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MARKOV CHAINS AND THEIR APPLICATION TO HYPERPLANE ARRANGEMENT

SEUNG-HO AHN AND BOONG-BI HAN

Dept. of Mathematics,
Chonnam National University, Kwangju, 500-757, Korea.

Abstract. In this paper, we introduce the MarKov chain and hyperplane arrangement. we prove some properties determined by a hyperplane arrangement and give an example as an application of them.

1. Introduction

An affine hyperplane in $V = \mathbb{R}^n$ is an $(n-1)$ dimensional affine subspace of $\mathbb{R}^n$. A finite set $\mathcal{A}$ of affine hyperplanes in $\mathbb{R}^n$ is called an affine hyperplane arrangement. $\mathcal{A}$ cuts V into regions called chambers. For an element $a \in \mathbb{R}^n$ and $c \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n | ax \leq c\}$ is called a half space. A polyhedron is a finite intersection of half space. A polytope is a bounded polyhedron. A face of a polyhedron is intersections with hyperplane for which the polyhedron is contained in one of two half spaces determined by the hyperplane. Let $\mathcal{A} = \{H_1, \cdots, H_n\}$ be an affine hyperplane arrangement in $\mathbb{R}^d$. The polyhedra determined by $\mathcal{A}$ are called chambers. We denote $\mathcal{F}$ by the collection of all faces of the chambers. Also,we denote by $\mathcal{C}$ the collection of all chambers. The arrangement $\mathcal{A}$ is called central if $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$. For $F \in \mathcal{F}$ and $C \in \mathcal{C}$, FC is defined by the nearest chamber to C having F as a face. The distance between chambers is defined by the number of hyperplanes in $\mathcal{A}$ separating C from FC. The operator $C \longmapsto FC$ will be called the action of F on $\mathcal{C}$. Let w be a probability measure on $\mathcal{F}$. Then a step in the walk is given by

From $C \in \mathcal{C}$, choose F from the measure w and move to FC.

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The random walk started at a chamber C_0 is the process $(C_l)_{l\geq 0}$ with $C_l = F_l \cdots F_1 C_0$, where $F_1, F_2, \cdots$ are picks from w. One can also describe the walk on $\mathcal{C}$ by giving its transition matrix K;

$$K(C, C') = \sum_{FC=C'} w(F).$$

Bidigare, Hanlon and Rockmore (denote by BHK) found all the eigenvalue of K([1]). And K.S.Brown and P.Diaconis got additional result.([6]). In this paper we study the properties of K(Thm 4.1 ,Thm 4.2) and give an example4.5 as application of them.

2. Arrangements of hyperplanes

Throughout this paper $\mathcal{A} = \{H_i\}_{i\in I}$ denotes a finite set of affine hyperplanes in $V = \mathbb{R}^n$. The intersection lattice $\mathcal{S}$ is the set of all nonempty affine subspaces $W \subseteq V$ of the form $W = \bigcap_{i\in I'} H_i$, where $I' \subseteq I$ is an arbitrary subset of I. Then $\mathcal{S}$ is a partially ordered set with the order inclusion. For $H \in \mathcal{A}$, we denote H^+ and H^- by the two open half spaces determined by H. The choice of which one to call H^+ is arbitrary. A face is defined by a nonempty set $F \subseteq V$ of the form

$$F = \bigcap_{i\in I} {H_i}^{\sigma_i},$$

where $\sigma_i \in \{+, -, 0\}$ and ${H_i}^0 = H_i$. Equivalently, if we choose for each i an affine function $f_i : V \longmapsto \mathbb{R}$ such that H_i is defined by $f_i = 0$, then a face is a nonempty set defined by equalities of the form $f_i > 0, f_i < 0$ or $f_i = 0$, one for each $i \in I$. The sequence $\sigma = (\sigma_i)_{i\in I}$ which encodes the definition of F is called the sign sequence of F and is denoted $\sigma(F)$.

The faces such that $\sigma_i \neq 0$ for each $i \in I$ are called chambers. A face F is open relative to its support, which is defined to be the affine subspace

$$\mathrm{supp}F = \bigcap_{\sigma_i(F)=0} H_i.$$

In fact, the faces F with a given support W form the chambers of the hyperplane arrangement $\mathcal{A}_W$ in W consisting of the intersections $H_i \bigcap W$ for those i such that $\sigma_i(F) \neq 0$. The arrangement $\mathcal{A}_W$ is called the restriction of $\mathcal{A}$ to W.

The face poset of $\mathcal{A}$ is the set $\mathcal{F}$ of faces, ordered as follows : Given $F, G \in \mathcal{F}$, we say that F is a face of G and write $F \leq G$ if for each $i \in I$ either $\sigma_i(F) = 0$ or $\sigma_i(F) = \sigma_i(G)$. Two chambers are said to be *adjacent* if they have a common codimension 1 face. The chamber graph of $\mathcal{A}$ has C as vertex set, with edges defined by the adjacency relation. We write $d(C, C')$ for the distance between C and C' in this graph. It is the minimal length l of a gallery

$$C = C_0, \cdots, C_l = C'$$

where C_{i-1} and C_i are adjacent for $1 \leq i \leq l$. It is also equal to the number of hyperplanes in $\mathcal{A}$ separating C from C' (cf[2]§I.4E).

Given $F, G \in \mathcal{F}$, their product FG is the face with sign sequence

$$\sigma_i(FG) = \begin{cases} \sigma_i(F), & \text{if } \sigma_i(F) \neq 0 \\ \sigma_i(G), & \text{if } \sigma_i(F) = 0 \end{cases}.$$

If $T = \bigcap_{i \in I} H_i \neq \emptyset$, we call $\mathcal{A}$ centered with center T. If $\mathcal{A}$ is centered, then the coordinates may be choosen so that each hyperplane contains the origin. In this case we call $\mathcal{A}$ central. We may assume that a hyperplane is essential, that is, that $\bigcap_{i \in I} H_i = \{0\}$. We can correspond $\mathcal{A}$ to a CW-complex $\sum = \sum_{\mathcal{A}}$. The cells of $\sum$ is the intersection $F \neq \{0\}$ with the sphere S^{n-1}. Note that $\sum$ is homeomorphic to S^{n-1}(fig 2-1).

It is also possible to realize $\sum$ as the boundary of a convex polytope $\widehat{\sum}$(fig 2-2). The hyperplane chambers walk can be view as a walk on the maximal cells of $\sum$. Each step consists of a cell e (possibly empty this corresponds to the face $F = \{0\} \in \mathcal{F}$) from some distribution on the cells, and then moving from the current chamber c to the nearest chamber

having e as a face.

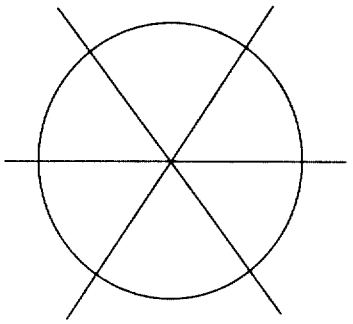

fig 2 - 1

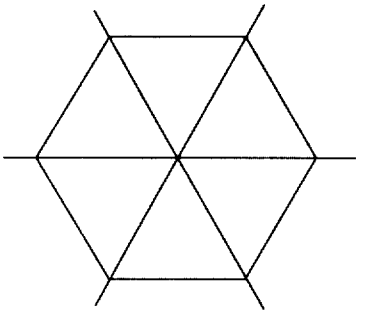

fig 2 - 2

A zonotope in a real vector space V is a Minkowski sum $Z = L_1 + \cdots + L_k$ of line segments, usually taken to be centered at the origin : $L_i = [-v_i, v_i]$. We may assume that the L_i are non-degenerate and that no two are parallel, i.e.,that the v_i are nonzero and pairwise independent. The L_i are then uniquely determined by Z; in fact, there is one for each parallelism class of edges of Z. The set of faces of the boundary of Z having an edge parallel to L_i is called the ith zone of Z. Note that Z is the image of the cube $[-1,1]^k$ under the linear map $\mathbb{R}^k \longmapsto V$ taking the k standard basis vectors $e_1, \cdots, e_k$ to $v_1, \cdots, v_k$. Thus Z is the convex hull of 2^k vectors $\sum_{1 \le i \le k} \pm v_i$, where the signs can be chosen arbitrarily. A simple example of a zonotope is a hexagon, obtained by projecting a cube in $\mathbb{R}^3$ onto a plane. see [3] or [4] for further information about zonotopes. Returning to our central hyperplane arrangement $\mathcal{A}$ in V, there is a zonotope $Z = Z_{\mathcal{A}}$ in the dual space V^*, with one zone for each hyperplane in $\mathcal{A}$, defined as follows: Choose $f_i \in V^*$ such that $H_i = \ker f_i$ and set

$$Z = \sum_{i \in I} [-f_i, f_i].$$

Equivalently, Z is the convex hull of the $2^{|I|}$ elements $\sum_{i \in I} \pm f_i$. The poset of nonempty faces of Z is anti-isomorphic to the face poset $\mathcal{F}$ of $\mathcal{A}$. This is proved in [3], Proposition 2.2.2, and [4],§7.3. It also follows by polarity theory([4],§2.3) from the results about

$\sum$ started above and proved in the appendix, since Z is in fact the polar of the polytope $\widehat{\sum}$ defined in the appendix. Thus Z has one vertex for each chamber C [that vertex being $\sum \sigma_i f_i$, where $\sigma = \sigma(C)$], one edge for each pair of adjacent chambers, ect. In particular, the 1-skeleton of Z is the chamber graph of $\mathcal{A}$. Figure 2-3 shows a simple example,in which $V = \mathbb{R}^2$ and V^* is identified with V.

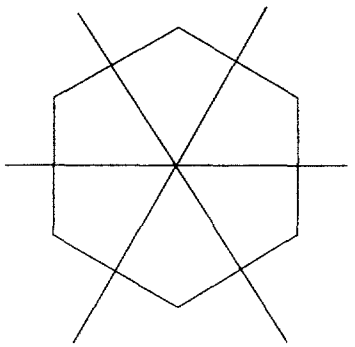

fig 2 - 3

Note that the hyperplane chamber walk can be viewed as a walk on the vertices of Z. Each step consists of choosing a random e of Z from some measure on the faces, and then moving from a vertex v to the unique vertex of e closest to v (in the usual edge-path metric on the 1-skeleton of Z).

REMARK. In some of the literature there is a slightly different definition of the zonotope associated to $\mathcal{A}$. Namely, one considers

$$Z' = \sum_{i \in I} [0, f_i],$$

or, equivalently, Z' is the convex hull of the $2^{|I|}$ elements $\sum_{i \in I} f_i$, where $s \subseteq I$ is an arbitrary subset. Note that Z' is obtained from Z by translating by $\sum_{i \in I} f_i$ and then multiplying by $\frac{1}{2}$. In particular, Z and Z' are combinatorially equivalent.

Möbius function. Finally, we recall the definition of the function $\mu = \mu_{\mathcal{S}}$.(cf. [7] §3.7,or [8] §2.2) This is defined inductively by

$\mu(V,V) = 1$ and, for $W \subsetneqq V$,

$$\mu(W,V) = -\sum_{W \subseteq U \subsetneqq V} \mu(W,U).$$

For example, if $\mathcal{A}$ consists of 3 lines L_i through the origin in $\mathbb{R}^2$.

The Möbius numbers are $\mu(\mathbb{R}^2,\mathbb{R}^2) = 1, \mu(L_i,\mathbb{R}^2) = -1$, and $\mu(\{0\},\mathbb{R}^2) = -\sum_{\{0\} \subseteq U \subsetneqq \mathbb{R}^2} \mu(\{0\},U) = -(\mu(\{0\},\{0\}) + \mu(\{0\}, L_1) + \mu(\{0\},L_2) + \mu(\{0\},L_3)) = 2$.

3. Markov chains

Let $U = \{a_1, \cdots, a_r\}$ be a set of logical possibilities. A probability measure for U is obtained by assigning to each element a_j a positive number $w(a_j)$, called a weight, in such a way that the weights assigned have sum 1. The measure of a subset A of U, defined by $m(A)$,is the sum of the weights assigned to elements of A. Let q be a certain statement relative to U. We change the probability set to the truth set Q of q. The conditional probability measure given q is a probability measure defined on Q, determined by weights.

$$\bar{w}(a_j) = \frac{w(a_j)}{m(Q)}.$$

Let p be a statement relative a set U having truth set P. The probability measure m is defined as $m(P)$.

Let p and q be two statements relative to a set U (q not a self contradiction) The conditional probability of p given q denoted by $\Pr[\, p \mid q \,]$ is the conditional probability measure given q.

PROPOSITION 3.1. *Under the above situations*

$$Pr[\, p \mid q \,] = \frac{Pr[\, p \wedge q \,]}{Pr[q]}$$

where $Pr[\, p \mid q \,]$ *and* $Pr[q]$ *are found from the measure* m.

Proof. Let $Q = \{a \in U | a \text{ is true for } q\}$ and $\Lambda = \{b \in U | b \text{ is true for } p \wedge q\}$. Then

$$\Pr[\, p \mid q \,] = \sum_{b \in \Lambda} \bar{w}(b) = \sum_{b \in \Lambda} \frac{w(b)}{m(Q)} = \frac{\sum_{b \in \Lambda} m(b)}{m(Q)} = \frac{\Pr[\, p \wedge q \,]}{\Pr[q]}.$$

We wish to give a probability measure to describe an experiment which takes place in stages. The outcome at the n-th stage is allowed to depend on the outcomes of the previous stages. It is assumed, however, that the probability for each possible outcome at a particular stage is known when the outcomes of all previous stages are known. From this knowledge we shall construct a possibility space and measure for the over-all experiment. We shall illustrate the construction of the possibility space and measure by a particular example. The general procedure will be clear from this.

EXAMPLE3.2. We choose at random one of two coins A or B. Coin A is a fair coin and B has heads on both sides. The coin chosen is tossed. If a tail comes up a die is rolled. If a head turns up the coin is thrown again. The first stage of the experiment is the choice of a coin. At the second stage, a coin is tossed. At the third stage a coin is tossed or a die is rolled, depending on the outcome of the first two stage. We indicate the possible outcomes of the experiment by a tree. The possibilities for the experiment are $t_1 = (A, H, H), t_2 = (A, H, T), t_3 = (A, T, 1), t_4 = (A, T, 2)$, etc. Each possibility may be identified with a path through the trees. Each path is made up of line segments called *branches*. In the tree we have just given, there are nine paths each having three branches. We know the probability for each outcome at a given stage when the previous stages are known. For example, if outcome A occurs on the first stage and T on the second stage, then the probability of a 1 for the branches and call them *branch probabilities*. We next assign weights to the paths equal to the product of the probabilities assigned to the components of the path. For example the path t_3 corresponds to outcome A on the first stage, T on the second, and 1 on the third. The weight assigned to this path is

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{24}$$

This procedure assigns a weight to each path of the tree and the sum of the weight assigned is 1. The set U of all paths may be considered a suitable possibility space for the consideration of any

statement whose truth value depends on the outcome of the total experiment. The measure assigned by the path weights is the appropriate probability measure.

The above procedure can be carried out for any experiment that takes place in stages. We require only that there be a finite number of possible outcomes at each stage and that we know the probabilities for any particular outcome at the j-th stage, given the knowledge of the outcome for the first $j-1$ stages. For each j we obtain a tree U_j. The set of paths of this tree serves as a possibility space for any statement relating to the first j experiments. On this tree we assign a measure to the set of all paths. We first assign branch probabilities. Then the weight assigned to a path is the product of all branch probabilities on the path. The tree measure are consistent in the following sense. A statement whose truth value depends only on the first j stages may be considered a statement relative to any tree U_i for $i \geq j$. Each of these trees has its own tree measure and the probability of the statement could be found from any one of these measures. However, in every case the same probability would be assigned. Assume that we have a tree for an n stage experiment. Let f_i be a function with domain the set of paths U_n and value the outcome at the j-th stage. Then the functions $f_1, f_2, \cdots, f_n$ are called *outcome functions*. The set of functions $f_1, f_2, \cdots, f_n$ is called a *stochastic process*. (In Markov chain theory it is convenient to denote the first outcome by f_0 instead of f_1.)

A stochastic process for which the outcome functions all have ranges which are of a given finite set is called a finite stochastic process.

The domain of f_n is the set U_n of paths in the tree of n-th stage and the range of it is the set V_n of possible outcomes for the n-th experiment.

A finite stochastic process is an independent process if

(I) For any statement p whose truth value depends only on the outcomes before the n-th,

$$\Pr[f_n = s_j \mid p] = \Pr[f_n = s_j].$$

A finite Markov process is a finite stochastic process such that

(II) For any statement p whose truth value depends only on the outcomes before the n-th,

$$\Pr[f_n = s_j \mid (f_{n-1} = s_i) \wedge p] = \Pr[f_n = s_j \mid f_{n-1} = s_i].$$

The n-th step transition probabilities for a Markov process, denoted by $\mathrm{p}_{ij}(n) = \Pr[f_n = s_j \mid f_{n-1} = s_i]$. A finite Markov chain is a finite Markov process such that the transition probabilities do not depend on n. In this case, they are denoted by p_{ij}. The transition matrix for a Markov chain is the matrix P with entries p_{ij}. The initial probability vector is the vector

$$\pi_0 = \{p_j^{(0)}\} = \{\Pr[f_0 = s_j]\}.$$

We give two examples of Markov chains;

EXAMPLE 3.3. We imagine a particle which moves in a straight line in unit steps. Each step is one unit to the right with probability p or one unit to the left with probability q. It moves unit it reaches one of two extreme points which are called boundary points. The possibilities for its behavior at these points determine several different kinds of Markov chains. The ststes are the possible positions. We take the case of 4 states, states s_1 and s_4 being the boundary states, and s_2, s_3 the interior states.

Assume that the process reaches states s_1 or s_4 it remains there from that time on. In this case, the transition matrix is given by

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Assume now that the particle is reflected when it reaches a boundary point and return to the be relay point from which it came. Thus if it ever step hits to s_1 it goes on the next step back

to s_2. If it hits s_4 it goes on the next step back to s_3. The matrix of transition probabilities becomes in this case

$$P = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

DEFINITION 3.4 (RANDOM WALK). Let P= (p_{ij}) be a transition matrix for a Markov chain with states $U = \{s_1, \cdots, s_r\}$. If $p_{ij} \neq 0$, we say that s_i can go to s_j by one step. If $p_{i_1 j_1} \neq 0, p_{j_1 k_2} \neq 0, \cdots, p_{j_{n-1} k_n} \neq 0$, s_{i_1} can go to s_{k_n} by n-steps. The walk on U is called a random walk.

A relation T is said to be a weak ordering if it satisfies reflexity and transibity. If T is a weak ordering, the relation R defined by xRy when $xTy \wedge yTx$ is the equivalence relation determined by it. Let T be a weak ordering defined on U. Define a new relation T^* on the set of equivalence classes by saying that uTv^* holds if every element of u bears the relation T to every element. This a partial ordering of the equivalence classes induced by T. Let us suppose r individuals are connected through a complex network. Each individual can pass a message on to a subset of the individuals. This we will call direct contact. These messages may be relayed, and relayed again. This will be indirect contact. Let aTb express that the individual a can contact b or that $a = b$. It is easy to see that T is a weak ordering of the set of individuals. It determines the equivalence relation $xTy \wedge yTx$,which may be read as x and y can communication each other, or $x = y$.

For i and j in the same equivalence classes, let N_{ij} be the set of n such that a message starting from member i can in member j's hands at the end of n-steps. We call one step the care which a message is sent from one member to any other member directly.

If the greatest common divisor of elements of N_{ii} is designated d_i, it can be shown that $d_i = d_j = d$ for elements i and j in the same equivalence class. We can see that $a \equiv b(mod\ d)$ for each elements a, b in N_{ij}. Thus for each element $a \in N_{ij}$, $a \equiv t_{ij}$

for some $0 \leq t_{ij} < d$. clearly, $t_{ii} \equiv 0(mod\ d)$ and $t_{ij} = 0$ is an equivalence relation on a equivalence class a cyclic class.

Let $U = \{s_1, s_2, \cdots s_r\}$ be a state of Markov chain. If we interpret $i\mathrm{T}j$ to mean that the process can go from s_i to s_j (not necessarily) one step or $s_i = s_j$, then all the results of the above are applicable. If there is only one cyclic class, then we say the equivalence class is regular, otherwise we that it is cyclic. The minimal elements of the partial ordering of equivalence classes are called ergodic sets. If the ergodic sets are regular, we call it a regular Markov chain. Note that the first case of the above example is a regular Markov chain and the 2nd case is not regular Markov chain because the only one equivalence class has two equivalence classes $\{s_1, s_3\}$ and $\{s_2, s_4\}$. The transition matrix for a regular Markov chain is called a regular transition matrix. By definitions it is easy to see that a transition matrix is regular if and only if P^N has no zero entries for some N. We have the following theorems for regular transition matix.

THEOREM 3.5. *If P is a regular transition matrix then*
(a) The power P^n approach a probability matrix A.
(b) Each row of A is the probability vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$,
(c) The components of α are positive.

Proof. [5]

THEOREM 3.6. *If P is a regular transition matrix and A and α are given in the above theorem, then*
(a)For any probability vector π, $\pi \cdot P^n$ approaches the vector α as n tends to infinity.
(b) The vector α is the unique probability vector such that $\alpha P = \alpha$.

Proof. [5]

4. Main theorems and applications

Returning to a hyperplane arrangement, let $\mathcal{C}$ be the set of all chambers for a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$. Let w be a

probability measure on the set of all faces $\mathcal{F}$. The matrix K with components

$$K(C,C') = \sum_{FC=C'} w(F) \ (C,C' \in \mathcal{C})$$

is the transition matrix of the Markov chain with states $\mathcal{C}$.

$$K^2(C,C') = \sum_{c'' \in \mathcal{C}} K(C,C'')K(C'',C')$$

is the chance of moving from C to C' in two steps. By theorem 3.5, $K^l(C,C')$ tends to a limit $\pi(C')$, independent of the starting chamber C:

$$K^l(C,C') \longmapsto \pi(C'). \text{ as } l \to \infty$$

Note that K is regular. π is called the stationary distribution of the walk. Also, π is the unique probability distribution satisfying

$$\sum_{C \in \mathcal{C}} \pi(C)K(C,C') = \pi(C')$$

for all C' by Theorem 3.6.

We will say that the measure w separates the hyperplanes in $\mathcal{A}$,or simply w is separating if for each $H \in \mathcal{A}$ there is a face F with $F \not\subseteq H$ and $w(F) > 0$.

THEOREM 4.1. *Let $\mathcal{A}$ be hyperplane arrangement in V, let $\mathcal{F}$ be the set of faces, let $\mathcal{S}$ be the intersection poset, and let w be a probability measure on $\mathcal{F}$. Then the matrix K with entries $K(C,C') = \sum_{FC=C'} w(F)$ is diagonalizable. For each $W \in \mathcal{S}$, there is an eigenvalue*

$$\lambda_W = \sum_{F \in \mathcal{F}, F \subseteq W} w(F)$$

with multiplicity

$$m_W = |\mu(W,V)| = (-1)^{codim(W,V)}\mu(W,V)$$

where μ is the Möbius function of $\mathcal{S}$ and codim(W,V) is the codimension of W in V.

THEOREM 4.2. *Let $\mathcal{A}$ be a hyperplane arrangement, let w be a probability measure on the set $\mathcal{F}$ of faces and K be the transition matrix. Then (a) K has a unique stationary distribution π if and only if the measure w is separation (b) Let w be separating. Then*

$$||{K_C}^l - \pi||_{TV} \leq \sum_{H \in \mathcal{A}} {\lambda_H}^l.$$

where $\lambda_H = \sum_{F \in \mathcal{F}, F \subseteq H} w(F)$ and TV is the total variation. Also ${K_C}^l$ is the probability measure;

$${K_C}^l = (K^l(C, C'))_{C' \in \mathcal{C}}.$$

Theorems are proved in [6] but we give a proof of theorem 4.1 here because it is interesting to prove it by using homology theory. Note that we give a proof of some unproved results during the proof. Before the proof we need some preliminaries.

For any finite set S, let $\mathbb{R}S$ denote the vector space of all real linear combinations $\sum_{s \in S} \alpha(s)s$ of elements of S. In particular, we have vector spaces $\mathbb{R}\mathcal{C}$ and $\mathbb{R}\mathcal{F}$ generated by the chambers and faces of a hyperplane arrangement.

Note that $\mathbb{R}\mathcal{F}$ is an $\mathbb{R}$-algebra (the semigroup algebra of $\mathcal{F}$) and $\mathbb{R}\mathcal{C}$ is an $\mathbb{R}\mathcal{F}$-module via the action of faces on chambers.

Given a probability measure w of $\mathcal{F}$, we have an element

$$T = T_w = \sum_{F \in \mathcal{F}} w(F)F.$$

of $\mathbb{R}\mathcal{F}$, which therefore acts as an operator on $\mathbb{R}\mathcal{C}$. Explicitly, given an element $\alpha = \sum_{C \in \mathcal{C}} \alpha(C)C \in \mathbb{R}\mathcal{C}$.

$$T(\alpha) = \sum_{F \in \mathcal{F}, C \in \mathcal{C}} w(F)\alpha(C)FC = \sum_{C' \in \mathcal{C}} \beta(C')C'$$

where

$$\beta(C') = \sum_{FC = C', F, C} w(F)\alpha(C) = \sum_{C \in \mathcal{C}} \alpha(C)K(C, C').$$

Here $K(C,C') = \sum_{FC=C'} w(F)$. Thus if element of $\mathbb{R}\mathcal{C}$ are viewed as row vectors indexed by $\mathcal{C}$, then T acts as right multiplication by the matrix K. In particular, the eigenvectors of T on $\mathbb{R}\mathcal{C}$ are the left eigenvectors of K.

We already have an eigenvector $\sum \pi(C)C$ with eigenvalue $1(= \lambda_{\mathbb{R}^n})$, where π is any stationary distribution for the chamber walk defined by K. Let

$$\partial_0 : \mathbb{R}\mathcal{C} \longmapsto \mathbb{R}$$

be defined by $\partial_0(C) = 1$ for all $C \in \mathcal{C}$. For each $F \in \mathcal{F}$ and for each $r \in \mathbb{R}$. If we define $Fr = r$, then $\mathbb{R}$ is as $\mathbb{R}\mathcal{F}$-module. It is easy to see that ∂_0 is a homomorphism of $\mathbb{R}\mathcal{F}$-modules. Also it is easy to see that $\mathrm{Ker}\partial_0$ is T-invariant, so T(and hence K) will be diagonalizable provided its restriction to Ker ∂_0 is diagonalizable.

Choose an orientation for the ambient space V. This means that we have a rule which associates a sign $\varepsilon = \pm 1$ to each ordered basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, in such a way that two ordered basis have the same (resp. opposite) sign if the matrix relating them has positive(resp. negative) determinant. Similarly, each hyperplane $H \in \mathcal{A}$ is itself a vector space and we choose arbitrary an orientation for it. Given a chamber C and a codimension 1 face A of C, define a sign $[A;C] = \pm 1$ as follows: choose a positively-oriented ordered basis $e_1, \ldots, e_{n-1}$ for $H = \mathrm{supp}A$, choose $v \in C$, and set

$$[A;C] = \varepsilon(e_1, \ldots, e_{n-1}, v).$$

If $A \in \mathcal{F}_1$ and C, C' are two chambers having A as a face, then $[A;C'] = -[A;C]$ because C and C' are on the opposite sides of $H = \mathrm{supp}A$. Define the surjective homomorphism

$$\partial_1 : \mathbb{R}\mathcal{F}_1 \longmapsto \ker\partial_0$$

is defined by $\partial_1(A) = [A;C]C + [A;C']C'$ for $A \in \mathcal{F}_1$, where C, C' are the chambers having A as a face. Define an action of $\mathcal{F}$ on $\mathbb{R}\mathcal{F}_1$. Given $F \in \mathcal{F}$ and $A \in \mathcal{F}_1$, consider the product FA in the semigroup $\mathcal{F}$. If $F \subseteq H = \mathrm{supp}A$, FA in $\mathcal{F}_1$, and we set $F * A = FA$. If $F \not\subseteq H$, then FA is a chamber and we set $F * A = 0$. This product makes $\mathbb{R}\mathcal{F}_1$ an $\mathbb{R}\mathcal{F}$- module. Also we can show that the map $\partial_1 : \mathbb{R}\mathcal{F}_1 \longmapsto \ker\partial_0$ is a homomorphism

of $\mathbb{R}\mathcal{F}$-modules. By induction on the dimension of the ambient spaces V. If we assume that T is diagonalizable on $\mathbb{R}\mathcal{F}_1$, then it is diagonalizable on the homomorphic image $\ker\partial_0$, hence also on $\mathbb{R}\mathcal{C}$. Thus the first part of theorem 4.1 is proves.

Now, we want to define a sequence of homomorphisms ∂_2

$$(*)\cdots\to\mathbb{R}\mathcal{F}_p\xrightarrow{\partial_p}\cdots\xrightarrow{\partial_2}\mathbb{R}\mathcal{F}_1\xrightarrow{\partial_1}\mathbb{R}\mathcal{C}\xrightarrow{\partial_0}\mathbb{R}\to 0$$

(which is eventually zero at the left) where $\mathcal{F}_p$ is the set of faces of codimension P in V. In order to define $\partial_p : \mathbb{R}_p\mathcal{F}\longmapsto\mathbb{R}\mathcal{F}_{p-1}$,we need numbers $[A;B]=\pm 1$ whenever A is a codimension 1 face of B. Choose an orientation for each W in the intersection lattice $\mathcal{S}$. Then if we restrict $\mathcal{A}$ to the support of B, the face B becomes a face of codimension 1, and our chosen orientation give us a number $[A;B]=\pm 1$. We define a linear map $\partial_p : \mathbb{R}\mathcal{F}_p\longmapsto\mathbb{R}\mathcal{F}_{p-1}$ by $\partial_p(A)=\sum_{B>A}[A:B]B$, where $B>A$ means that A is a codimension 1 face of B. We can show that $\partial_{p-1}\partial_p=0$ for each $p\geq 0$([6]). Thus $(*)$ is a chain complex. Next define an action of $\mathcal{F}$ on $\mathbb{R}\mathcal{F}_p$. Given $F\in\mathcal{F}$ and $A\in\mathcal{F}_p$, set

$$F*A=\begin{cases} FA, & \text{if } F\subseteq \mathrm{Supp}A \\ 0, & \text{otherwise.}\end{cases}$$

This makes $\mathbb{R}\mathcal{F}_p$ an $\mathbb{R}\mathcal{F}$-module, which we may decompose according to supports:

$$\mathbb{R}\mathcal{F}_p=\bigoplus_{W\in\mathcal{S}_p}\mathbb{R}\mathcal{C}_W,$$

where $\mathcal{S}_p=\{W\in\mathcal{S}:\mathrm{codim}(W,V)=p\}$ and $\mathcal{C}_W$ is the set of faces with support W. The complex $(*)$ now becomes

$$\cdots\to\bigoplus_{W\in\mathcal{S}_2}\mathbb{R}\mathcal{C}_W\to\bigoplus_{H\in\mathcal{A}}\mathbb{R}\mathcal{C}_H\to\mathbb{R}\mathcal{C}\to\mathbb{R}\to 0.$$

and it is a chain comlpex of $\mathbb{R}\mathcal{F}$-modules.

LEMMA 4.3. *The sequence* $(*)$ *is exact.*

Proof. The zonotope Z associated to $\mathcal{A}$ is a contractible regular cell-complex. The operators ∂_p are precisely the boundary operators of the augmented cellular chain complex of Z ([9]). Since every contractible space is a cyclic, the augmented chain complex $(*)$ is exact.

LEMMA 4.4. *Let*

$$0 \to V_m \xrightarrow{\partial_m} \cdots \to V_1 \xrightarrow{\partial} V_0 \to 0$$

be an exact sequence of finite dimensional vector spaces. Let T_i *be a linear operator on* V_i *such that* $\partial_i T_i = T_{i-1}\partial_i$ *for* $1 \le i \le M$, *and let* φ_i *be the characteristic polynomial of* T_i. *Then*

$$\varphi_0 {\varphi_1}^{-1} \varphi_2 {\varphi_3}^{-1} \cdots = 1.$$

Proof. Let $m = 2$. Then $V_0 = V_1/V_2$. If $V_2 =< v_1, ..., v_d >$, then we can put $V_1 =< \partial v_1, ..., \partial v_d, s_1, ..., s_l >$ and $V_0 =< \partial s_1, ..., \partial s_l >$, where are denoted by $\partial_2 = \partial_1 = \partial$. Let $T_2(v_i) = \sum_{j=1}^{n} a_{ij} v_j$, $1 \le i \le d$, and let $A = (a_{ij})$. Then $\varphi_2 = \det(\lambda I - A)$. $T_i(\partial v_i) = \partial T_2(v_i) = \partial(\sum_{j=1}^{d} a_{ij} v_j) = \sum_{j=1}^{d} a_{ji} \partial(v_j)$. Let $T_1(s_\alpha) = \sum_{\gamma=1}^{d} b_{\gamma\alpha} \partial v_\gamma + \sum_{\beta=1}^{d} c_{\beta\alpha} s_\beta$, $1 \le \alpha \le l$. Let $B = (b_{\gamma\alpha})$ and $C = (c_{\beta\alpha})$. Then the matrix of T_1 is

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

and thus $\varphi_1 = \det(\lambda I - A) \cdot (\lambda I - C)$. Also

$$\begin{aligned} T_0(\partial s_\alpha) = \partial_1 T_1(s_\alpha) &= \partial_1 \Big(\sum_{\gamma=1}^{d} b_{\gamma\alpha} \partial v_\gamma + \sum_{\beta=1}^{l} c_{\beta\alpha} s_\beta\Big) \\ &= \partial_1 \Big(\sum_{\beta=1}^{l} c_{\beta\alpha} s_\beta\Big) = \sum_{\beta=1}^{l} c_{\beta\alpha} \partial_1 s_\beta. \end{aligned}$$

So $\varphi_0 = \det(\lambda I - C)$. Hence we have

$$\varphi_1 = \varphi_2 \varphi_0$$

and so $\varphi_0 \varphi_2{}^{-1} \varphi_2 = 1$. Now assume that the lemma is true for $k < 2m+1$ Let $V_{2m+1} =< v_1, ..., v_d >$ and $V_{2m} =< \partial v_1, ..., \partial v_d, s_1, ..., s_l >$. Consider the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_{2m}/V_{2m+1} & \longrightarrow & V_{2m} & \longrightarrow & V_{2m-1} & \longrightarrow & \cdots \\ & & \downarrow \bar{T}_{2m} & & \downarrow T_{2m} & & \downarrow T_{2m-1} & & \\ 0 & \longrightarrow & V_{2m}/V_{2m+1} & \longrightarrow & V_{2m} & \longrightarrow & V_{2m-1} & \longrightarrow & \cdots \end{array}$$

Then, by the induction hypothesis,

$$\varphi_0 \varphi_1{}^{-1} \cdots \bar{\varphi}_{2m} = 1.$$

Also it is easy to shows that

$$\bar{\varphi}_{2m} = \varphi_{2m+1}{}^{-1} \varphi_{2m}.$$

Therefore $\varphi_0 \varphi_1{}^{-1} \cdots \varphi_{2m} \varphi_{2m+1}{}^{-1} = 1$ and so the conclusion is true for this case. Similarly, we can show the conclusion is true for the case of $2m+2$.

Now, let's return to the proof of the remaining part of theorem 4.1. Applying lemma 4.4 to the operator $T = \sum w(F)F$ on $(*)$, we may assume inductively that we have a decomposition of characteristic polynomial φ_U of T acting on $\mathbb{R}\mathcal{C}_U$ for each $U \neq V$ in $\mathcal{S}$,say

$$\varphi_U(\lambda) = \prod_{W \in \mathcal{S}; W \subseteq U} (\lambda - \lambda_W)^{m(W,U)},$$

because $\varphi_U(\lambda) = \lambda - \lambda_{\{0\}}$ if $U = \{0\} = \bigcap H_i$.

Note that

$$T : \mathbb{R}\mathcal{C}_{\{0\}} \longmapsto \mathbb{R}\mathcal{C}_{\{0\}}$$

is the map $T(r) = rw\{0\}$. Thus the characteristic polynomial is $\lambda - w\{0\} = \lambda - \lambda_{\{0\}}$. By lemma 4.3 and lemma 4.4 the characteristic polynomial $\varphi = \varphi_V$ for T is

$$\varphi(\lambda) = (\lambda - \lambda_V) \prod_{W,U,W\subseteq U\subsetneq V} (\lambda - \lambda_W)^{-(-1)^{codim(U,V)m(W,U)}},$$

because φ_i correspond to characteristic polynomial of the operator between the vector space of codimension $i-1$ generators. If we put $m(V,V) = 1$ and for $W \subsetneq V$,

$$m(W,V) = -\sum_{U;W\subseteq U\subsetneq V} (-1)^{codim(U,V)} m(W,U),$$

then $\varphi(\lambda) = \prod_{W\in\mathcal{S}} (\lambda - \lambda_W)^{m(W,V)}$.

It is enough to show that

$$m(W,V) = (-1)^{\mathrm{codim}(W,V)}\mu(W,V)$$

We prove this by induction on the $\mathrm{codim}(W,V)$. If $W = V$ (i.e $\mathrm{codim}(W,V) = 0$),

$$m(V,V) = \mu(V,V) = 1.$$

We assume that the equation is true for $\mathrm{codim}(W,V) \le k$.

Let $\mathrm{codim}(W,V) = k+1$ Then

$$\begin{aligned} m(W,V) &= -\sum_{W\subseteq U\subsetneq V} (-1)^{\mathrm{codim}(U,V)} m(W,U) \\ &= -\sum_{W\subseteq U\subsetneq V} (-1)^{\mathrm{codim}(U,V)}(-1)^{\mathrm{codim}(W,U)}\mu(W,U) \\ &= -\sum_{W\subseteq U\subsetneq V} (-1)^{\mathrm{codim}(U,V)+\mathrm{codim}(W,U)}\mu(W,U) \\ &= -\sum_{W\subseteq U\subsetneq V} (-1)^{\mathrm{codim}(W,V)}\mu(W,U) \\ &= (-1)^{\mathrm{codim}(W,V)}\mu(W,V) \end{aligned}$$

Thus we proved theorem 4.1 for the central case. We abbreviate the proof for the non-central case. A proof of it is proved in ([6]).

Now we give an example as an application of the theorem

EXAMPLE 4.5. Let $\mathcal{A}$ consist of m lines through the origin in $\mathbb{R}^2$. There are $4m+1$ faces $2m$ chambers, $2m$ rays, and the origin. Suppose,for this exposition, that the measure w is supported on the set of rays. One can then picture the walk as follows; There are $2m$ rooms in a circular house. A mouse lives in the walks $\mathbb{R}$(The rays), occupying these with propensity $w(R)$. At each step of the walk, a cat is in one of the rooms and the mouse picks a wall ;the cat moves to nearest adjacent to that wall. The 1-dimensional spherical complex $\sum$ is $2m$-gon in this example. The chambers of the hyperplane arrangement correspond to the edges of $\sum$, and the rays correspond to the vertices of a $2m$-gon. If the rays are chosen uniformly, $w(R) = \frac{1}{2m}$ for all R, then the stationary distribution π is of course uniform. For general weights, w is separating unless it is supported on a pair $\pm R$ of opposite rays.

The eigenvalues given by Theorem 4.1 are as follows: Each of the m lines L contributes an eigenvalue $\lambda_L = w(R) + w(-R)$ of multiplicity $m_L = -\mu(L, V) = 1$, where $\pm R$ are the rays in L. The whole plane $V = \mathbb{R}^2$ contributes the eigenvalue $\lambda_V = 1$ with multiplicity $\mu(V, V) = 1$. Finally, the trivial subspace $\{0\}$ contributes the eigenvalue $\lambda_{\{0\}} = 0$ with $\mu(\{0\}, V) = m - 1$.

Consider now the bound of Theorem 4.2 in two simple cases. Suppose first that $w(R) = \frac{1}{2m}$ for each R, so that π is uniform, $\pi(C) = \frac{1}{2m}$ for all C. Here $\lambda_L = \frac{1}{m}$ so the bound becomes

$$||K_C{}^l - \pi||_{TV} \leq \sum_L \lambda_L{}^l = \frac{1}{m^{l-1}} = \frac{m}{m^l}$$

It follows that for large m the distance to stationarity is small after two or three steps.

As a second example, suppose one weight is large and the other are small, e,g., $w(R_1) = \frac{1}{2}, w(R) = \frac{1}{2(2m-1)}$ for $R \neq R_1$. Then the bound becomes

$$\begin{aligned} ||K_C{}^l - \pi||_{TV} &\leq \left(\frac{1}{2} + \frac{1}{2(2m-1)}\right)^l + (m-1)\left(\frac{1}{2m-1}\right)^l \\ &= \frac{m^l + m - 1}{(2m-1)^l}. \end{aligned}$$

Again, a few steps suffice for convergence to stationarity, but the result is not quite as quick as in the uniform case.

References

1. T.P.Bidigare, P.Hanlon and D.Rockmore, *A combinatorial description of the spectrum for the Tsetlin library and its generalization to hyperplane arrangements*, preprint, 1997.
2. K.S.Brown, *Buildings*, Springer-Verlag, 1989. (New York-Berlin)
3. A.Bj örner,M.Las Vergnas B.Sturmfels,N.White,and G.Ziegler, *oriented matroids Encyclopedia of Mathematics and its Applications,46*, Cambridge University Press, 1993. (Cambridge)
4. G.M.Ziegler, *Lectures on polytopes*, Graduate Text in Mathematics, 152, Springer-Verlag, 1995. (New York)
5. John G.Kemeny, J.Laurie Snell, *Finite Markov chains*, Spring-Verlag, 1960. (New York, Heidelberg Berlin)
6. Kenneth S. Brown. Persi Diaconis, *Random walk and hyperplane arrangements*, Ann. prabab to appear, 1998.
7. R.P.Stanley, *Enumerative combinatorics,Vol 1*, Corrected reprint of the 1986 orignal. Cambridge studies in Advanced Mathematics, 49., Cambridge university press, Cambridge, 1997.
8. P. Orlik and H. Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen wissenschaften, 300,Springer-verlag, Berlin, 1992.
9. James R.Munkres, *Topology a first coure*, 1975.