평면대수곡선을 기반으로 한 스테레오 비젼
(Stereo Vision based on Planar Algebraic Curves)

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요 약 최근 원추곡선에 기반한 스테레오 비젼에 대한 연구가 주목을 받고 있는데, 이는 원추곡선이 행렬표현, 대응관계설정의 용이성, 그리고 실세계에서 쉽게 찾을 수 있다는 좋은 성질을 갖는다는 점에서 당연한 현상이라 여겨진다. 하지만, 일반적인 고차의 대수곡선에 대한 확장은 아직 성공적으로 이루어지지 못하고 있는 실정이다. 기약인 대수곡선 (irreducible algebraic curve)은 실세계에서 많지 않지만, 직선과 원추곡선은 무수히 많고, 따라서 이들의 곱으로 주어지는 높은 차수의 대수곡선도 무수히 많다. 본고에서는 2이상의 임의의 차수를 가지는 대수곡선을 calibration된 두 대의 카메라를 가지고 스테레오 문제를 푼다. 대응관계설정과 복원, 두 가지 문제 모두에 대한 closed form solution을 제시한다. f_1, f_2, π를 각각 두 이미지 곡선, 공간상의 평면이라 하고, $VC_P(g)$를 평면곡선 g와 점 P로 만들어지는 원추곡선이라 하면, $VC_{O_1}(f_1) = VC_{O_1}(VC_{O_2}(f_2) \cap \pi)$ 의 관계를 이용하여 미지수인 평면 π의 계수들, d_1, d_2, d_3 에 대한 다항 방정식들을 얻을 수 있다. 약간의 변형을 통하여 d_1에 대한 다항 방정식을 얻을 수 있고, 이 방정식의 유일한 양수해는 나머지 과정에서 매우 중요한 역할을 한다. 그 이후에는 $O(n^2)$개의 일변수 다항식에 대한 계산만으로 모든 스테레오 문제를 해결한다. 이는 과거의 여러 개의 다변수 다항식의 공통근을 구해야 했던 방법에 비교된다. synthetic 데이터와 실제 이미지에 대한 실험은 우리의 알고리듬이 옳음을 보여준다.

Abstract Recently the stereo vision based on conics has received much attention by many authors. Conics have many features such as their matrix expression, efficient correspondence checking, abundance of conical shapes in real world. Extensions to higher algebraic curves met with limited success. Although irreducible algebraic curves are rather rare in the real world, lines and conics are abundant whose products provide good examples of higher algebraic curves. We consider plane algebraic curves of an arbitrary degree $n \geq 2$ with a fully calibrated stereo system. We present closed form solutions to both correspondence and reconstruction problems. Let f_1, f_2, π be image curves and plane and $VC_P(g)$ the cone with generator (plane) curve g and vertex P. Then the relation $VC_{O_1}(f_1) = VC_{O_1}(VC_{O_2}(f_2) \cap \pi)$ gives polynomial equations in the coefficient d_1, d_2, d_3 of the plane π. After some manipulations, we get an extremely simple polynomial equation in a single variable whose unique real positive root plays the key role. It is then followed by evaluating $O(n^2)$ polynomials of a single variable at the root. It is in contrast to the past works which usually involve a simultaneous system of multivariate polynomial equations. We checked our algorithm using synthetic as well as real world images.

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1. Introduction

The goal of stereo vision is to recover 3D information of a real world object in space from the pair of its images using a stereo rig. It consists of three stages:

1. Decide on primitives.
2. Solve the correspondence problem between the two images.

3. Reconstruct the object in space.

Traditionally points and lines have been used as primitives in which the difficult problem of correspondence has to be solved [1,4,7,9]. Recently these primitives were replaced by conics and plane algebraic curves of higher degree [2,8,12]. These are more compact and require only set-to-set correspondence which is easier to solve. Other advantages of using conics as primitives have been well described in [2,8,12] and hence we will not discuss on this any further. A number of papers suggested desirabilitiy of algebraic curves of higher degree [2,3,5,8,16,18] as they contain more information, but more work seems needed in order that they be as successful as conics. One obvious fact of usefulness is that a line and a conic together can be regarded as a cubic curve represented by the product of their equations. This indeed was the case in [5] where the authors used the well known cubic invariants to solve pose problem.

In this paper we consider the problems of correspondence and reconstruction of a plane algebraic curve of degree ≥ 2 in space from their two stereo images. We assume that our stereo system consists of a pair of identical pinhole cameras and that the system is fully calibrated. A good introduction to this problem is in the paper by Ma [8]. There is another kind of reconstruction problem which has been addressed in a number of papers, namely projective reconstruction requiring no calibration. A good comparative summary of recent works on this is well described by Rothwell et al. in [15].

We are interested in an exact (euclidean) reconstruction rather than a projective reconstruction. For our problem we construct two polynomials with coefficients which are polynomials in the coefficients of the unknown plane that contains the object curve. The two polynomials must be equal if and only if the pair of the stereo images are those of the same object curve in space. From the constraint of the equality, we derive under a certain non-vanishing condition a number of equations in a single unknown whose solution leads to the recovery of the object plane containing the object curve in space and hence the object curve itself.

The major work along this line is the paper by Ma [8] which presented two methods for solving the stereo problem: the one based on Proposition 1 in his paper involves too many multivariate polynomial equations whereas his alternative method based on elimination theory involves polynomial equations in several variables of high degree as well as certain unresolvable variable. Another paper in the same vein as ours is Xu [18] which did not even fully analyze the case of conics. Our analytical results are tested using synthetic as well as real world images. Most recently the independent work by Lili [6] shows the similar results as ours.

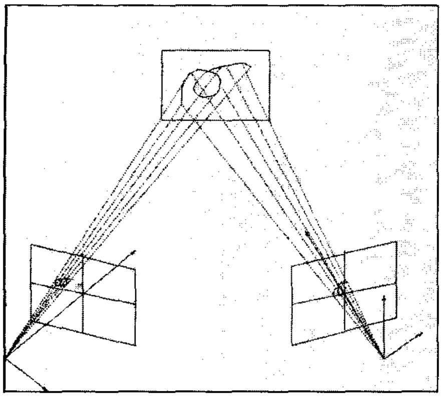
The rest of our paper is organized as follows. Section 2 sets up notation and preliminary observations, section 3 states basic constraints and define a certain number ρ which plays a key role, section 4 presents our main results on reconstruction and correspondence, section 5 reports on experimental results on cubic and quartic curves, one synthetic and the other real world images, and the last section discusses advantages and disadvantages of our method and further research works.

2. Notation and preliminary observation

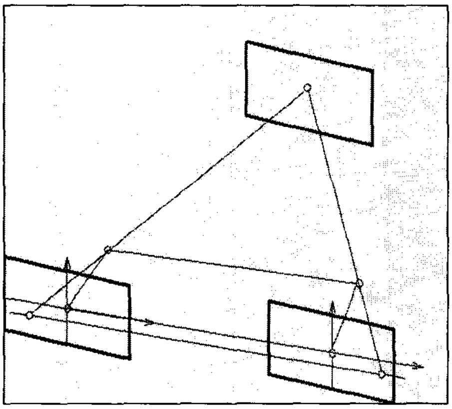
A stereo system usually consists of two identical pinhole cameras set apart at a certain distance and oriented differently (Figure 1a). We assume that our system is fully calibrated. In particular the transformation consisting of rotation and translation from one camera coordinate frame to the other is known.

Our primitives will be plane algebraic curves of degree $n \geq 2$. It is easy to see that a projective transformation of a plane algebraic curve is also an algebraic curve of the same degree. In particular the stereo images of a plane algebraic curve are also plane algebraic curves of the same degree.

For computational simplicity we work with simpler stereo rig in which the two cameras, C_1 and C_2, are oriented identically, that is, their imaging planes lie on the same plane, known as a *non-convergent* system. (Figure 1b)



(a) Convergent Stereo Set-up



(b) Non-convergent Stereo Set-up

Fig. 1 A general stereo set-up and viewing cones.

Conical surfaces

Suppose we are given a pair of stereo images represented by polynomials f_1 and f_2 of degree n in the coordinates of the imaging planes of the two cameras C_1 and C_2 respectively,

$$f_1 = \sum_{i+j+k=n} \binom{n}{ijk} a_{ijk} u_1^i v_1^j \quad (1)$$

$$f_2 = \sum_{i+j+k=n} \binom{n}{ijk} b_{ijk} u_2^i v_2^j \quad (2)$$

where the multinomial coefficient $\binom{n}{ijk} = \frac{n!}{i!j!k!}$ is coupled for computational convenience (Figure 2).

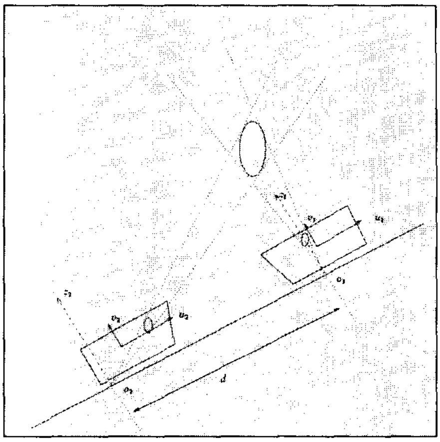


Fig. 2 Nonconvergent Set-up with baseline distance d

Let (x_s, y_s, z_s) denote the coordinate system for the camera C_s. Let d be the distance between the two cameras C_1 and C_2. We assume that the coordinate systems for C_1 and C_2 differ only in the x-axis so that the coordinates (x_1, y, z) in the C_1 frame and the coordinates $(x_1 + d, y, z) = (x_2, y, z)$ in the C_2 frame represent the same point where $y = y_1 = y_2$ and $z = z_1 = z_2$. We also assume that for each camera C_s the z-axis is the optical axis, the origin is the optical center o_s and the imaging plane is the plane parallel to the $x-y$ plane located at the unit distance from the origin in the positive direction.

A planar curve in space and a point P not on the plane containing the curve spans a surface consisting of lines through the point P and points of the curve. This surface will be referred to as the *(conical) surface* (or simply *cone*) spanned by the curve and the point P (*the vertex*). If the vertex is at the origin then such a surface is represented by a homogeneous polynomial in x, y, z. Suppose the polynomial f_1 (1) and f_2 (2) represent the images on the cameras C_1 and C_2 of a planar algebraic curve f in space. Then the surfaces F_1 and F_2 spanned by these curves through the respective vertices at o_1 and o_2 are given as follows.

$$F_1 : \sum_{i+j+k=n} \binom{n}{ijk} a_{ijk} x_1^i y^j z^k = 0 \quad (3)$$

$$F_2 : \sum_{i+j+k=n} \binom{n}{ijk} b_{ijk} (x_1 + d)^i y^j z^k = 0 \quad (4)$$

Note that equation (3) is homogeneous whereas equation (4) is not because it is expressed in the C_1 coordinate frame. If the rotation part of the transformation between C_1 and C_2 was not assumed to be trivial then the equation for F_2 would be more complicated without any theoretical advantage.

We use C_1 coordinate frame for the world coordinate frame and drop the subscript 1 from x_1.

Object plane in space

To reconstruct the object curve in space from its stereo images is to detect the one that projects onto the given stereo images. In the case of conic Quan [14] shows that there are two distinct conics in space which project onto the same pair of the stereo images. In other words there are two distinct planes each containing a conic which projects onto the same pair of images. It turns out that one of them separates the two optical centers and the other does not. Although one might expect the same phenomena for arbitrary algebraic curves it is not true in general. Bézout's Theorem says that the intersection of two planar curves of degrees m, n consists of mn points. Since the degree of a space curve is the number of intersection points of the curve and a generic hyperplane, the intersection of two surfaces of degree m, n is a space curve of degree mn. Hence the intersection of the two cones in our case is a space curve of degree n^2. Moreover the curve has a planar component of degree n by the assumption and the remaining component (not necessarily irreducible) of degree $n^2 - n$ lying on a surface of degree n^2. If $n=2$, the remaining component of the intersection of the two cones is of degree 2 which is a conic and hence is contained in a plane. This explains why there are two distinct planes each containing a conic which projects onto the same pair of images. However, if the planar surface containing the object curve is not transparent then the one having the optical center on the same side is the correct plane and the object curve is the intersection of this plane and the conical surface F_1 or equivalently F_2.

A plane in space is represented by $d_1x + d_2y + d_3z + d_4 = 0$. We assume that the plane is visible from the two optical centers, one of them is the origin of the world coordinate frame. Such a plane cannot pass through any of these points. This implies $d_4 \neq 0$ and hence we can choose d (the distance between the two cameras) for d_4. Thus such a plane can be represented by,

$$\pi: d_1x + d_2y + d_3z + d = 0 \quad (5)$$

The assumption of visibility from the other optical point implies $d_1 \neq 1$ since the point has the coordinates $(-d, 0, 0)$. This fact will be very important in this paper.

Suppose the solution plane π is given by $\mathbf{n}^t \mathbf{x} + d = 0$ where $\mathbf{n} = (d_1, d_2, d_3)^t$ and $\mathbf{x} = (x, y, z)^t$. Let $\gamma = (\mathbf{n}^t \mathbf{o}_1 + d)(\mathbf{n}^t \mathbf{o}_2 + d)$. The two viewpoints $\mathbf{o}_1$ and $\mathbf{o}_2$ are on the same side of π if and only if $\gamma > 0$. Since $\mathbf{o}_1 = (0,0,0)^t$, $\mathbf{o}_2 = (-d, 0, 0)^t$, we have

$$\gamma = (d)(d_1(-d) + d) = d^2(1 - d_1)$$

which implies $\gamma > 0$ if and only if $\rho = 1 - d_1 > 0$.

3. Basic constraints and the reconstruction key

Let Γ be the intersection of the plane π given by equation (5) and the conical surface given by equation (4) and let F_3 be the conical surface spanned by the curve Γ and the vertex at the origin o of the world coordinate frame. In Figure 2, F_1 is the left cone, F_2 is the right cone, and π is the plane containing the big ellipse in the center of figure. Then Γ is the intersection curve of F_2 and π, and F_3 is the cone generated by o and Γ.

Lemma 1 *The conical surface* F_3 *is given by*

$$F_3: \sum_{i+j+k=n} \binom{n}{ijk} \sum_{\alpha+\beta+\gamma=i} \binom{i}{\alpha\beta\gamma} b_{ijk} \rho^{\alpha} (-d_2)^{\beta} (-d_3)^{\gamma} x^{\alpha} y^{\beta+j} z^{\gamma+k} = 0 \qquad (6)$$

where $\rho = 1 - d_1$.

This is proved by eliminating the variable d from equations (4) and (5) which is not difficult.

The following self-evident fact provides the key constraint in this paper.

Lemma 2 *f_1 and f_2 are the images of the same algebraic curve lying on the plane π if and only if*

$$F_1 = \eta F_3 \quad \text{for some constant } \eta \qquad (7)$$

The equation in this lemma has only four unknowns, η, $\rho = 1 - d_1, d_2, d_3$. Later in this section we will derive a quadratic equation in the single variable ρ on which all the other variables depend linearly. This is where we differ from Ma [8] in which his factorization condition provides equations of degree n in the same set of four unknowns with unresolvable constant.

Coefficient comparison

By comparing the coefficients of the term $x^i y^j z^k$ of the two sides of the above equation (7), we obtain the following basic formula.

· **Correspondence checking equations**

$$a_{ijk} = \eta \rho^i \sum_{\substack{0 \le \beta \le j, \\ 0 \le \gamma \le k}} \binom{j}{\beta} b_{i+\beta+\gamma\, j-\beta\, k-\gamma} (-d_2)^{\beta} (-d_3)^{\gamma} \qquad (8)$$

where $\rho = 1 - d_1$

For a closed form solution to the reconstruction problem, we need only determine the plane π containing the object curve in space, or the coefficients d_1, d_2, d_3 of the plane. We do this simply by solving a first few equations in (8). As it turns out, the first two equations express d_2 and d_3 as first degree polynomials of the variable $\rho = 1 - d_1$, and the next three equations are of the form $A\rho^2 + B = 0$. If the images f_1 and f_2 correspond to each other then these three equations must have a pair of common real solutions, one positive and the other negative. The positive solution is the correct one if the planar surface containing the object curve is not transparent.

Here are the details. From formula (8) for the coefficients of the terms x^n, $x^{n-1}y$, $x^{n-1}z$, we obtain the following three formulas.

$$\begin{aligned} a_{n00} &= \eta \rho^n b_{n00} \\ a_{n-1\,1\,0} &= \eta \rho^{n-1} (b_{n-1\,1\,0} - d_2 b_{n00}) \\ a_{n-1\,0\,1} &= \eta \rho^{n-1} (b_{n-1\,0\,1} - d_3 b_{n00}) \end{aligned}$$

We assume $b_{n00} \neq 0$. Since $\rho \neq 0$, this implies $a_{n00} \neq 0$. From the above three equations, we obtain the following two equations.

$$\begin{aligned} \frac{a_{n-1\,1\,0}}{a_{n00}} &= \frac{\rho^{n-1}(b_{n-1\,1\,0} - d_2 b_{n00})}{\rho^n b_{n00}} = \frac{1}{\rho}\left(\frac{b_{n-1\,1\,0}}{b_{n00}} - d_2\right) \\ \frac{a_{n-1\,0\,1}}{a_{n00}} &= \frac{\rho^{n-1}(b_{n-1\,0\,1} - d_3 b_{n00})}{\rho^n b_{n00}} = \frac{1}{\rho}\left(\frac{b_{n-1\,0\,1}}{b_{n00}} - d_3\right) \end{aligned}$$

Let $A_{ijk} = a_{ijk}/a_{n00}$ and $B_{ijk} = b_{ijk}/b_{n00}$. Then we have

$$\begin{aligned} d_2 &= B_{n-1\,1\,0} - \rho A_{n-1\,1\,0} \\ d_3 &= B_{n-1\,0\,1} - \rho A_{n-1\,0\,1} \end{aligned} \qquad (9)$$

Therefore once ρ is determined, the coefficients d_1, d_2, d_3 are determined by these expressions, and hence so is the plane π.

From formula (8) for the coefficients of the terms, $x^{n-2}y^2$, $x^{n-2}yz$ and $x^{n-2}z^2$, we obtain the following three quadratic equations in ρ only.

$$(A^2_{n-1\,1\,0} - A_{n-2\,2\,0})\rho^2 + B_{n-2\,2\,0} - B^2_{n-1\,1\,0} = 0 \qquad (10)$$

$$(A_{n-2\,1\,1} - A_{n-1\,0\,1} A_{n-1\,1\,0})\rho^2 + B_{n-1\,0\,1} B_{n-1\,1\,0} - B_{n-2\,1\,1} = 0 \qquad (11)$$

$$(A^2_{n-1\,0\,1} - A_{n-2\,0\,2})\rho^2 + B_{n-2\,0\,2} - B^2_{n-1\,0\,1} = 0 \qquad (12)$$

All these equations have the form $A\rho^2 + B = 0$. Solve the first non-vanishing equation for ρ. If they all vanish it can be shown that the next equation takes the form $A\rho^3 + B = 0$. More generally we can prove the following lemma.

Lemma 3 *The first non-vanishing equation in ρ*

takes the form $A\rho^r + B = 0$.

Proof See appendix A.

In the last paragraph of the section 2 we assumed the non-transparency for the object plane, hence the negative root is meaningless. It follows from lemma 3 that there exists one or no positive real root ρ of the first non-vanishing equation.

Definition The *reconstruction key* of the two images f_1 and f_2 is the unique positive real root of the first non-vanishing polynomial in lemma 3 if it exists.

If all the equations vanish then the object curve can be shown to be the set of n overlapping lines, that is, a single line in reality. Details are in appendix A. In this case we define the reconstruction key to be ∞.

We considered the case $b_{n00} \neq 0$. Lili [6] stated that ''We can always choose a canonical pair of coordinate system such that $a_{n00}\, b_{n00} \neq 0$.'' However, we found it impossible in general. This phenomenon is due to the difference between rotating the object and rotating the image [11]. However the subset defined by the equation $b_{n00} = 0$ in the space of all coefficient vectors is a lower dimensional subspace and hence the exact probability of $b_{n00} = 0$ is zero. Hence we may assume that $b_{n00} = 0$ does not vanish for real-world images.

4. Correspondence and reconstruction

In this section we present our main results on closed form solutions to the fundamental problems of stereo vision, correspondence and reconstruction of plane algebraic curves of degree $n \geq 2$. From the discussions in the preceding two sections, we have the following theorem.

Theorem Suppose that f_1, f_2 (cf. (1), (2)) are algebraic curves of degree n each on C_1, C_2, respectively. Then f_1 and f_2 are the corresponding images of the same plane algebraic curve in space if and only if the reconstruction key ρ exists and satisfies all the equations in (8). In this case,

(1) the planar surface π is given by (5)

$$(1-\rho)x + d_2 y + d_3 z + d = 0$$

with $d_1 = 1-\rho, d_2, d_3$ defined by (9), and

(2) the object curve is the intersection of π and the conical surface F_1 (cf.(3)).

We formulate the above theorem into an algorithm (Algorithm 1) for checking correspondence and reconstructing the object curve.

Algorithm 1

Suppose two curves f_1 and f_2 are given on the imaging planes of C_1 and C_2,

$$f_1 = \sum_{i+j+k=n} a_{ijk} u_1^i v^j \quad \text{with}\, a_{n00} \neq 0$$
$$f_2 = \sum_{i+j+k=n} b_{ijk} u_2^i v^j \quad \text{with}\, b_{n00} \neq 0$$

1. **For all** i,j,k **do** 1 and 2 below.

$$a_{ijk} := a_{ijk} \binom{n}{ijk}^{-1}, \quad b_{ijk} := b_{ijk} \binom{n}{ijk}^{-1}$$

(to express in the form of (1) and (2))

2. **Put** $A_{ijk} := a_{ijk}/a_{n00}$, $B_{ijk} := b_{ijk}/b_{n00}$
3. **Find** the reconstruction key $\rho \leq \infty$.
   If $\rho = \infty$ **goto** 6.
   If ρ does not exist then **goto** 7.

4 **Put** $d_1 := 1-\rho$ and **compute** d_2 and d_3 using (9).

5 **For all** i,j,k,

$$E_{ijk} := \rho^{n-i} A_{ijk} - \sum_{\beta,\gamma} \binom{j}{\beta}\binom{k}{\gamma} B_{i+\beta+\gamma\ j-\beta\ k-\gamma}\,(-d_2)^{\beta}(-d_3)^{\gamma} \qquad (13)$$

Compute $E = \frac{2}{(n+1)(n+2)} \sum_{ijk} E_{ijk}^2$.

If $E \leq \varepsilon$ (preassigned tolerance) then **return** ''*the two image curves* f_1 *and* f_2 *are the images of the same planar curve in space, and the curve is the intersection of the plane* π *in the above theorem and the conical surface given by (3).*'' **Stop.**

6. **If** $A_{n-1\,1\,0}B_{n-1\,0\,1} - A_{n-1\,0\,1}B_{n-1\,1\,0} < \varepsilon$

then ''*Two lines are correspondent and their common preimage is given intersection of the* *following two planes.''* **Stop.**

$$\begin{cases} \pi_1 : x_1 + A_{n-1\,1\,0}\, y + A_{n-1\,0\,1}\, z = 0 \\ \pi_2 : d + x_1 + B_{n-1\,1\,0}\, y + B_{n-1\,0\,1}\, z = 0 \end{cases}$$

Otherwise goto 7.

7. **Return** ''f_1 *and* f_2 *do not correspond to each other.''* **Stop.**

The above algorithm takes time for the steps 3 and 5. To perform step 3, we have to check equation (10), (11), (12), ... sequentially to see if they vanish. We arrive at the first non-vanishing equation which will be of the form $A\rho^r + B = 0$. In the worst case, they all vanish and we have checked $\frac{(n+1)(n+2)}{2} - 3$ equations in which case the object curve is simply n overlapping lines. (Appendix A) The equation $A\rho^r + B = 0$ has the unique single positive root or none. In the latter case, the answer is ''no'' to the correspondence problem. In the former case, the root is the reconstruction key. The step 5 checks the remaining $\frac{(n+1)(n+2)}{2} - \frac{(r)(r+1)}{2} - 1$ equations at the reconstruction key. We required $b_{n\,0\,0} \neq 0$ which is almost always true theoretically since in the space of all coefficients the measure of the subspace defined by $b_{n\,0\,0} = 0$ is zero.

5. Experimental Results

In this section we show four experimental results, two on cubic curves and two on quartic curves. The first one on cubic is to check correctness of our mathematical analysis, hence it is a synthetic curve. We start with an object curve f given by a cuspidal cubic in space, compute its stereo images f_1 and f_2. We show how to reconstruct f from f_1 and f_2 using our algorithm. This avoids measurement errors in the image processing on real images. The second one uses the real image of a 5.25 inch floppy diskette from which we obtain a cubic by multiplying the central circle by one of the edge line. This experiment is about reconstruction for reducible (singular) curve. For reconstruction for irreducible curve, we use a pair of stereo images of postscript image of spiril curve. Spiril curve is a quartic curve which is a section of torus by plane parallel to the rotational axis. For correspondence test, we take a pair of stereo images with many coins (*i.e.* conics) and by multiplying conics we obtain many quartics. Our third experiment is on these quartics.

5.1 Cuspidal Cubic

Our coordinate systems are as in the preceding sections with the focal length 1. The object curve under consideration is given as the intersection of the cuspidal cylinder and the plane given as follows.

$$f : \begin{cases} x^3 - y^2 = 0 \\ \frac{1}{2}x - 5y - z + d = 0 \end{cases}$$

The image curves are given by,

$$\begin{cases} f_1 : -2du_1^3 - u_1v^2 + 10v^3 + 2v^2 = 0 \\ f_2 : (-2du_2 + 10dv + 2d)^3 - (u_2 - 10v - 2)d^2v^2 = 0 \end{cases}$$

Assuming that f_1, f_2 are given without knowing f in advance we want to show how to recover f using the algorithm described in the proceeding section. Computation shows the preprocessed coefficients a_{ijk} and b_{ijk} are

$$\begin{array}{lll} a_{3\,0\,0} = -2d & a_{2\,1\,0} = 0 & a_{2\,0\,1} = 0 \\ a_{1\,2\,0} = -\frac{1}{3} & a_{1\,1\,1} = 0 & a_{1\,0\,2} = 0 \\ b_{3\,0\,0} = -8d^3 & b_{2\,1\,0} = 40d^3 & b_{2\,0\,1} = 8d^3 \\ b_{1\,2\,0} = -200d^3 - \frac{d^2}{3} & b_{1\,11} = -40d^3 & b_{1\,0\,2} = -8d^3 \end{array}$$

From equation (11),

$$\begin{aligned} (A_{2\,1\,0}^2 - A_{1\,2\,0})\rho^2 + B_{1\,2\,0} - B_{2\,1\,0}^2 &= -\frac{1}{6d}\rho^2 + 25 + \frac{1}{24d} - (-5)^2 \\ &= -\frac{1}{6d}(\rho^2 - \frac{1}{4}) = 0. \end{aligned}$$

$$\text{Hence } \rho = \frac{1}{2} \geq 0 \text{ (reconstruction key)}, \quad d_1 = \frac{1}{2},\ d_2 = -5,\ d_3 = -1$$

Thus the plane π is given by $\frac{1}{2}x_1 - 5y - z = -d$, and hence F_1 is given by,

$$\begin{aligned} -2dx^3 - x_1y^2 + 10y^3 + 2y^2z &= -2dx^3 - 2y^2(\frac{1}{2}x - 5y - z) \\ &= -2d(x^3 - y^2) \end{aligned}$$

Thus the original curve is the intersection of π and F_1 and *hence the original curve is completely* recovered.

5.2 Decomposable Cubic

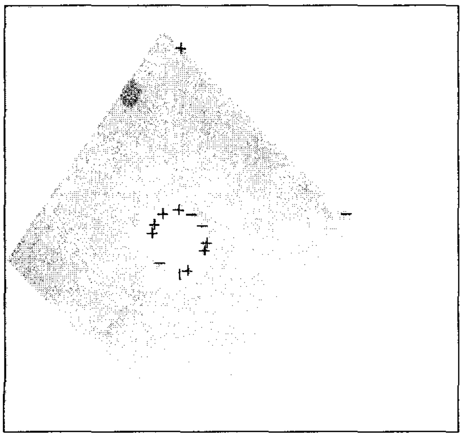
We consider the cubic curve f consisting of a conic c and a line l, that is, $f=lc$. We proceed as follows.

1. Make PostScript File.
2. Take stereo images using two identical cameras (PULNIX TM-7CN line scan cameras with focal length 16mm).
3. Find the best fit curves by minimizing their algebraic distances.
4. Follow the stereo algorithm.

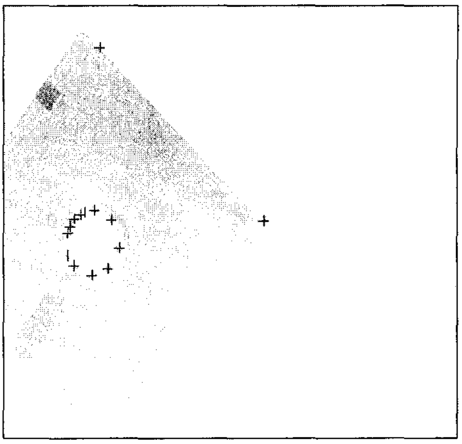
We tested on a 5.25 inch floppy diskette image in figure 3. Our conic is the center circle and the line is an edge. As before we use subscript $s=1,2$ to indicate the image on the camera C_s. The equations of lines and conics are as follows.

$$\begin{cases} l_1: & 0.0563597+1.31038x+y=0 \\ l_2: & 0.131947+1.26278x+y=0 \\ c_1: & -0.0560636+1.61007x-8.884442x^2 \\ & +0.0828622y-0.203743xy-6.541348y^2=0 \\ c_2: & 0.0101725+0.454067x-8.746502x^2 \\ & +0.142588y-0.06565xy-6.466102y^2=0 \end{cases}$$

The resulting cubics are



(a) Left Image



(b) Right Image

Marked points are data points for fitting procedure

Fig. 3 Diskette Images

$$\begin{cases} f_1: & -0.00315973+0.0172785x+1.60908x^2 \\ & -11.642x^3-0.0513935y+1.70717xy \\ & -9.15142x^2y-0.285806y^2 \\ & -8.77539xy^2-6.541348y^3=0 \\ f_2: & 0.00134222+0.0727582x-0.580683x^2 \\ & -11.0449x^3+0.0289865y+0.625463xy \\ & -8.8294x^2y-0.710592y^2 \\ & -8.23094xy^2-6.466102y^3=0 \end{cases}$$

From equation (11), $0.177402-0.182601\ \rho^2=0$. Hence

$$\rho=0.985661, \quad \text{and}$$
$$d_1=0.014339 \quad d_2=0.00820288 \quad d_3=0.0629354$$

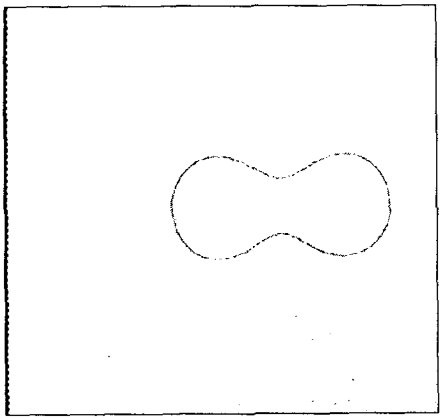
Computation shows the total error $E=0.000171477$. We had placed the object plane parallel to the image planes, that is, π is given by $z=z_0$ and hence $d_1=0$ and $\rho=1$. Our experimental result shows that our method is correct.

5.3 Spiril Curve

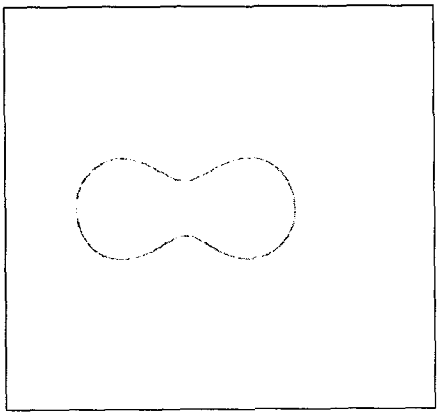
We show an experimental result with an irreducible quartic curve. Our primitive curve is called *spiril curve* [16] which is a section of torus by plane parallel to the rotational axis. The implicit equation of spiril curve is

$$(x^2+y^2+p^2+d^2-R^2)^2=4d^2(x^2+p^2).$$

We followed the procedure given in section 5.2. Figure 4 shows images which are taken by CCD camera. We binarized the image with a certain threshold and used all black pixels for fitting procedure.



(a) Left Image



(b) Right Image

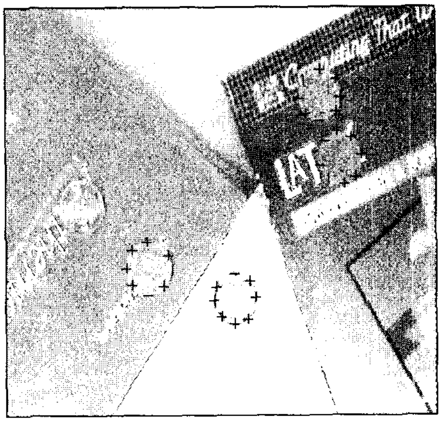
Spiril curve which is a section of torus by plane parallel to the rotational axis.

Fig. 4 Peanut like Quartic Curve

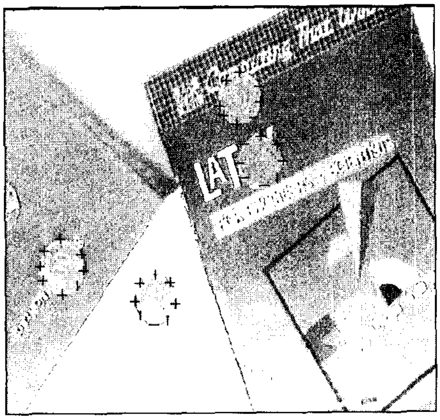
The fitted quartics are

$$
\begin{cases}
f_1: & -0.00150304-0.00512004x+1.55474x^2+29.6532x^3 \\
& +137.630243x^4+0.022119y+0.334202xy+3.6193x^2y \\
& +0.481874x^3y+0.768963y^2+11.5783xy^2+108.517703x^2y^2 \\
& +0.940521y^3+2.331289xy^3+11.897231y^4=0 \\
f_2: & -0.00043553+0.0249506x-0.648047x^2-8.63508x^3 \\
& +137.286683x^4+0.0135216y-0.101736xy+3.12007x^2y \\
& -1.757671x^3y+0.491873y^2-3.66731xy^2+108.900601x^2y^2 \\
& +0.724412y^3-0.299074xy^3+11.455089y^4=0
\end{cases}
$$

From equation (11), $0.132196-0.131411\rho^2=0$ Hence, $\rho=1.00298$, and
$d_1=-0.00298$ $d_2=-0.00407864$ $d_3=-0.069748$



(a) Left Image



(b) Right Image

Marked points are data points which are used in curve-fitting procedure.

Fig. 5 Coins Image for Matching

Computation shows the total error $E=0.00530749$. We had placed the object plane parallel to the image planes, that is, π is given by $z=z_0$ and hence $d_1=0$ and $\rho=1$. Our experimental result shows that our method is correct.

5.4 Decomposable Quartic (Correspondence)

We test our matching constraints for a set of decomposable quartics. See figure 5. In each image, we number coins from 1 to 5 in anti-clockwise.

Table 1 Matching Errors

		q^r_{ij}									
		22	23	24	25	33	34	35	44	45	55
	22	**0.0000001**	0.0000165	0.0002237	0.0005001	0.0000707	0.0001036	0.0003135	0.0007790	0.0012931	0.0019904
	23	0.0000169	**0.0000003**	0.0003130	0.0006119	0.0000180	0.0001759	0.0004145	0.0008951	0.0014037	0.0020816
	24	0.0001635	0.0002749	**0.0000004**	0.0000578	0.0004190	0.0000174	0.0000190	0.0001623	0.0003992	0.0007751
	25	0.0003906	0.0005535	0.0000483	**0.0000001**	0.0007451	0.0001141	0.0000177	0.0000343	0.0001679	0.0004320
q^l_{ij}	33	0.0000548	0.0000125	0.0004058	0.0007151	**0.0000002**	0.0002555	0.0005103	0.0009991	0.0014940	0.0021429
	34	0.0000739	0.0001498	0.0000185	0.0001039	0.0002510	**0.0000008**	0.0000400	0.0002238	0.0004729	0.0008458
	35	0.0002367	0.0003603	0.0000192	0.0000148	0.0005068	0.0000488	**0.0000005**	0.0000682	0.0002253	0.0004987
	44	0.0005369	0.0007070	0.0001360	0.0000308	0.0009004	0.0002278	0.0000833	**0.0000005**	0.0000344	0.0001654
	45	0.0008863	0.0010978	0.0003334	0.0001462	0.0013324	0.0004712	0.0002463	0.0000554	**0.0000002**	030000377
	55	0.0013237	0.0015740	0.0006145	0.0003492	0.0018477	0.0007981	0.0004966	0.0001979	0.0000584	**0.0000003**

Denote c^l_i (c^r_i) for i-th conic in left (right) image. To test the performance of our matching algorithm, we make a quartic q^l_{ij} (q^r_{ij}) by multiplying c^l_i and c^l_j (c^r_i and c^r_j). In table 1, ij means q^l_{ij} (or q^r_{ij}). Diagonal entires are remarkably less than off-diagonal ones. This means that diagonal entries are correctly matched.

6. Concluding Remarks

We presented an algorithm for closed form solutions to the problems of correspondence and reconstruction of a planar algebraic curve in space. We derived our algorithm based on simple observation using elementary algebra in contrast to the past works using the elimination theory of algebraic geometry solving a system of multivariate polynomial equations. However in order to use our algorithm the image polynomials are required to satisfy certain non-vanishing condition. This in not a serious obstacle since the exact probability of satisfying the requirement is 1. There is another method which can overcome this problem[10]. One other restriction was that we worked with non-convergent stereo system, but this is a trivial simplification since data on any stereo system can be easily converted into such a system by simple computations. We have not seriously looked into the problem of numerical stability and robustness of our algorithm which requires further research in the future. It should be also interesting to investigate projective analog of our methods.

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A. Vanishing of the equations

We have three quadratic equations in ρ (10), (11), (12). If all these equations vanish then we obtain the following identities.

$$
\begin{aligned}
A_{n-2\,2\,0} &= A^2_{n-1\,1\,0}\\
A_{n-2\,1\,1} &= A_{n-1\,1\,0}\,A_{n-1\,0\,1}\\
A_{n-2\,0\,2} &= A^2_{n-1\,0\,1}\\
B_{n-2\,2\,0} &= B^2_{n-1\,1\,0}\\
B_{n-2\,1\,1} &= B_{n-1\,1\,0}\,B_{n-1\,0\,1}\\
B_{n-2\,0\,2} &= B^2_{n-1\,0\,1}
\end{aligned}
$$

By equation (8),

$$
\begin{aligned}
\rho^3 A_{n-3\,3\,0} &= \sum_{\beta=0}^{3}\binom{3}{\beta} B_{n-3+\beta\;3-\beta\;0}(-d_2)^\beta\\
&= B_{n-3\,3\,0} - 3B_{n-2\,2\,0}d_2 + 3B_{n-1\,1\,0}d_2^2 - d_2^3\\
&= B_{n-3\,3\,0} - 3B^2_{n-1\,1\,0}d_2 + 3B_{n-1\,1\,0}d_2^2 - d_2^3\\
&= B_{n-3\,3\,0} - B^3_{n-1\,1\,0} + (B_{n-1\,1\,0} - d_2)^3\\
&= B_{n-3\,3\,0} - B^3_{n-1\,1\,0} + A^3_{n-1\,1\,0}\rho^3
\end{aligned}
$$

$$
\begin{aligned}
\rho^3 A_{n-3\,2\,1} &= \sum_{\beta=0}^{2}\sum_{\gamma=0}^{1}\binom{2}{\beta}\binom{1}{\gamma} B_{n-3+\beta+\gamma\;2-\beta\;1-\gamma}(-d_2)^\beta(-d_3)^\gamma\\
&= \sum_{\beta=0}^{2}\binom{2}{\beta} B_{n-3+\beta\;2-\beta\;1}(-d_2)^\beta - d_3\sum_{\beta=0}^{2}\binom{2}{\beta} B_{n-2+\beta\;2-\beta\;0}(-d_2)^\beta\\
&= B_{n-3\,2\,1} - B^2_{n-1\,1\,0}B_{n-1\,0\,1} + \sum_{\beta=0}^{2}\binom{2}{\beta} B^{2-\beta}_{n-1\,1\,0}B_{n-1\,0\,1}(-d_2)^\beta\\
&\quad - d_3\sum_{\beta=0}^{2}\binom{2}{\beta} B^{2-\beta}_{n-1\,1\,0}(-d_2)^\beta\\
&= B_{n-3\,2\,1} - B^2_{n-1\,1\,0}B_{n-1\,0\,1} + B_{n-1\,0\,1}(A_{n-1\,0\,1}\rho)^2\\
&\quad - (B_{n-1\,0\,1} - A_{n-1\,0\,1}\rho)(A_{n-1\,1\,0}\rho)^2\\
&= B_{n-3\,2\,1} - B^2_{n-1\,1\,0}B_{n-1\,0\,1} + A_{n-1\,0\,1}A^2_{n-1\,1\,0}\rho^3
\end{aligned}
$$

$$
\begin{aligned}
\rho^3 A_{n-3\,1\,2} &= \sum_{\beta=0}^{2}\sum_{\gamma=0}^{1}\binom{1}{\beta}\binom{2}{\gamma} B_{n-3+\beta+\gamma\;1-\beta\;2-\gamma}(-d_2)^\beta(-d_3)^\gamma\\
&= \sum_{\gamma=0}^{2}\binom{2}{\gamma} B_{n-3+\gamma\;1\;2-\gamma}(-d_3)^\gamma - d_2\sum_{\gamma=0}^{2}\binom{2}{\gamma} B_{n-2+\gamma\;0\;2-\gamma}(-d_3)^\gamma\\
&= B_{n-3\,1\,2} - B^2_{n-1\,0\,1}B_{n-1\,1\,0} + \sum_{\gamma=0}^{2}\binom{2}{\gamma} B^{2-\gamma}_{n-1\,0\,1}B_{n-1\,1\,0}(-d_3)^\gamma\\
&\quad - d_2\sum_{\gamma=0}^{2}\binom{2}{\gamma} B^{2-\gamma}_{n-1\,0\,1}(-d_3)^\gamma\\
&= B_{n-3\,1\,2} - B_{n-1\,1\,0}B^2_{n-1\,0\,1} + B_{n-1\,1\,0}(A_{n-1\,1\,0}\rho)^2\\
&\quad - (B_{n-1\,1\,0} - A_{n-1\,1\,0}\rho)(A_{n-1\,0\,1}\rho)^2\\
&= B_{n-3\,1\,2} - B_{n-1\,1\,0}B^2_{n-1\,0\,1} + A^2_{n-1\,0\,1}A_{n-1\,1\,0}\rho^3
\end{aligned}
$$

$$
\begin{aligned}
\rho^3 A_{n-3\,0\,3} &= \sum_{\gamma=0}^{3}\binom{3}{\gamma} B_{n-3+\gamma\;3-\gamma\;0}(-d_3)^\gamma\\
&= B_{n-3\,0\,3} - 3B_{n-2\,0\,2}d_3 + 3B_{n-1\,0\,1}d_3^2 - d_3^3\\
&= B_{n-3\,0\,3} - 3B^2_{n-1\,0\,1}d_3 + 3B_{n-1\,0\,1}d_3^2 - d_3^3\\
&= B_{n-3\,0\,3} - B^3_{n-1\,0\,1} + (B_{n-1\,0\,1} - d_3^3)\\
&= B_{n-3\,0\,3} - B^3_{n-1\,0\,1} + A^3_{n-1\,0\,1}\rho^3
\end{aligned}
$$

So, we have four cubic equations in ρ with real coefficients. Since the cubic equation is of the form $A\rho^3+B=0$, the number of (real) solutions is always one in general.

In this way we can prove the following.

Claim If the equations obtained from comparing the coefficients of $x_2^{n-j-k}y^jz^k$ vanish for $j+k \le l-1$ for some l, then

$$
\begin{aligned}
A_{i\,j\,k} &= A^j_{n-1\,1\,0}\,A^k_{n-1\,0\,1}\\
B_{i\,j\,k} &= B^j_{n-1\,1\,0}\,B^k_{n-1\,0\,1}\\
&\text{for } j+k \le l
\end{aligned}
$$

Proof We use the mathematical induction on l. If $l \le 2$, the statement is true.

Assume the statement holds for $l-1 \ge 2$, then by the equation (8),

$$
\begin{aligned}
\rho^l A_{n-l\,j\,k} &= \sum_{\beta=0}^{j}\binom{j}{\beta}(-d_2)^\beta\Big[\sum_{\gamma=0}^{k}\binom{k}{\gamma} B_{n-l+\beta+\gamma\;j-\beta\;k-\gamma}(-d_3)^\gamma\Big]\\
&= \sum_{\beta=1}^{j}\binom{j}{\beta}(-d_2)^\beta\Big[\sum_{\gamma=0}^{k}\binom{k}{\gamma} B_{n-l+\beta+\gamma\;j-\beta\;k-\gamma}(-d_3)^\gamma\Big]\\
&\quad + \sum_{\gamma=0}^{k}\binom{k}{\gamma} B_{n-l+\gamma\;j\;k-\gamma}(-d_3)^\gamma\\
&= \sum_{\beta=1}^{j}\binom{j}{\beta}(-d_2)^\beta\Big[\sum_{\gamma=0}^{k}\binom{k}{\gamma} B^{j-\beta}_{n-1\,1\,0}B^{k-\gamma}_{n-1\,0\,1}(-d_3)^\gamma\Big]\\
&\quad + \sum_{\gamma=1}^{k}\binom{k}{\gamma} B^j_{n-1\,1\,0}B^{k-\gamma}_{n-1\,0\,1}(-d_3)^\gamma + B_{n-l\,j\,k}\\
&= \sum_{\beta=1}^{j}\binom{j}{\beta}(-d_2)^\beta B^{j-\beta}_{n-1\,1\,0}A^k_{n-1\,0\,1}\rho^k\\
&\quad + B^j_{n-1\,1\,0}\,[A^k_{n-1\,1\,0}\rho^k - B^k_{n-1\,0\,1}] + B_{n-l\,j\,k}\\
&= A^k_{n-1\,0\,1}\rho^k\,[A^j_{n-1\,1\,0}\rho^j - B^j_{n-1\,0\,1}]\\
&\quad + B^j_{n-1\,1\,0}\,[A^k_{n-1\,1\,0}\rho^k - B^k_{n-1\,0\,1}] + B_{n-l\,j\,k}\\
&= A^j_{n-1\,1\,0}A^k_{n-1\,0\,1}\rho^l + B_{n-l\,j\,k} - B^j_{n-1\,1\,0}B^k_{n-1\,0\,1}
\end{aligned}
$$

Hence

$$
(A_{n-l\,j\,k} - A^j_{n-1\,1\,0}A^k_{n-1\,0\,1})\,\rho^l = (B_{n-l\,j\,k} - B^j_{n-1\,1\,0}B^k_{n-1\,0\,1}) \quad (14)
$$

From the vanishing condition, we get the following identities.

$$A_{ijk} = A^{j}_{n-1\,1\,0}A^{k}_{n-1\,0\,1}$$
$$B_{ijk} = B^{j}_{n-1\,1\,0}B^{k}_{n-1\,0\,1}$$

This completes the proof.□

Proof of (Lemma 3) By equation (14) in the proof of Claim, if all the equations in ρ
of degree less than l vanish then all the polynomials of degree l are given by equation (14).
Hence the first non-vanishing polynomial is of the form $A\rho^{l}+B=0$.□

Additionally, note that if all the equations vanish, then

$$
\begin{aligned}
f_1: &\sum_{i+j+k=n}\binom{n}{i\ j\ k} b_{ijk}x_1^i y^j \\
&= b_{n00}\sum_{i+j+k=n}\binom{n}{i\ j\ k} A_{ijk}\, x_1^i y^j \\
&= b_{n00}\sum_{i+j+k=n}\binom{n}{i\ j\ k} A^{j}_{n-1\,1\,0}A^{k}_{n-1\,0\,1}\, x_1^i y^j \\
&= b_{n00}\,(x_1+A_{n-1\,1\,0}y+A_{n-1\,0\,1})^n
\end{aligned}
$$

$$
\begin{aligned}
f_2: &\sum_{i+j+k=n}\binom{n}{i\ j\ k} a_{ijk}x_2^i y^j \\
&= a_{n00}\sum_{i+j+k=n}\binom{n}{i\ j\ k} B_{ijk}\, x_2^i y^j \\
&= a_{n00}\sum_{i+j+k=n}\binom{n}{i\ j\ k} B^{j}_{n-1\,1\,0}B^{k}_{n-1\,0\,1}\, x_2^i y^j \\
&= a_{n00}\,(x_2+B_{n-1\,1\,0}y+B_{n-1\,0\,1})^n
\end{aligned}
$$

So, image curves f_1 and f_2 are lines and the viewing cones C_1 and C_2 are planes. The plane π is any plane containing the intersection line of planes C_1 and C_2. Note that in this case the object curve is a line. Though the number of possible planes is infinite, the possible object curve is uniquely determined as the intersection line of two planes C_1 and C_2.

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