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HEDGING OPTION PORTFOLIOS WITH TRANSACTION COSTS AND BANDWIDTH

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ABSTRACT. Black-Scholes equation arising from option pricing in the presence of cost in trading the underlying asset is derived. The transaction cost is chosen precisely and generalized to reflect the trade in the real world. Furthermore the concept of the bandwidth is introduced to obtain the better rehedging. The model with bandwidth derived in this paper can be used to calculate the more accurate option price numerically even if it is nonlinear and more complicated than the models shown before.

1. INTRODUCTION

The option pricing model, derived by Black and Scholes [3], assumes perfect markets. This assumption is obviously unrealistic. In recent years, there are several works investigating the effects of incorporating more realistic assumptions about the underlying markets in which trading occurs into the option valuation process. The most important effects are arising from the inclusion of nonzero transaction costs of the underlying assets. This approach was started by Leland [9] and extended by Boyle & Vorst [4], Hoggard, Whalley & Wilmott [8], Avellaneda & Panas [2], Toft [10], Whalley & Wilmott [11], and Henrotte [6]. The first five of these assume hedging takes place at given discrete time intervals and the last two assume flexible but prescribed trading rules. These involve a band around the ideal value of Δ, within which the number of assets actually held in the portfolio is allowed to vary. These can be also expressed by partial differential equations for the value of option, V hereafter, which are similar to the Black-Scholes equation except an extra term representing the effect of the transaction costs.

$$V_t + \frac{\sigma^2 V^2}{2} V_{SS} - rV + rSV_S = K(\nu, S),$$

where ν is generally a nonlinear functional of the Gamma, $\Gamma = V_{SS}$, the second derivative of the option with respect to the asset price, and consequently these equations are nonlinear.

Leland [9] included transaction costs proportional to the value of the shares traded into the Black-Scholes model using a fixed revision interval between trades by including the transaction cost term in a modified value for the variance. Boyle and Vorst [4] incorporate proportional transaction costs directly into a binomial framework, and obtain a limiting variance adjustment to Black-Scholes in the case of a long call option

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that is similar to but different from Leland's due to the different assumptions made about the distribution of movements in the underlying asset.

As a different direction, the global-in-time models have been introduced by Hodges & Neuberger [7] and Davis, Panas & Zariphopoulou [5]. Because they are using the utility maximization function, the optimality can be achieved [1]. But such models are slow to compute since they usually result in three- or four-dimensional free boundary problems [12].

In this paper, we generalize the Hoggard, Whalley and Wilmott [8] and Henrotte [6] which are concentrated on the analysis of the transaction costs and bandwidth rehedging policy. In section 2, we derive the nonlinear partial differential equation which the option price with more general and precise transaction cost satisfies. In section 3, we introduce the bandwidth for the better rehedging and obtain the partial differential equation under the given bandwidth.

2. A Generalized Transaction Cost

In this section we derive the model with dividends taxed at rate τ under the specific transaction costs. This model is the generalization of Hoggard, Whalley, and Wilmott model [8]. Before deriving the mathematical model which gives the value of the option with transaction costs, we need to mention the assumptions.

- The portfolio is revised every δt where δt is a finite, fixed and small interval.

$$\delta S = \mu S \delta t + \sigma S \phi \delta t^{1/2} \tag{1}$$

  where ϕ is drawn from a standardized normal distribution and μ is the stock price's instantaneous expected return and σ is the instantaneous variance of stock price's return.
- Short selling is allowed and the assets are divisible.
- The risk-free interest rate r and the asset volatility σ are known, deterministic functions of time over the life of the option as constant.
- No arbitrage opportunities (The hedged portfolio has an expected return equal to that from a bank deposit).
- The constant dividend yield is η and this dividend is taxed at rate τ.

We proceed as in the Black-Scholes model and consider a portfolio Π consisting of derivative products, whose total value is V, and a number Δ, which will be determined later, of shares.

$$\Pi = V - \Delta S.$$

Using Itô's lemma, we obtain the change in value of this portfolio, $\delta\Pi$, by expanding V about (S, t) with a transaction costs $K(\nu, S)$ where ν is the number of shares traded, as

$$\begin{aligned} \delta\Pi &= \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \phi^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S\right)\delta t \\ &\quad + \sigma S \left(\frac{\partial V}{\partial S} - \Delta\right)\phi \delta t^{1/2} - K(\nu, S) - \eta(1-\tau)\Delta S\, \delta t \end{aligned} \tag{2}$$

Eliminating the risk associated with the stochastic movements or adopting a strategy of delta hedging, we choose the number of shares as

$$\Delta = \frac{\partial V}{\partial S}.$$

Hence we hold the number of assets as

$$-\frac{\partial V}{\partial S}(S,t).$$

After a time step δt and rehedging, the number of assets we hold becomes

$$-\frac{\partial V}{\partial S}(S+\delta S, t+\delta t).$$

Hence we can find the number of assets we trade, ν, which is given by

$$\nu = -\frac{\partial V}{\partial S}(S+\delta S, t+\delta t) + \frac{\partial V}{\partial S}(S,t).$$

Using Itô's lemma, this number can be obtained as

$$\nu = \sigma S \, \frac{\partial^2 V}{\partial S^2} \, \phi \, \delta t^{1/2}. \tag{3}$$

We do not know beforehand how many shares will be traded, but we can calculate the expected number.

The transaction costs $K(\nu, S)$ is expressed as the sum of three terms, a fixed cost, a cost proportional to volume traded and a cost proportional to the value traded.

$$K(\nu, S) = k_1 + k_2|\nu| + \left(\sum_{i=1}^{n} (\zeta_i - \zeta_{i-1}) U(|\nu|S - x_i) \right) |\nu| S \tag{4}$$

where k_i and ζ_i are constant, and x_i represents the level of the amount $|\nu|S$ which has the proportional constant ζ_i and U is the Heaviside function as

$$U(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Theorem 1. *The value of the option with the transaction cost function as*

$$K(\nu, S) = k_1 + k_2|\nu| + \left(\sum_{i=1}^{n} (\zeta_i - \zeta_{i-1}) U(|\nu|S - x_i) \right) |\nu| S$$

satisfies the following nonlinear partial differential equation:

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\eta)(1-\tau) S \frac{\partial V}{\partial S} - r(1-\tau)V \\ = \; & \frac{k_1}{\delta t} + \sqrt{\frac{2}{\pi \, \delta t}} k_2 \, \sigma S \left| \frac{\partial^2 V}{\partial S^2} \right| \\ & + \sqrt{\frac{2}{\pi \, \delta t}} \, \sigma S^2 \left| \frac{\partial^2 V}{\partial S^2} \right| \sum_{i=1}^{n} (\zeta_i - \zeta_{i-1}) \exp \left[- \frac{x_i^2}{2\sigma^2 S^4 \left(\frac{\partial^2 V}{\partial S^2} \right)^2 \delta t} \right]. \end{aligned} \tag{5}$$

Proof. ¿From the assumption on no arbitrage opportunities,

$$E[\delta\Pi] = r(1-\tau)\Pi\delta t = r(1-\tau)(V - S\frac{\partial V}{\partial S})\delta t. \tag{6}$$

Equation (2) becomes

$$E[\delta\Pi] = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)\delta t - E[K(\nu, S)] - \eta(1-\tau)S\frac{\partial V}{\partial S}\delta t.$$

Here

$$\begin{aligned} E[K(\nu,S)] &= E\left[k_1 + k_2|\nu| + \left(\sum_{i=1}^{n}(\zeta_i - \zeta_{i-1})U(|\nu|S - x_i)\right)|\nu|S\right] \\ &= E[k_1] + k_2 E[|\nu|] + S\sum_{i=1}^{n}(\zeta_i - \zeta_{i-1})E[|\nu|U(|\nu|S - x_i)]. \end{aligned}$$

We know the fact that

$$\begin{aligned} E[|\nu|] &= E\left[\sigma S \left|\frac{\partial^2 V}{\partial S^2}\right| |\phi| \delta t^{1/2}\right] \\ &= \sigma S \left|\frac{\partial^2 V}{\partial S^2}\right| \delta t^{1/2} E[|\phi|] \\ &= \sqrt{\frac{2}{\pi}}\sigma S \left|\frac{\partial^2 V}{\partial S^2}\right| \delta t^{1/2}. \end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned} & E[|\phi|U(|\nu|S - x_i)] \\ = & \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|y|U(R|y| - x_i)e^{-\frac{1}{2}y^2}dy \qquad \text{where } R = \sigma S^2\left|\frac{\partial^2 V}{\partial S^2}\right|\delta t^{1/2} \\ = & \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x_i/R}|y|e^{-\frac{1}{2}y^2}dy + \frac{1}{\sqrt{2\pi}}\int_{x_i/R}^{\infty}|y|e^{-\frac{1}{2}y^2}dy \\ = & \sqrt{\frac{2}{\pi}}\int_{x_i/R}^{\infty}|y|e^{-\frac{1}{2}y^2}dy \\ = & \sqrt{\frac{2}{\pi}}\exp\left[-\frac{1}{2}\frac{x_i^2}{R^2}\right]. \end{aligned}$$

Hence

$$E[|\nu|SU(|\nu|S - x_i)] = \sqrt{\frac{2}{\pi}}\sigma S^2\left|\frac{\partial^2 V}{\partial S^2}\right|\delta t^{1/2}\exp\left[-\frac{x_i^2}{2\sigma^2 S^4\left(\frac{\partial^2 V}{\partial S^2}\right)^2\delta t}\right]. \tag{8}$$

¿From (7) and (8), the expected value of the transaction cost is given by

$$\begin{aligned} E[K(\nu,S)] &= k_1 + k_2\sqrt{\frac{2}{\pi}}\sigma S \left|\frac{\partial^2 V}{\partial S^2}\right| \delta t^{1/2} \\ &+\sum_{i=1}^{n}(\zeta_i-\zeta_{i-1})\sqrt{\frac{2}{\pi}}\sigma S^2 \left|\frac{\partial^2 V}{\partial S^2}\right| \delta t^{1/2} \exp\left[-\frac{x_i^2}{2\sigma^2 S^4\left(\frac{\partial^2 V}{\partial S^2}\right)^2 \delta t}\right]. \end{aligned}$$

Finally we have

$$\begin{aligned} (9)\quad E[\delta\Pi] &= \left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}-\eta(1-\tau)S\frac{\partial V}{\partial S}\right)\delta t \\ &+k_1+\sqrt{\frac{2}{\pi}}k_2\,\sigma S\left|\frac{\partial^2 V}{\partial S^2}\right|\delta t^{1/2} \\ &+\sqrt{\frac{2}{\pi}}\,\sigma S^2\left|\frac{\partial^2 V}{\partial S^2}\right|\delta t^{1/2}\sum_{i=1}^{n}(\zeta_i-\zeta_{i-1})\exp\left[-\frac{x_i^2}{2\sigma^2 S^4\left(\frac{\partial^2 V}{\partial S^2}\right)^2\delta t}\right]. \end{aligned}$$

The equations (6) and (9) complete the proof. □

It is difficult to obtain the approximate solution of the nonlinear partial differential equation in Theorem 1 because of its complicated nonlinearity although we generalize the transaction costs here. But the procedure of this section will be used to derive the partial differential equation with bandwidth in the next section, from which we have the more accurate numerical solution easier than from that of Theorem 1. If $n = 1$ and $x_i = 0$, then we have the simple partial differential equation

$$\begin{aligned} &\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV \\ =\;& \frac{k_1}{\delta t}+\sqrt{\frac{2}{\pi\,\delta t}}\,\sigma S(k_2+\zeta_1 S)\left|\frac{\partial^2 V}{\partial S^2}\right| \end{aligned}$$

which is similar to the model derived in [8] with no dividends and the simple transaction cost

$$K(\nu,S) = k_1 + k_2|\nu| + \zeta_1|\nu|S.$$

3. Hedging to the Bandwidth

In the previous section we derive the model of option price with realistic transaction costs when hedging takes place at fixed intervals of time. But the better strategy is to rehedge whenever the position becomes too far out of line with the perfect hedge position. Hedging has to take place discretely while prices of options are monitored continuously.

The perfect Black-Scholes hedge is

$$\Delta = \frac{\partial V}{\partial S}.$$

Because we can not perfectly hedge the portfolio, we will try to reduce the risk from mishedging. Suppose that we hold $-\Xi$ of the underlying asset but we do not want to accept the extra cost of trading to reposition the hedge. Then the portfolio is given by

$$\Pi = V - \Xi S.$$

Using Itô's lemma, we can write the change in value of this portfolio from $(S + \delta S, t + \delta t), \delta\Pi$, as

$$\begin{aligned} \delta\Pi \;=\; & \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \phi^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Xi S\right)\delta t \\ & + \sigma S \left(\frac{\partial V}{\partial S} - \Xi\right)\phi \delta t^{1/2} - \eta(1-\tau)\Xi S\,\delta t. \end{aligned} \tag{10}$$

The risk, as measured by the variance over a timestep δt of the imperfectly hedged position is, to leading order,

$$\sigma^2 S^2 \left(\frac{\partial V}{\partial S} - \Xi\right)^2 \delta t \tag{11}$$

since

$$E[\delta\Pi] = \left(\frac{1}{2}\sigma^2 S^2 \phi^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial S} - \Xi\right)\mu S - \eta(1-\tau)\Xi S\right)\delta t$$

and

$$E[\delta\Pi^2] = \sigma^2 S^2 \left(\frac{\partial V}{\partial S} - \Xi\right)^2 \delta t + O(\delta t^2).$$

When $\Xi = \partial V/\partial S$, the above variance (11) becomes zero. It is a natural hedging strategy that bounding the variance within a given tolerance and it can be expressed by

$$\sigma S \left|\frac{\partial V}{\partial S} - \Xi\right| \le \Lambda \tag{12}$$

where Λ is a measure of the maximum expected risk in the portfolio. Whenever (12) is violated (i.e. the perfect hedge $\partial V/\partial S$ and the current hedge Ξ go out of line), the position must be rebalanced. Consequently this inequality (12) defines the bandwidth of the hedging position.

We now specify the maximum risk as below,

$$\Lambda = S^2 \left(\frac{\partial V}{\partial S} - \Xi\right)^2. \tag{13}$$

This gives us the number we trade when the bandwidth is violated,

$$|\nu| = \left|\frac{\partial V}{\partial S} - \Xi\right| = \frac{\Lambda^{1/2}}{S}. \tag{14}$$

Theorem 2. *The option's price V with the transaction cost (4) and the bandwidth (13) satisfies the nonlinear partial differential equation*

$$\begin{aligned} & \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\eta)(1-\tau) S \frac{\partial V}{\partial S} - r(1-\tau)V \\ = \quad & \frac{\sigma^2 S^4 \Gamma^2}{\Lambda} \left(k_1 + \left(k_2 + S \sum_{i=1}^{n} (\zeta_i - \zeta_{i-1}) U(\Lambda^{1/2} - x_i) \right) \frac{\Lambda^{1/2}}{S} \right) \end{aligned}$$

where Γ is the option's gamma $\partial^2 V / \partial S^2$.

Proof. ¿From (3), we have

$$\nu^2 = \sigma^2 S^2 \phi^2 \Gamma^2 \, \delta t.$$

Using (14),

$$\nu^2 = \frac{\Lambda}{S^2}.$$

These two equations give us

$$\delta t = \frac{\Lambda}{\sigma^2 S^4 \phi^2 \Gamma^2},$$

hence we have the expected value of $1/\delta t$ as

$$E\left[\frac{1}{\delta t}\right] = \frac{\sigma^2 S^4 \Gamma^2}{\Lambda}.$$

Following the procedure of the proof in Theorem 1,

$$\begin{aligned} E\left[\frac{K(\nu, S)}{\delta t}\right] &= E\left[\frac{k_1 + k_2|\nu| + (\sum_{i=1}^{n} (\zeta_i - \zeta_{i-1}) U(|\nu| S - x_i)) \, |\nu| S}{\delta t}\right] \\ &= \frac{\sigma^2 S^4 \Gamma^2}{\Lambda} \left(k_1 + \left(k_2 + S \sum_{i=1}^{n} (\zeta_i - \zeta_{i-1}) U(\Lambda^{1/2} - x_i) \right) \frac{\Lambda^{1/2}}{S} \right). \end{aligned}$$

□

References

[1] C. Atkinson, P. Wilmott, Portfolio management with transaction costs: an asymptotic analysis. Math. Fin. 5 (1995) 357-367.

[2] M. Avellaneda, A. Paras, Optimal hedging portfolios for derivative securities in the presence of large transaction costs, Appl. Math. Fin. 2 (1995)

[3] F. Black, M. Scholes, The pricing of options and corporate liabilities. J. Pol. Eco. 81 (1973) 637-654.

[4] P. Boyle, T. Vorst, Option replication in discrete time with transaction costs, J. Fin. 47 (1992) 271-293.

[5] M.H. Davis, V. Panas, T. Zariphopoulou, European option pricing with transaction costs, SIAM J. Cont. & Opt. 31 (1993) 470-493.

[6] P. Henrotte, Transaction costs and duplication strategies, Working papers Stanford Univ. et Hec (1993).

[7] S.D. Hodges, A. Neuberger,Optimal replication of contingent claims under transaction costs, Rev. Futures Markets 8 (1989) 222-239.

[8] T. Hoggard, A.E. Whalley, P. Wilmott, Hedging option portfolios in the presence of transaction costs, Adv. in Fut. and Opt. Res. 7 (1994) 21-35.

[9] H.E. Leland, option pricing and replication with transaction costs, J. Fin. 40 (1985) 1283-1301.

[10] K.B. Toft, On the mean variance tradeoff in option replication, J. Fin. & Quant. Anal. 31 (1996).
[11] A.E. Whalley, P. Wilmott, A comparison of hedging strategies, ECMI Proc. (1993) 427-434.
[12] A.E. Whalley, P. Wilmott, An asymptotic analysis of the Davis, Panas & Zariphopoulou model for option pricing with transaction costs, Math. Fin. 7 (1997) 307-324.

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