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# Relativistic Hydrodynamics and Quasiperiodic Oscillations

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## ABSTRACT

We present preliminary numerical simulations of tilted-disk accretion around a rotating black hole. Our goal is to explore whether hydrodynamic instabilities near the Bardeen-Petterson radius could be responsible for generating moderate-frequency quasi-periodic oscillations in X-ray binaries. We review the relevant general relativistic hydrodynamic equations, and discuss preliminary results on the structure and dynamics of a thin, Keplerian disk.

*Key Words :* accretion, accretion disks – black hole physics

## I. INTRODUCTION

Accretion disks may occur around neutron stars or low-mass ($\sim 3-10\ M_\odot$) black holes in X-ray binaries or around very massive black holes in active galactic nuclei (AGN). Differential Lense-Thirring precession, or frame dragging, can have an important influence on the structure of such disks if the central object is rotating. Bardeen & Petterson (1975) showed that if the angular momentum axes of the outer disk and the central object are not aligned, then the disk is expected to relax to a new configuration, a result we refer to as the Bardeen-Petterson effect. In this new configuration, the inner region of the disk is aligned with the equitorial plane of the central object out to a transition radius, beyond which the disk remains misaligned.

The Bardeen-Petterson transition radius is expected to occur approximately where the differential precession frequency matches the viscous time-scale. The orbital frequency at this transition radius thus becomes another fundamental frequency for the disk. We have conjectured (Fragile et al. 2001) that excess power may be seen near this frequency in the X-ray power spectra of low-mass X-ray binaries (LMXBs) due to direct interactions between the outer and the inner disk. This may provide an explanation for one class of moderate-frequency quasi-periodic oscillations (QPOs). Perturbations in the properties of the gas in the transition region could also excite other frequency modes (Psaltis & Norman, 2000). The Bardeen-Petterson effect could thus generate a spectrum of tightly correlated QPOs, such as have been observed in a large number of low-mass X-ray binaries (Psaltis et al. 1999).

Here we provide a preliminary report on a detailed numerical study of tilted-disk accretion onto rapidly rotating low-mass black holes. Our goal with this study is to characterize the dynamics of the gas in the transition region and look for quasi-periodic behavior. Although we still are in preliminary stages, some interesting behavior is already apparent.

## II. RELATIVISTIC HYDRODYNAMICS

To treat this problem correctly requires some generalization beyond much of the discussion we have heard at this conference. First one must treat the problem in a true four-dimensional general-relativistic curved space time. To do this we utilize the ADM (Arnowitt et al. 1962) metric where by the proper space-time interval between any two events is conveniently split into a three space and a time-like coordinate normal to the hypersurface of the three-space,

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j \ , \quad (1)$$

where $\alpha$ is the lapse function describing the lapse of proper time from one hypersurface to the next and $\beta^i$ is the shift vector specifying how the coordinates connect from one times slice to the next. This metric has the nice feature that the evolution of the system can be represented as a series of space-like hypersurfaces along a time-like coordinate. Hence, it matches one's Newtonian intuition.

Having made this choice, the hydrodynamic equations of motion can be made to assume a form which is reminiscent of their Newtonian counterparts. There are, however, some important modifications of the usually assumed perfect fluid equations of motion.

Since viscosity can play an important role in fixing the Bardeen-Petterson radius, one must incorporate viscousity and heat flow into the stress energy tensor. Thus, we write the stress-energy tensor for an imperfect (i.e. viscous) fluid in contravariant form as

$$T^{\mu\nu} = \rho(1+\epsilon)U^\mu U^\nu + Pg^{\mu\nu} + \Sigma^{\mu\nu} \ , \quad (2)$$

where now $\Sigma^{\mu\nu}$ is the viscous stress which includes both the damping and heat flow terms appropriate to viscous fluids (Mathews & Wilson 2001). Specifically we can write,

$$\Sigma^{\mu\nu} = -\eta h^{\mu\rho} h^{\nu\sigma} W_{\rho\sigma} - \chi\Big(h^{\mu\rho}U^{\nu} + h^{\nu\rho}U^{\mu}\Big)Q_{\rho} - \zeta h^{\mu\nu}U^{\rho}_{\ ;\rho} \ , \tag{3}$$

where $\eta$ is the *shear viscosity coefficient*, $\chi$ is the *heat conduction coefficient*, and $\zeta$ is the *bulk viscosity coefficient.* The tensor $h^{\mu\nu}$ in this equation is a projection operator on the hyperplane normal to $U^{\mu}$,

$$h^{\mu\nu} = g^{\mu\nu} + U^{\mu}U^{\nu} \ . \tag{4}$$

The shear tensor $W^{\mu\nu}$ describes the velocity gradient for viscosity,

$$W_{\mu\nu} = U_{\mu;\nu} + U_{\nu;\mu} - \frac{2}{3}g_{\mu\nu}U^{\sigma}_{\ ;\sigma} \ . \tag{5}$$

The quantity $Q_{\mu}$ in Eq. (3) is called the *heat flow vector*,

$$Q_{\mu} = T_{;\mu} + TU^{\nu}U_{\mu;\nu} \ , \tag{6}$$

where $T$ is the temperature. The second term in this equation is required to keep the entropy increasing in time. It is an odd term in that it drives an acceleration of the fluid.

The three scalar quantities, $\rho$, $\epsilon$, and $P$ in Eq. (2), are the baryon density, internal energy, and the isotropic pressure, respectively. The density, pressure, specific internal energy $\epsilon$ are related - through an equation of state (EOS) index, $\Gamma$,

$$P = \rho\epsilon(\Gamma - 1) \ . \tag{7}$$

Note that $\Gamma$ is not necessarily a constant of density, although in the preliminary results described here we adopt $\Gamma = 5/3$.

The four-velocity $U^{\mu}$ in Eq. (2) satisfies the usual normalization condition:

$$U^{\mu}U_{\mu} = -1 \ . \tag{8}$$

In ADM coordinates it can be related to the coordinate three-velocity of the fluid $V^i$

$$V^i = \frac{\gamma^{ij}U_j}{U^t} - \beta^i \ . \tag{9}$$

For the hydrodynamic equations of motion it is convenient (Wilson 1972) to introduce Lorentz-contracted coordinate state variables:

$$D = \rho W \ , \quad E = \rho\epsilon W \ , \quad S_i = \rho_h W U_i \ , \tag{10}$$

where $\rho_h = [\rho(1+\epsilon) + P]$ and $W \equiv \alpha U^t$. We also let $\gamma \equiv \sqrt{Det(\gamma_{ij})}$

One more important generalization which we introduce is the possibility of allowing the hydrodynamic coordinate grid to move along with the fluid flow independently of the choice for the shift vector $\beta^i$. For our purposes the shift vector will be fixed by the background Kerr metric and is not necessarily aligned with fluid flow near the Bardeen-Petterson radius. Hence, we adopt the Euler-Lagrange approach. This is a powerful tool for either Newtonian of relativistic hydrodynamics which can be of immense value in stabilizing advection on the grid. In this approach, a grid velocity $V_g^i$ can be exploited to minimize fluid motion with respect to the grid and therefore maintain high accuracy in the advection scheme. For some systems, this method rivals other methods we have heard discussed at this conference (Norman 2001).

With the above choices, the fully general-relativistic Euler-Lagrange, Navier-Stokes equations of motion can be specified. First, the conservation of baryon flux ,

$$\Big(\rho U^{\mu}\Big)_{;\mu} = \Big(DV^{\mu}\Big)_{;\mu} = 0 \ , \tag{11}$$

leads to a relativistic continuity equation

$$\dot{D} + D\frac{\dot{\gamma}}{\gamma} + \frac{1}{\gamma}\frac{\partial}{\partial x^i}(\gamma D(V^i - V_g^i)) + \frac{D}{\gamma}\frac{\partial}{\partial x^i}\Big(\gamma V_g^i\Big) = 0 \ . \tag{12}$$

Next, taking the four-divergence of the spatial part of the stress-energy tensor,

$$(T_i^{\ \nu})_{;\nu} = 0 \ , \tag{13}$$

we can obtain an equation for the momentum density

$$\begin{aligned} \dot{S}_i &+ S_i\frac{\dot{\gamma}}{\gamma} + \frac{1}{\gamma}\frac{\partial}{\partial x^j}(S_i(V^j - V_g^j)\gamma) + \frac{S_i}{\gamma}\frac{\partial}{\partial x^j}\Big(\gamma V_g^j\Big) \\ &+ \frac{\alpha \partial P}{\partial x^i} - S_j\frac{\partial \beta^j}{\partial x^i} + \rho_h\frac{\partial \alpha}{\partial x^i} \\ &+ \rho_h\alpha\Big(U^2\frac{\partial \ln\alpha}{\partial x^i} + \frac{U_jU_k}{2}\frac{\partial \gamma^{jk}}{\partial x^i}\Big) \\ &= -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\nu}}\Big(\sqrt{g}\Sigma_i^{\ \nu}\Big) + \Gamma^{\lambda}_{i\nu}\Sigma_{\lambda}^{\ \nu} \ . \end{aligned} \tag{14}$$

Similarly, by projecting the conservation of stress-energy into the frame of the fluid,

$$U^{\mu}(T_{\mu}^{\ \nu})_{;\nu} = 0 \ , \tag{15}$$

we can obtain an equation for the conservation of energy. Ultimately, this gives an equation for internal energy

$$\begin{aligned} \dot{E} &+ \Gamma E\frac{\dot{\gamma}}{\gamma} + \frac{1}{\gamma}\frac{\partial}{\partial x^i}\Big(E(V^i - V_g^i)\gamma\Big) + \frac{\Gamma E}{\gamma}\frac{\partial}{\partial x^i}\Big(\gamma V_g^i\Big) \\ &+ (\Gamma - 1)E\Big[\frac{\dot{W}}{W} + \frac{1}{\gamma W}\frac{\partial}{\partial x^i}\Big(W(V^i - V_g^i)\gamma\Big)\Big] \\ &= \Big(U^{\mu}\Sigma_{\mu}^{\ \nu}\Big)_{;\nu} - \Sigma_{\mu}^{\ \nu}U^{\mu}_{\ ;\nu} \ . \end{aligned} \tag{16}$$

### (a) The Metric Tensor

For the simulations reported here, we have chosen to work in a cylindrical three-space coordinate system. This is a convenient system for considering accretion flow around the central compact object.

For our first studies we have chosen a fixed background Kerr metric. This is reasonable since thin-disk accretion flow contributes negligibly to the mass-energy of the system. This approach does, however, omit the self gravity of the accreting fluid, which we will have to consider later. Nevertheless, this is a reasonable and commonly employed assumption, which greatly simplifies the problem. In the usual Boyer-Lindquist coordinates $r$, $\theta$, and $\phi$, the Kerr metric can be written

$$\begin{aligned} ds^2 &= -(1-2Mr/\Sigma)dt^2 - 4Mar\sin^2\theta/\Sigma dt d\phi \\ &+ \Sigma/\Delta dr^2 + \Sigma d\theta^2 + A\sin^2\theta/\Sigma d\phi^2 \ , \end{aligned} \quad (17)$$

where $a = J/M$ is the black hole angular momentum, $\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2\cos^2\theta$, and $A = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$.

It is straightforward to convert from spherical Boyer-Lindquist coordinates, $r$, $\theta$, and $\phi$, into cylindrical coordinates, $R$, $\phi$, and $z$, via the relations: $R \equiv r\sin\theta$ and $z \equiv r\cos\theta$.

As a further simplification for our first studies we have also assumed that the three-space can be approximated by one which is is conformally flat. That is, we restrict the three-metric tensor to have the form

$$\gamma_{ij} = \psi^4 \hat{\gamma_{ij}} \ , \quad (18)$$

where, in cylindrical coordinates,

$$\hat{\gamma_{ij}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \quad (19)$$

The conformal factor $\psi^4$ is a positive scalar function describing the ratio between the scale of distance in the curved space relative to our flat space manifold. This approximate gauge condition is motivated by the fact that it considerably simplifies the solution without substantially compromising the accuracy. Of particular importance in the present application is the work of Cook et al. (1996) who have shown that this approximation works remarkably well for axisymmetric systems. Thus, we rewrite the three-space metric in the form of Eq. (18). Under these restrictions, the desired metric components are:

$$\begin{aligned} \alpha^2 &= \frac{\Sigma\Delta}{A} \ , \\ \beta^\phi &= \frac{-2arM}{A} \ , \\ \psi^4 &= \frac{A}{r^2\Sigma} \ . \end{aligned} \quad (20)$$

The two remaining shift vectors, $\beta^r$ and $\beta^\theta$, are taken to be zero in this gauge.

### (b) Code Validation and Testing

Our code is based upon the Euler-Lagrange finite differencing methods (e.g. Wilson, 1972). We have utilized an adaptation (cf. Wilson and Mathews 2001) of the monotonicity advection scheme (Van Leer 1979; Roe 1981). The code performs well when compared with the common analytic solutions, such as the 1-D Riemann shock tube, for mildly relativistic velocities. The deviation between the analytic and numerical solution was $\sim 5\%$ for $\gamma \approx 1.5$, but grew to $\gtrsim 20\%$ for $\gamma \gtrsim 2$.

## III. PRELIMINARY RESULTS

Our goals in this first preliminary study was simply to be sure that Lense-Thirring precession is indeed present in the simulations and secondly to obtain a first glimpse of the kind of instabilities likely to be present. Consequently, our first simulation is that of a slightly tilted thin non-viscous accretion disk around a maximally rotating black hole. We have also replaced a full radiative transport calculation, with a volume integration of the Steffan-Boltzmann luminosity. This should give an indication of the anticipated luminosity in the limit of an optically thin disk. Full radiation transport will ultimately be important in a study of tilted-disk accretion. Nevertheless, this work provides a foundation upon which to proceed.

### (a) Initial Model

We adopt an initial model similar to that of Nelson & Papaloizou (2000). That is, we begin with a stationary thin Keplerian accretion disk tilted $\approx 1°$ with respect to the angular momentum axis of the black hole. We then allow this disk to evolve under the influence of accretion flow and Lense-Thirring precession.

For initial conditions, we use the standard thin-disk model as outlined in Pringle (1981). This model considers a thin gaseous disk of surface density $\Sigma$ and central density $\rho_c = \Sigma/H$, where $H$ is the thickness of the disk. The gas in the disk orbits around the black hole with a Keplerian angular velocity $\Omega'_{Kep} = V^{\phi'} = (GM/R'^3)^{1/2}$, where primed coordinates refer to the equitorial plane of the disk. The thickness of the disk is regulated by the pressure, $P$, which supports the gas against the vertical component of gravity.

In the standard disk equation, matter in the disk is assumed to gradually drift inward due to a viscous stress $t_{R\phi} = -\alpha P$, where $\alpha$ is the Shakura & Sunyaev (1973) viscosity parameter. In a steady accretion disk, the inward drift of material is governed by the angular momentum equation,

$$V^{R'} R'\Sigma \frac{\partial}{\partial R'}(\Omega'_{Kep}R'^2) = \frac{3}{2}\frac{\partial}{\partial R'}(\nu\Sigma\Omega'_{Kep}R'^2) \ , \quad (21)$$

where $V^{R'} = -\dot{M}/2\pi R'\Sigma$ is the drift velocity, $\nu = \alpha c_s H$ is the kinematic viscosity coefficient, and $c_s =$

$(dP/d\rho)^{1/2}$ is the local sound speed in the gas.

For a compact body such as a black hole or neutron star, orbital motion becomes unstable in the vicinity of the object and gas falls onto the object with a constant angular momentum. Our standard disk model treats the transition to free fall by assuming $t_{R\phi} = 0$ at the radius of the last marginally stable orbit $R_{ms}$. For a Schwarzschild black hole, $R_{ms} = 6R_{GR}$, while for a maximally rotating Kerr black hole, $R_{ms} = R_{GR}$.

In this model, we initially assume that the orbital energy is dissipated through viscous heating in the disk and is then efficiently radiated away from the two faces of the disk. The initial disk is therefore cold and thin, $c_s \ll R'V^{\phi'}$, $H \ll R'$, and the accretion is slow, $V^{R'} \ll R'V^{\phi'}$.

Although radiation pressure may in fact dominate over gas pressure near the inner edge of the disk (Shakura & Sunyaev 1973), such a configuration may not be stable (Lightman & Eardley 1974; Shakura & Sunyaev 1976). For simplicity, we assume that the gas pressure is dominant throughout (i.e. $P_g \gg P_r$).

### (b) Disk Evolution

For this work we have considered a $10M_\odot$ maximally rotating ($a = M$) black hole as the central compact object. For optimum speed in these initial simulations, we limited the computational grid size to approximately 200,000 grid zones in our full simulation. Because of the very thin, extended nature of the disk being modeled, we had to concentrate most of this resolution in the $R$ and $\phi$ directions. This left us with fairly coarse resolution in the $z$ direction. This also restricted us to relatively modest tilt angles for the disk. Here we consider a tilt angle of $\theta \approx 1°$. Although this should be sufficient for us to study the Bardeen-Petterson effect numerically, such a small tilt angle may not be sufficient to generate observable quasi-periodic oscillations in the resultant X-ray power spectrum as discussed in (Fragile, Mathews & Wilson 2001).

Two remaining parameters need to be specified in order to solve the initial disk equations outlined above. These are the viscosity parameter $\alpha$ and the accretion rate $\dot{M}$. We chose these initially to be $\alpha = 0.1$ and $\dot{M} = \dot{M}_{Edd}$. Note, however, that these parameters are only oresently used to define the disk initial model. They were not included in the fluid equations of motion for this initial study.

From these initial conditions, this disk was evolved under the influence of the Kerr geometry using the hydrodynamic equations. We followed the temporal evolution for about 12 ms. For comparison, the orbital period at the marginally stable orbit for this configuration is $\approx 0.6$ ms, so we evolved material for about 20 orbits. We observed that disk became highly excited and heated as material accumulated near the centrifugal barrier. This is consistent with other results presented at this conference (Molteni 2001). This resulted in a sequence of flares exiting vertically from the disk. A rough volume integration of the internal energy during this instability suggests roughly periodic outbursts in which orders of magnitude variations in the radiated energy can be emitted as this instability develops. All of this is preliminary, but at least suggestive that QPO's may ultimately emerge from this simulation.

## IV. DISCUSSION

An observed association of a QPO with the orbital frequency at the Bardeen-Petterson transition radius could provide important constraints on the mass and angular momentum of the accreting body and possibly on the properties of the accretion disk. Furthermore, identification of such QPOs in several LMXBs could provide information about the relative abundance of tilted-disk accretion systems, which is important in understanding their formation and evolution.

This study has allowed us to get an initial glimpse at tilted-disk accretion. Thus far we have not conclusively identified an instability with the characteristics of the speculated moderate frequency QPO. Nevertheless, we have clearly seen that a tilted accretion disk around a rapidly-rotating compact object will undergo severe twisting due to relativistic frame dragging. Longer, more detailed simulations should allow more thorough characterization of the effects studied here. Further improvements to the code, such as incorporating viscosity and radiative transport, will help determine the observational consequences of our results.

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## REFERENCES

Arnowitt, R., Deser, S., & Misner, C. W. 1962, in Gravitation: An Introduction to Current Research, ed. L. Witten (New York: Wiley), 227

Bardeen, J. M. & Petterson, J. A. 1975, ApJ, 195, L65

Cook, G. B., Shapiro, S. L., & Teukolsky, S. A. 1996, Phys. Rev. D, 53, 5533

Fragile, P. C., Mathews, G. J., & Wilson, J. R. 2001, ApJ, 553, 955

Hawley, J. F., Smarr, L. L., & Wilson, J. R. 1984, ApJ, 277, 296

Lightman, A. P. & Eardley, D. M. 1974, ApJ, 187, L1

Molteni, D. 2001, this conference proceedings

Nelson, R. P. & Papaloizou, J. C. B. 2000, MNRAS, 315, 570

Norman, M. L. 2001, this conference proceedings.

Pringle, J. E. 1981, ARA&A, 19, 137

Psaltis, D., Belloni, T., & van der Klis, M. 1999, ApJ, 520, 262

Psaltis, D. & Norman, C. 2000, ApJ, submitted

Roe, P. L. 1981, J. Comp. Phys, 43, 537.

Shakura, N. I. & Suniaev, R. A. 1976, MNRAS, 175, 613

Shakura, N. I. & Sunyaev, R. A. 1973, A&A, 24, 337

Van Leer, B. 1979, J. Comp Phys., 32, 101.

Wilson, J. R. 1972, ApJ, 173, 431

Wilson, J. R. & Mathews, G. J. 2001, *General Relativistic Numerical Hydrodynamics*, (Cambridge University Press; Cambridge), in press (2001).