_{ji} - \nabla_j K_{ki} + l_k A_{ji} - l_j A_{ki} - n_k L_{ji} + n_j L_{ki} = 0, \tag{1.17}$$

$$\nabla_k L_{ji} - \nabla_j L_{ki} + m_k A_{ji} - m_j A_{ki} + n_k K_{ji} - n_j K_{ki} = 0, \tag{1.18}$$

where R_{kjih} is covariant components of the Riemann-Christoffel curvature tensor of M, and those of the Ricci by

$$\nabla_k l_j - \nabla_j l_k + A_{kr}K_j{}^r - A_{jr}K_k{}^r + m_k n_j - m_j n_k = 0, \tag{1.19}$$

$$\nabla_k m_j - \nabla_j m_k + A_{kr}L_j{}^r - A_{jr}L_k{}^r + n_k l_j - n_j l_k = 0, \tag{1.20}$$

$$\text{(1.21)}\quad \nabla_k n_j - \nabla_j n_k + K_{kr}L_j{}^r - K_{jr}L_k{}^r + l_k m_j - l_j m_k = \frac{c}{2}\phi_{kj}.$$

In the following we need the following definition. The normal connection of a semi-invariant submanifold M of codimension 3 in a complex space form is said to be *L-flat* if it satisfies $dn = \frac{c}{2}\omega$, that is, $\nabla_j n_i - \nabla_i n_j = \frac{c}{2}\phi_{ji}$, where $\omega(X, Y) = g(X, \phi Y)$ for any vectors X and Y on M (p514, [17]).

Differentiating $A\xi = \alpha\xi$ covariantly along M, and using (1.5), we find

$$\text{(1.22)}\qquad \xi^r\nabla_k A_{jr} = A_{jr}A_{ks}\phi^{rs} - \alpha A_{kr}\phi_j{}^r + (\nabla_k\alpha)\xi_j,$$

which together with (1.8) and (1.16) yields

(1.23)
$$\begin{aligned} 2A_{jr}A_{ks}\phi^{rs} - \alpha(A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r) + \frac{c}{2}\phi_{kj} \\ = \xi_k\nabla_j\alpha - \xi_j\nabla_k\alpha + 2(m_k l_j - m_j l_k). \end{aligned}$$

Transvecting ξ^k to this and using $A\xi = \alpha\xi$, (1.8) and (1.10), we obtain

$$\text{(1.24)}\qquad \nabla_j\alpha - (\xi^t\nabla_t\alpha)\xi_j = 2kl_j.$$

2. The third fundamental forms of semi-invariant submanifolds

In the rest of this paper we shall suppose that M is a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$ and that the third fundamental form n satisfies $dn = 2\theta\omega$ for a certain scalar θ on M, that is,

$$(2.1) \qquad \nabla_j n_i - \nabla_i n_j = 2\theta\phi_{ji}.$$

Then we have by (1.21)

$$K_{jr}L_i{}^r - K_{ir}L_j{}^r + l_j m_i - l_i m_j = -2(\theta - \frac{c}{4})\phi_{ji},$$

or, using (1.13)

$$(2.2) \qquad K_{jr}L_i{}^r + l_j m_i = -(\theta - \frac{c}{4})\phi_{ji},$$

which together with (1.8), (1.9) and (1.10) yields

$$(2.3) \qquad K_{jr}l^r = kl_j, \quad L_{jr}m^r = 0.$$

REMARK. To write our formulas in a convention form, in the sequel we denote by $h_{(2)} = A_{ji}A^{ji}$, $h = g^{ji}A_{ji}$, $\alpha = A_{ji}\xi^j\xi^i$, $K_{(2)} = K_{ji}K^{ji}$ and $L_{(2)} = L_{ji}L^{ji}$.

Multiplying (2.2) with ϕ^{ji} and summing for j and i, and using (1.6), (1.8) and (1.11), we find

$$K_{(2)} - k^2 = 2(n-1)(\theta - \frac{c}{4}),$$

which together with (1.8) implies that

$$(2.4) \qquad \| K_{ji} - k\xi_j\xi_i \|^2 = 2(n-1)(\theta - \frac{c}{4}),$$

where $\| F \|^2 = g(F, F)$ for any tensor field F on M.

In the same way, we have from (1.7), (1.10), (1.12) and (2.2)

$$(2.5) \qquad L_{(2)} = 2(n-1)(\theta - \frac{c}{4}).$$

Differentiating (2.1) covariantly along M and using (1.4), we obtain

$$\nabla_k(\nabla_j n_i - \nabla_i n_j) = 2(\nabla_k\theta)\phi_{ji} + 2\theta(A_{ki}\xi_j - A_{kj}\xi_i),$$

or, using the first Bianchi identity,

$$(\nabla_k\theta)\phi_{ji} + (\nabla_j\theta)\phi_{ik} + (\nabla_i\theta)\phi_{kj} = 0,$$

which implies $(n-2)\nabla_k\theta = 0$. Thus $\theta(\geq \frac{c}{4})$ is constant if $n > 2$.
First of all, we prove

LEMMA 2.1. *Let M be a semi-invariant submanifold of codimension 3 with L-flat normal connection in a complex projective space $M_{n+1}(c)$. If the structure vector ξ is an eigenvector of the shape operator A in the direction of the distinguished normal, then we have $A_{(2)} = A_{(3)} = 0$ and $\nabla_j{}^{\perp}C = 0$.*

Proof. This lemma was proved in [11] when the ambient space is a complex space form.

Transforming (2.2) by $\phi_k{}^i$ and taking account of (1.6) and (1.12), we have

$$K_{jk}^2 + \xi_j(K_{kr}m^r) + l_j l_k = (\theta - \frac{c}{4})(g_{jk} - \xi_j\xi_k), \tag{2.6}$$

which unable us to obtain

$$\xi_j(K_{kr}m^r) - \xi_k(K_{jr}m^r) = 0.$$

Therefore we have

$$K_{kr}m^r = -(m_r m^r)\xi_k \tag{2.7}$$

because of (1.8). Thus it follows that

$$K_{ji}{}^2 + l_j l_i - (m_r m^r)\xi_j\xi_i = (\theta - \frac{c}{4})(g_{ji} - \xi_j\xi_i). \tag{2.8}$$

In the same way, we have from (2.2)

$$L_{jr}l^r = km_j + (l_t l^t + k^2)\xi_j. \tag{2.9}$$

Transvecting (2.2) with m^i and making use of (1.11) and (2.3), we obtain

$$(\theta - \frac{c}{4} - m_r m^r)l_j = 0.$$

Similarly, we verify, using (2.2) and (2.9), that

$$(\theta - \frac{c}{4} - l_r l^r - k^2)(m_t m^t - k^2) = 0.$$

Now, let Ω be a set of points such that $l_t l^t \neq 0$ on M and suppose that Ω be non-empty. Then we have

$$m_r m^r = \theta - \frac{c}{4}, \quad l_r l^r + k^2 = \theta - \frac{c}{4} \tag{2.10}$$

on Ω. From now on, we discuss our arguments on the open subset Ω of M. Then (2.8) turns out to be

$$K_{ji}{}^2 = (\theta - \frac{c}{4})g_{ji} - l_j l_i. \tag{2.11}$$

Differentiating this covariantly along Ω, we find

$$K_j{}^r \nabla_k K_{ir} + K_i{}^r \nabla_k K_{jr} + l_j \nabla_k l_i + l_i \nabla_k l_j = 0. \tag{2.12}$$

From this it is verified that (see [11])

$$\nabla_k K_{ji} = n_k L_{ji} + l_i A_{jk} + l_j A_{ik}. \tag{2.13}$$

If we differentiate the first equation of (1.8) covariantly and take account of (1.5), (1.6) and (2.13), then we obtain

$$(2.14) \qquad \nabla_k m_j = -n_k l_j - A_{kr} L_j{}^r.$$

Differentiating the first equation of (1.9) covariantly and using (1.11) and (2.14), we also find

$$(2.15) \qquad \nabla_j k = 2A_{jr} l^r.$$

Substituting (2.13) into (2.12), we obtain

$$(2.16) \qquad \nabla_k l_j = n_k m_j - A_{kr} K_j{}^r - kA_{jk}.$$

Differentiating (2.15) covariantly along Ω and making use of (2.16), we get

$$\begin{aligned} \nabla_k \nabla_j k = {} & 2(\nabla_k A_{jr}) l^r + 2A_j{}^r (n_k m_r - A_{ks} K_r{}^s - kA_{kr}) \\ & + n_j (2A_{kr} m^r - kn_k), \end{aligned}$$

from which, taking the skew-symmetric part and making use of (1.11), (1.16), (2.3) and (2.9),

$$(\theta - \frac{c}{2})(m_k \xi_j - m_j \xi_k) = 0.$$

Therefore it follows that $(\theta - \frac{c}{2})(m_j + k\xi_j) = 0$. If we suppose that $c < 0$, then we have $\theta \neq \frac{c}{2}$ on Ω because of (2.10). Thus we have $m_j = -k\xi_j$.

From this and (2.10), we see that $l_j = 0$, a contradiction. Hence Ω is empty, namely, we have $l_j = 0$ on whole space M. Thus we have

LEMMA 2.2. *Let M be a semi-invariant submanifold of codimension 3 in $H_{n+1}\mathbb{C}$ satisfying (2.1). Then we have $\nabla_j{}^{\perp} C = -k\xi_j E$ on M.*

3. Further properties of the third fundamental forms

We continue now, our arguments under the same hypotheses (2.1) as in section 2. Furthermore suppose, throughout this section, that the structure vector ξ satisfies $A_{jr}\xi^r = \alpha\xi_j$. Then we have

$$l_j = 0 \tag{3.1}$$

and hence

$$m_j = -k\xi_j \tag{3.2}$$

because of (1.2). Thus (1.6), (1.7), (1.8), (1.13) and (1.14) are reduced respectively to

$$L_{jr}\phi_i{}^r = -K_{ji} + k\xi_j\xi_i, \tag{3.3}$$

$$K_{jr}\phi_i{}^r = L_{ji}, \tag{3.4}$$

$$K_{jr}\xi^r = k\xi_j, \quad L_{jr}\xi^r = 0, \tag{3.5}$$

$$L_{jr}K_i{}^r + L_{ir}K_j{}^r = 0, \tag{3.6}$$

$$L_{ji}{}^2 = K_{ji}{}^2 - k^2\xi_j\xi_i. \tag{3.7}$$

From (3.2) we have

$$\nabla_k m_j = -\xi_j\nabla_k k + kA_{kr}\phi_j{}^r,$$

from which, taking the skew-symmetric part and using (1.20), (3.1) and (3.2),

$$A_{kr}L_j{}^r - A_{jr}L_k{}^r + k(A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r) = \xi_j\nabla_k k - \xi_k\nabla_j k.$$

Since we have $A\xi = \alpha\xi$, we then have

$$\nabla_k k = \lambda\xi_k \tag{3.8}$$

because of (3.5), where $\lambda = \xi^t\nabla_t k$.
From the last two equations, it is clear that

$$A_{kr}L_j{}^r - A_{jr}L_k{}^r = k(A_{jr}\phi_k{}^r - A_{kr}\phi_j{}^r). \tag{3.9}$$

Similarly, we also have from (1.19), (3.1) and (3.2)

$$k(n_j - \mu\xi_j) = 0, \tag{3.10}$$

$$A_{kr}K_j{}^r - A_{jr}K_k{}^r = 0, \tag{3.11}$$

where $\mu = kn_t\xi^t$.

Using (1.15) $\sim$ (1.24) and (3.1) $\sim$ (3.11), we can verify that k vanishes identically on M (see [11]). Thus we have

LEMMA 3.1. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in $H_{n+1}\mathbb{C}$. If it satisfies $dn = 2\theta\omega$ and $A\xi = \alpha\xi$. Then $\nabla_j{}^{\perp}C = 0$, namely, the distinguished normal is parallel in the normal bundle.*

4. Parallel distinguished normal vectors

In this section, we consider a semi-invariant submanifold of codimension 3 satisfying $dn = 2\theta\omega$ in a complex hyperbolic space.

Suppose that the distinguished normal C is parallel in the normal bundle. Then we have $l_j = m_j = 0$. Thus, (1.16), (1.17), (1.19) and (1.20) turn out respectively to

$$\nabla_k A_{ji} - \nabla_j A_{ki} = \frac{c}{4}(\xi_k \phi_{ji} - \xi_j \phi_{ki} - 2\xi_i \phi_{kj}), \tag{4.1}$$

$$\nabla_k K_{ji} - \nabla_j K_{ki} = n_k L_{ji} - n_j L_{ki}, \tag{4.2}$$

$$A_{jr} K_i{}^r - A_{ir} K_j{}^r = 0, \quad A_{jr} L_i{}^r - A_{ir} L_j{}^r = 0. \tag{4.3}$$

Since we have $dn = 2\theta\omega$, relationships (2.2) and (2.8) are reduced respectively to

$$K_{jr} L_i{}^r = -(\theta - \frac{c}{4})\phi_{ji}, \tag{4.4}$$

$$K_{ji}{}^2 = (\theta - \frac{c}{4})(g_{ji} - \xi_j \xi_i). \tag{4.5}$$

Since we have $K_{ir}\xi^r = 0$, by differentiating covariantly along M and using (1.7) with $l_j = 0$, we find

$$(\nabla_k K_{ir})\xi^r = -L_{ir} A_k{}^r. \tag{4.6}$$

Differentiating (4.5) covariantly along M and using (1.5), we have

(4.7)
$$K_j{}^r(\nabla_k K_{ir}) + K_i{}^r(\nabla_k K_{jr}) = (\theta - \frac{c}{4})(\xi_j A_{kr}\phi_i{}^r + \xi_i A_{kr}\phi_j{}^r).$$

Using the quaite same method as that used to (2.13) from (2.12), we can derive from (4.7) the following :

(4.8)
$$2K_j{}^r\nabla_k K_{ir} = (\theta - \frac{c}{4})\{2n_k\phi_{ij} + (A_{ir}\phi_j{}^r - A_{jr}\phi_i{}^r)\xi_k + (A_{kr}\phi_j{}^r - A_{jr}\phi_k{}^r)\xi_i + (A_{kr}\phi_i{}^r + A_{ir}\phi_k{}^r)\xi_j\},$$

where we have used (4.2) and (4.4).

In the following, we are going to prove $A_{(2)} = 0$. By means of (4.5), we may only consider the case where $\theta - \frac{c}{4} \neq 0$ because it is already seen that θ is constant. By (4.2) we can, using $k = l = 0$, verify that $\nabla_r K_j{}^r = L_{jr}n^r$. Thus, multiplying (4.8) with g^{ki} and summing for k and i, we find

$$K_j{}^r L_{rs}n^s = (\theta - \frac{c}{4})(\phi_{rj}n^r + \xi^s A_{sr}\phi_j{}^r),$$

which together with (4.4) implies that $\xi^s A_{sr}\phi_j{}^r = 0$ and hence

(4.9) $$A\xi = \alpha\xi.$$

Therefore, if we transvect (4.8) with ξ^j and take account of (1.8) and (4.9), then we obtain

(4.10) $$A\phi = \phi A.$$

From this and (4.1) we can prove the followings (cf. [6], [16]) :

(4.11) $$A_{ji}{}^2 = \alpha A_{ji} + \frac{c}{4}(g_{ji} - \xi_j\xi_i),$$

(4.12) $$\nabla_k A_{ji} = -\frac{c}{4}(\xi_j \phi_{ki} + \xi_i \phi_{kj}).$$

By means of (4.10), the equation (4.8) can be written as

$$K_j{}^r \nabla_k K_{ir} = (\theta - \frac{c}{4})(n_k \phi_{ij} + \xi_k A_{ir} \phi_j{}^r + \xi_i A_{kr} \phi_j{}^r).$$

Transforming by $K_h{}^j$ and using (1.7), (4.3), (4.5) and (4.6), we obtain

(4.13) $$\nabla_k K_{ji} = n_k L_{ji} - \xi_k A_{jr} L_i{}^r - \xi_i A_{kr} L_j{}^r - \xi_j A_{ir} L_k{}^r,$$

Differentiating (1.7) with $l_j = 0$ covariantly and using (1.4) and (4.13), we have

(4.14) $$\nabla_k L_{ji} = -n_k K_{ji} + \xi_k A_{jr} K_i{}^r + \xi_i A_{kr} K_j{}^r + \xi_j A_{ir} K_k{}^r,$$

which together (1.8) with $l_j = 0$ and (4.9) implies that

(4.15) $$T_r(AA_{(2)}) = 0, \quad T_r(A^2 A_{(2)}) = 0$$

because of (4.11).

On the other hand, we have $A_{(2)}\xi = 0$ and $T_r A_{(2)} = 0$ and (4.5), the shape operator $A_{(2)}$ has at most three distinct constant eigenvalues $0, \sqrt{\theta - \frac{c}{4}}, -\sqrt{\theta - \frac{c}{4}}$ with multiplicities $1, n-1, n-1$ respectively.

By (4.9), (4.10) and (4.11), we also see that A has at most three distinct constant eigenvalues $\alpha, (\alpha + \sqrt{D})/2$, $(\alpha - \sqrt{D})/2$ with multiplicities $1, r, s$ respectively, where $D = \alpha^2 + c, r + s = 2n - 2$.

Since we have $AA_{(2)} = A_{(2)}A$, it follows that A and $A_{(2)}$ are diagonalizable at the same time. Because of (4.15), we have $(\theta - \frac{c}{4})r(\alpha^2 + c) = 0$. Thus $s = 2(n-1)$ and consequently A has two constant eigenvalues α and $(\alpha - \sqrt{D})/2$ with multiplicities 1, $2(n-1)$ repectively. Accordingly the trace h of A is given by

$$h = n\alpha - (n-1)\sqrt{D}. \tag{4.16}$$

Using (1.5) $\sim$ (1.8), (1.15), (2.1) and (4.1) $\sim$ (4.13), it is seen that (see [11])

$$(h + 3\alpha)(h - \alpha) = 2(n-1)\{2(n+1)\theta - (n+2)c\}, \tag{4.17}$$

$$(\theta - \frac{3}{4}c)(h - \alpha) = 2(n-1)\alpha(\theta - \frac{c}{2}). \tag{4.18}$$

By the way, we have from (4.16) and (4.17)

$$\alpha(\alpha - \sqrt{D}) = 2(\theta - \frac{3}{4}c).$$

Combining (4.16), (4.18) and the last equation, we see that

$$(\theta - \frac{3}{4}c)^2 = \alpha^2(\theta - \frac{c}{2}).$$

From this, (2.5) and (4.5) we have

LEMMA 4.1. *Let M be a real (2n-1)-dimensional ($n > 2$) semi-invariant submanifold of codimension 3 satisfying $dn = 2\theta\omega$ for a certain scalar $\theta(\leq \frac{c}{2})$ in a complex hyperbolic space $H_{n+1}\mathbb{C}$. If the distinguished normal is parallel in the normal bundle, then we have $A_{(2)} = A_{(3)} = 0$.*

Let $N_0(p) = \{\eta \in {T_p}^{\perp}(M) \mid A_\eta = 0\}$ and $H_0(p)$ the maximal J-invariant subspace of $N_0(p)$. As a consequence of Lemma 4.1, we have $A_{(2)} = A_{(3)} = 0$, the orthogonal complement of $H_0(p)$ is invariant under parallel translation with respect to the normal connection because of ${\nabla_j}^{\perp}C = 0$. Thus, by the reduction theorem in [5], [18] and by Lemma 2.2 and Lemma 3.1 we have

THEOREM 4.2. *Let M be a real (2n-1)-dimensional $(n > 2)$ semi-invariant submanfold of codimension 3 in a complex hyperbolic space $H_{n+1}\mathbb{C}$. If the structure vector ξ is an eigenvector for the shape operator in the direction of the distinguished normal and the third fundamental tensor n satisfies $dn = 2\theta\omega$ for a certain scalar $\theta(\leq \frac{c}{2})$, then M is a real hypersurface in a complex hyperbolic space $H_n\mathbb{C}$.*

Owing to Theorem B and Theorem 4.2, we have

THEOREM 4.3. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanfold of codimension 3 in a complex hyperbolic space $H_{n+1}\mathbb{C}$ such that the third fundamental tensor satisfies $dn = 2\theta\omega$ for a certain scalar $\theta(\leq \frac{c}{2})$, where $\omega(X, Y) = g(X, \phi Y)$ for any vectors X and Y on M. Then M has constant eigenvalues corresponding the shape operator A in the direction of distinguished normal and the structure vector ξ is an eigenvector of A if and only if M is locally congruent to one of the type A_0, A_1, A_2 or B in $H_n\mathbb{C}$.*

Finally, we prove

THEOREM 4.4. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ minimal semi-invariant submanifold of codimension 3 in a complex hyperbolic space $H_{n+1}\mathbb{C}$. If the third fundamental form satisfied $dn = 2\theta\omega$ for a certain scalar θ, then M is a minimal real hypersurface in $H_n\mathbb{C}$.*

Proof. Since M is minimal, Lemma 2.2 tells us that the distinguished normal is parallel in the normal bundle. Thus (4.1) $\sim$ (4.11) are valid. Taking the trace of (4.11), we have

$$h_{(2)} - \frac{c}{4}(n-1) = 0$$

because of $\mathrm{Tr}A = 0$. It is contradictory by virtue of $c < 0$. Hence we obtain $\theta = \frac{c}{4}$ and therefore $A_{(2)} = A_{(3)} = 0$. This completes the proof.

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