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# $T$-FUZZY INTEGRALS OF SET-VALUED MAPPINGS

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Abstract. In this paper we define $T$-fuzzy integrals of set-valued mappings, which are extensions of fuzzy integrals of the single-valued functions defined by Sugeno. And we discuss their properties.

## 1. Introduction

Since Aumann [1] introduced integrals for set-valued mappings, several kinds of integrals for set-valued mappings have been studied by many authors [3,5,6,7]. In fact, they are all based on the classical Lebesgue integral.

Sugeno [9] introduced the concepts for fuzzy measures and fuzzy integrals for single-valued mappings, which are useful in several applied fields like mathematical economics, optimal control theory and engineering. In particular, they have been studied by Ralescu and Adams [8], Wang [11] and others.

On the other hand, using the approaches of Aumann, Zhang and Wang [14] and Zhang and Gou [12,13] extended fuzzy integrals of Sugeno to set-valued mappings and considered many properties.

In this paper, we extend fuzzy integrals of set-valued mappings to $T$-fuzzy integrals of set-valued mappings, which are different from those by Zhang and Guo [12]. And we discuss properties of our integrals.

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In the sequel we will use the following concepts and notations. $(\Omega, \Sigma, m)$ is a probability measure space. Let $\mu : \Sigma \longrightarrow [0,1](:= I)$ be a fuzzy measure in the sense of Sugeno [9], and in addition, we assume $\mu$ satisfies the following two conditions; for $A, B \in \Sigma$

(i) $\mu$ is null-additive, i.e., $\mu(A) = 0$ implies $\mu(A \cup B) = \mu(B)$,

(ii) $\mu \ll m$, i.e., $m(A) = 0$ implies $\mu(A) = 0$.

A set-valued mapping is a mapping $F$ from $\Omega$ to $2^I \setminus \{\emptyset\}$ and it is measurable if its graph is measurable, i.e.,

$GrF = \{(\omega, r) \in \Omega \times I : r \in F(\omega)\} \in \Sigma \times \mathcal{B}$,

where $\mathcal{B}$ is the Borel algebra of $I$.

$S(F)$ is the family of $m$-a.e. measurable selections of $F$. It is known that $S(F)$ is a closed subset of $I^\Omega$.

## 2. $T$-fuzzy integrals of set-valued mappings

In this section we give the definition of fuzzy integrals of set-valued mappings and investigate their properties.

**Definition 2.1[10].** *A binary operation $T$ on $[0,1]$ is called a t-norm if*

*(1) $T(a,1) = a$,*

*(2) $T(a,b) \le T(a,c)$ whenever $b \le c$,*

*(3) $T(a,b) = T(b,a)$,*

*(4) $T(a, T(b,c)) = T(T(a,b),c)$*

*for all $a, b, c \in [0,1]$.*

**Definition 2.2.** *Let $F : \Omega \longrightarrow 2^I \setminus \{\emptyset\}$ be a measurable set-valued mapping and $A \in \Sigma$. The $T$-fuzzy integral of $F$ on $A$ is defined as*

$$(T)\int_A F d\mu = \vee_{\alpha \in I} T(\alpha, \mu(A \cap F_\alpha)),$$

*where* $F_\alpha = \{\omega \in \Omega : F(\omega) \cap [\alpha, 1] \neq \emptyset\}$.

*Remark A.* Definition 2.2 is different from Definition 3.1 [13] and is a generalization of the following definition to set-valued mapping:

The fuzzy integral of a measurable single-valued function $f : \Omega \longrightarrow I$ on $A$ is defined as

$$(T)\int_A f d\mu = \vee_{\alpha \in I} T(\alpha, \mu(A \cap f_\alpha)),$$

where $f_\alpha = \{\omega \in \Omega : f(\omega) \geq \alpha\},\ A \in \Sigma$.

This definition is a generalization of Definition 3.1 [9] and is similar to Definition 2.2 [13].

**Proposition 2.3.** $(T)\int_A F d\mu = (T)\int_\Omega \chi_A \cdot F d\mu$, *where*

$$(\chi_A \cdot F)(\omega) = \begin{cases} F(\omega), \text{ if } \omega \in A \\ \{0\}, \text{ if } \omega \notin A. \end{cases}$$

*Proof.*

$$\begin{aligned} (T)\int_A F d\mu &= \vee_{\alpha \in I} T(\alpha, \mu(A \cap F_\alpha)) \\ &= \vee_{\alpha \in I \setminus \{0\}} T(\alpha, \mu(A \cap F_\alpha)) \vee T(0, \mu(A \cap F_0)) \\ &= \vee_{\alpha \in I \setminus \{0\}} T(\alpha, \mu((\chi_A \cdot F)_\alpha)) \vee T(0, \mu((\chi_A \cdot F)_0)) \\ &= \vee_{\alpha \in I} T(\alpha, \mu((\chi_A \cdot F)_\alpha)) \\ &= (T)\int \chi_A \cdot F d\mu \end{aligned}$$

**Proposition 2.4.** *Let $F$ be a measurable set-valued mapping. If $\mu(A) = 0$, then* $(T)\int_A F d\mu = 0$.

*Proof.* It is clear from Definition 2.2.

By Proposition 2.3, sometimes we only discuss the integral on $\Omega$. And instead of $(T)\int_\Omega F d\mu$, we will write $(T)\int F d\mu$.

**Definition 2.5.** *Let $F$ and $G$ be measurable set-valued mappings. If $F(\omega) = G(\omega)$ for $\omega \in \Omega$, $m$-a.e., then we say $F$ is $m$-a.e. equal to $G$, simply write by $F = G$ $m$-a.e..*

**Lemma 2.6.** *Let $F$ and $G$ be measurable set-valued mappings such that $F = G$ $m$-a.e. Then $\mu(F_\alpha) = \mu(G_\alpha)$.*

*Proof.* Suppose that $H = \{\omega \in \Omega : F(\omega) \neq G(\omega)\}$. Then $m(H) = 0$. Since $\mu \ll m$, $\mu(H) = 0$. Since $\mu$ is null-additive, $\mu(F_\alpha) = \mu(H \cup F_\alpha) = \mu(H \cup G_\alpha) = \mu(G_\alpha)$. This completes the proof.

From Lemma 2.6 we can obtain the following theorem.

**Theorem 2.7.** *Let $F$ and $G$ be measurable set-valued mappings. If $F = G$ $m$-a.e., then $(T)\int F d\mu = (T)\int G d\mu$.*

**Theorem 2.8.** *Let $F : \Omega \longrightarrow 2^I \setminus \{\emptyset\}$ be a measurable set-valued mapping with closed values. Then the following hold:*

*(i) $(T)\int F d\mu = T(\beta, \mu(F_\beta))$ for some $\beta \in I$.*

*(ii) $\sup_{f \in S(F)} f(\omega) \geq \beta$ for all $\omega \in F_\beta$.*

*Proof.* (i) Let $(T)\int F d\mu = A$. Then there exists $\{\alpha_n\} \subset I$ such that $\lim_n\{T(\alpha_n, \mu(F_{\alpha_n}))\} = A$. Without loss of generality, we can choose a subsequence of $\{\alpha_n\}$ monotonically converging to some $\beta \in I$. Without confusion, we also denote it as $\{\alpha_n\}$. Since $\alpha_n \longrightarrow \beta$, monotonically, $\alpha_n \nearrow \beta$ or $\alpha_n \searrow \beta$. If $\alpha_n \nearrow \beta$, then $F_{\alpha_n} \searrow \cap_{\alpha_n} F_{\alpha_n} = F_\beta$. Thus $\lim \mu(F_{\alpha_n}) = \mu(\cap F_{\alpha_n}) = \mu(F_\beta)$. If $\alpha_n \searrow \beta$, then $F_{\alpha_n} \nearrow \cup_{\alpha_n} F_{\alpha_n} \subset F_\beta$. Therefore

$$\begin{aligned} A &= \lim_n T(\alpha_n, \mu(F_{\alpha_n})) \\ &\leq \lim_n T(\alpha_n, \mu(F_\beta)) \\ &= T(\beta, \mu(F_\beta)) \\ &\leq \vee_{\alpha \in I} T(\alpha, \mu(F_\alpha)) \\ &= A. \end{aligned}$$

Hence $A = T(\beta, \mu(F_\beta))$.

(ii) Since $(T)\int F d\mu = T(\beta, \mu(F_\beta))$ for some $\beta \in I$, for each $\omega \in F_\beta$, by the Castaing representation [3] there exists $\{f_n\} \subset S(F)$ such that $\lim f_n(\omega) \geq \beta$. We can choose a subsequence $\{f_{n_j}(\omega)\}$ of $\{f_n(\omega)\}$ such that $\{f_{n_j}(\omega)\}$ is monotone increasing or monotone decreasing. Suppose that $\{f_{n_j}(\omega)\}$ is monotone increasing. Then $f_{n_j}(\omega) \nearrow \lim f_n(\omega)$. Therefore

$$\sup_{f\in S(F)} f(\omega) \geq \sup_{n_j} f_{n_j}(\omega) = \lim f_n(\omega) \geq \beta.$$

In case that $\{f_{n_j}(\omega)\}$ is monotone decreasing, we can similarly show that $\sup_{f\in S(F)} f(\omega) \geq \beta$. Hence $\sup_{f\in S(F)} f(\omega) \geq \beta$ for all $\omega \in F_\beta$.

**Theorem 2.9.** *Let $F : \Omega \longrightarrow 2^I \setminus \{\emptyset\}$ be a measurable set-valued mapping with closed values. Then*

$$(T)\int F d\mu = (T)\int \sup_{f\in S(F)} f d\mu.$$

*Proof.* Let $A_\alpha = \{\omega : (\sup_{f\in S(F)} f)(\omega) \geq \alpha\}$ for $\alpha$, $0 \leq \alpha \leq 1$. Then $A_\alpha \subset F_\alpha$. In fact, for $\omega \in A_\alpha$, $(\sup_{f\in S(F)} f)(\omega) \geq \alpha$. We can choose a sequence $\{f_n\}$ in $S(F)$ such that $f_n(\omega) \longrightarrow (\sup_{f\in S(F)} f)(\omega)$ as $n \longrightarrow \infty$. Since $f_n(\omega) \in F(\omega)$ and $F(\omega)$ is closed, $(\sup_{f\in S(F)} f)(\omega) \in F(\omega)$. Since $(\sup_{f\in S(F)} f)(\omega) \geq \alpha$, $F(\omega) \cap [\alpha, 1] \neq \emptyset$, i.e., $\omega \in F_\alpha$. Since $A_\alpha \subset F_\alpha$ for each $\alpha \in [0,1]$, $(T)\int \sup_{f\in S(F)} f d\mu \leq (T)\int F d\mu$.

Let's show the reverse inequality. By Theorem 2.8 (i) we can choose $\beta \in I$ such that $(T)\int F d\mu = T(\beta, \mu(F_\beta))$. Then $\mu(F_\beta) \geq \beta$ or $\mu(F_\beta) < \beta$. Suppose that $\mu(F_\beta) \geq \beta$. By Theorem 2.8 (ii) $(\sup_{f\in S(F)} f)(\omega) \geq \beta$ for all $\omega \in F_\beta$. Thus $\{\omega : (\sup_{f\in S(F)} f)(\omega) \geq \beta\} \supset F_\beta$. Therefore

$$\begin{aligned}(T)\int \sup_{f\in S(F)} f d\mu &= \vee_{\alpha\in I} T(\alpha, \mu(\{\omega : (\sup_{f\in S(F)} f)(\omega) \geq \alpha\})) \\ &\geq T(\beta, \mu(\{\omega : (\sup_{f\in S(F)} f)(\omega) \geq \beta\})) \\ &\geq T(\beta, \mu(F_\beta)) \\ &= (T)\int F d\mu.\end{aligned}$$

Suppose that $\mu(F_\beta) < \beta$. By Theorem 2.8 (ii) $(\sup_{f\in S(F)} f)(\omega) \geq \beta$ for all $\omega \in F_\beta$. Thus $\{\omega : (\sup_{f\in S(F)} f)(\omega) \geq \beta\} \supset F_\beta$. Therefore

$$\begin{aligned}(T)\int \sup_{f\in S(F)} f d\mu &= \vee_{\alpha\in I} T(\alpha, \mu(\{\omega : (\sup_{f\in S(F)} f)(\omega) \geq \alpha\}))\\ &\geq T(\beta, \mu(\{\omega : (\sup_{f\in S(F)} f)(\omega) \geq \beta\}))\\ &\geq T(\beta, \mu(F_\beta))\\ &= (T)\int F d\mu.\end{aligned}$$

This completes the proof.

**Corollary 2.10.** *Let $F : X \longrightarrow 2^I \setminus \{\emptyset\}$ be a measurable set-valued mapping with closed values. Then there exists $g \in S(F)$ such that $(T)\int F d\mu = (T)\int g d\mu$.*

*Proof.* Since $S(F)$ is a closed subset of $I^\Omega$, $S(F)$ is also compact. Thus $\sup_{f\in S(F)} f = g$ for some $g \in S(F)$. By Theorem 2.9 $(T)\int F d\mu = (T)\int \sup_{f\in S(F)} f d\mu = (T)\int g d\mu$.

**Example 2.11.** *Let $T(a,b) = a\cdot b$ and define a set-valued mapping $F : [0,1] \longrightarrow 2^I$ by*

$$F(\omega) = \begin{cases} \frac{3}{4} & \text{if } \omega \in [0,\frac{1}{4}) \cup (\frac{1}{2}, 1] \\ [\frac{1}{2}, 1] & \text{if } \omega = \frac{1}{4}, \omega = \frac{1}{2} \\ \{\frac{1}{2}, 1\} & \text{otherwise.} \end{cases}$$

*Then $S(F)$ is compact and $(T)\int F d\mu = \frac{9}{16} = (T)\int f d\mu$, where $f : [0,1] \longrightarrow [0,1]$ is a function such that*

$$f(\omega) = \begin{cases} \frac{9}{16} & \text{if } \omega \in [0,\frac{1}{4}] \cup [\frac{1}{2}, 1] \\ 1 & \text{otherwise.} \end{cases}$$

**Propsoition 2.12.** *Let $F : \Omega \longrightarrow 2^I \setminus \{\emptyset\}$ be a measurable set-valued mapping with closed values and $c \in [0,1]$. Then*

$$(T)\int (c \vee F) d\mu = (T)\int c d\mu \vee \ (T)\int F d\mu.$$

*Proof.* By Corollary 2.10 $(T)\int(c\vee F)d\mu = (T)\int g d\mu$ for some $g\in S(c\vee F)$. Since $(c\vee F)(x)=\{c\vee f(x)|f(x)\in F(x)\}$,

$$S(c\vee F)=\{g|g(x)\in(c\vee F)(x)\}=\{g|g(x)\in\{c\vee f(x)|f(x)\in F(x)\}\}.$$

Therefore

$$\begin{aligned}(T)\int(c\vee F)d\mu &= (T)\int g d\mu\\ &= (T)\int(c\vee f)d\mu \qquad \text{for some } f\in S(F)\\ &= (T)\int c d\mu \vee\ (T)\int f d\mu \qquad \text{by Theorem 2.1[11]}\\ &= (T)\int c d\mu \vee\ (T)\int F d\mu.\end{aligned}$$

**Proposition 2.13.** *Let $F:\Omega\longrightarrow 2^I\setminus\{\emptyset\}$ be a measurable set-valued mapping with closed values. If $F_1\subset F_2$ (i.e. $F_1(\omega)\subset F_2(\omega)$ for each $\omega\in\Omega$), then $(T)\int F_1 d\mu\le (T)\int F_2 d\mu$.*

*Proof.* By Corollary 2.10 there exists an $f_1\in S(F_1)$ such that $(T)\int F_1 d\mu=(T)\int f_1 d\mu$. Let

$$GrE=GrF_2\cap\{(\omega,i)\in\Omega\times I|f_1(\omega)\le i\le 1\}.$$

Then $GrE\neq\emptyset$. Since $F_2$ is measurable, $GrE$ is measurable. Therefore $E$ is a measurable set-valud mapping with closed values. Since $E$ is measurable, there exists $f_2\in S(E)$. Thus $f_2\in S(F_2)$ and $(T)\int f_1\le (T)\int f_2 d\mu$. Hence

$$\begin{aligned}(T)\int F_1 d\mu &= (T)\int f_1 d\mu\\ &\le (T)\int f_2 d\mu\\ &\le (T)\int F_2 d\mu.\end{aligned}$$

## 3. Convergence theorems

In this section we give the convergence theorems for set-valued mappings.

Let $\{A_n\} \subset 2^I$ be a sequence. Then $\limsup A_n = \{\omega : \omega = \lim_{k\longrightarrow\infty} \omega_{n_k}, \omega_n \in A_n\}$ and $\liminf A_n = \{\omega : \omega = \lim \omega_n, \omega_n \in A_n\}$ are closed sets [2]. If $\limsup A_n = \liminf A_n = A$, then we say $\{A_n\}$ is convergent to $A$.

Using above definition, let $\{F_n\}$ be a sequence of set-valued mappings, we can define $\limsup F_n$, $\liminf F_n$ and $\lim F_n$ by pointwise way. For example:

$(\limsup F_n)(\omega) = \limsup F_n(\omega)$ for $\omega \in \Omega$, $m$-a.e.

**Theorem 3.1 (Fatou's Lemma).** *Let $\{F_n\}$ be a sequence of measurable set-valued mappings with closed values. Then the following hold:*

*(i)* $\limsup\ (T) \int F_n d\mu \leq (T) \int \limsup F_n d\mu$.

*(ii)* $(T) \int \liminf F_n d\mu \leq \liminf\ (T) \int F_n d\mu$.

*Proof.* (i) Let $y = \limsup\ (T) \int F_n d\mu$ and $y_n = (T) \int F_n d\mu$. Then there exist a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y = \lim y_{n_k}$. By Corollary 2.10 there exist $f_n \in S(F_n)$ such that $y_n = (T) \int f_n d\mu$. Thus $y_{n_k} = (T) \int f_{n_k} d\mu$. Since $\{f_{n_k}\} \subset I^\Omega$, there exists a subsequence $\{f_m\}$ of $\{f_{n_k}\}$ such that $\{f_m\}$ is convergent. So $\lim y_m = \lim (T) \int f_m d\mu = y$. Therefore

$$\begin{aligned} y &= \lim\ (T) \int f_m d\mu \\ &= (T) \int \lim f_m d\mu \qquad \text{by Theorem 2.3 [11]} \\ &\leq (T) \int \limsup F_n d\mu. \end{aligned}$$

(ii) Let $y = (T) \int \liminf F_n d\mu$. Then by Corollary 2.10 there exists $f \in S(\liminf F_n)$ such that $y = (T) \int f d\mu$. Write $I^\infty = I \times I \times \cdots$, then $I^\infty$ is a complete metric space (with the metric induced by the usual product topology). For each $\omega \in \Omega$, define $G(\omega)$ of $I^\infty$ by

$$G(\omega) = \{(y_1, y_2, \cdots) : y_n \in F_n(\omega), \lim y_n = f(\omega)\}.$$

Then $G$ is a measurable set-valued mapping [14]. By the Castaing representation [3] there exists $g \in S(G)$. In fact $g$ is a sequence of measurable functions $\{f_n\}$ such that $f_n \in S(F_n)$. Moreover $\lim f_n = f$. Hence $y = (T)\int f d\mu = \lim\ (T)\int f_n d\mu \leq \liminf\ (T)\int F_n d\mu$.

*Remark B.* The proof of Theorem 3.1 (ii) is similar to the proof of Theorem 3.2 [14].

From Fatou's Lemma, we can obtain the following Lebesgue Convergence Theorem.

**Theorem 3.2.** *Let $\{F_n\}$ be a sequence of measurable set-valued mappings with closed values and $F$ a measurable set-valued mapping with closed values. If $\lim F_n = F$, then* $\lim\ (T)\int F_n d\mu = (T)\int F d\mu$.

*Proof.* Since $F = \liminf F_n = \limsup F_n$,

$$\begin{aligned}(T)\int F d\mu &= (T)\int \liminf F_n d\mu \\ &\leq \liminf\ (T)\int F_n d\mu \qquad \text{by Theorem 3.1(ii)} \\ &\leq \limsup\ (T)\int F_n d\mu \\ &\leq (T)\int \limsup F_n d\mu \qquad \text{by Theorem 3.1(i)} \\ &= (T)\int F d\mu.\end{aligned}$$

Hence $\lim\ (T)\int F_n d\mu = (T)\int F d\mu$.

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