J. KSIAM **Vol.4**, No.2, 11-20, 2000

RELATIONSHIPS BETWEEN AMERICAN PUTS AND CALLS ON FUTURES CONTRACTS

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ABSTRACT. This paper presents a formula that relates the optimal exercise boundaries of American call and put options on futures contract. It is shown that the geometric mean of the optimal exercise boundaries for call and put written on the same futures contract with the same exercise price is equal to the exercise price which is time invariant. The paper also investigates the properties of American calls and puts on futures contract.

1. INTRODUCTION

The seminal paper of Black (1976) provided a groundwork for the valuation of options written on futures contracts and showed how to extend the Black-Scholes (1973) option pricing formula to the case of European options on futures contracts. However, all of the publicly traded futures options on the organized exchanges are of the American type; i.e., they allow the holder to exercise them before the expiration date. Assuming that the risk-free rate of interest is positive, American futures options are always subject to early exercise and so American futures options must be worth strictly more than their European counterparts. Therefore Black's formula does not provide a correct value and there is no closed-form solution for American futures options.

Working from McKean's (1965) formulation of the free boundary problem, integral formulations of American option values were independently derived by Kim (1990), Jacka (1991), and Carr, Jarrow, and Myneni (1992). Kim (1990) derived the valuation formulas by considering an option exercisable at a finite number of points in time and evaluating its continuous limit as the time intervals shrink to zero. Jacka (1991) obtains the same valuation formulas using the probability theory applied to the optimal stopping problem. Carr, Jarrow, and Myneni (1992) also obtain the formulas by considering the trading strategy which converts an American option into a European one. Kim and Yu (1996) gives a more simpler and intuitive proof of the valuation formulas. Valuation formulas for American futures options can be obtained as a special case of these formulas. (One very comprehensive reference on options is that by Duffie 1992;

Key words and phrases. American option valuation, futures options, optimal exercise boundary, optimal stopping problem.

it contains a considerable amount of background material on option valuation. For a rigorous survey on the theory of American option pricing, see Myneni 1992).

The integral formulations of American option values yield a pair of nonlinear integral equations, one is for calls and the other is for puts, that must be satisfied by the optimal exercise boundaries. The optimal exercise boundary is defined by the critical futures price at or above (below) which it is optimal to exercise the American calls (puts) on futures contracts. This paper presents a formula that relates the optimal exercise boundaries of American call and put futures options, by exploiting the special structure of these integral equations. It is shown that the geometric mean of the optimal exercise boundaries for American call and put options written on the same futures contract with the same exercise price is equal to the exercise price which is a time invariant constant. This relationship between the optimal exercise boundaries enables us to investigate the relationships between American call and put futures options.

The paper is organized as follows. Section 2 examines European options on futures contracts. Following Black (1976), this section gives an identity between Black's formulas for European call and put futures options. In Section 3, American options on futures contracts are examined and the relationships between American call and put futures options are presented. Concluding remarks are in Section 4.

2. European Options on Futures Contracts

Consider European call and put options written on a futures contract with exercise price K and expiry date T. The underlying futures contract expires at or after time T. Throughout the paper, the usual conditions are assumed that the markets are perfect with continuous trading, there are no-arbitrage opportunities, the risk-free rate of interest, r, is a positive constant, and the futures price F_t satisfies a stochastic differential equation:

$$dF_t = (\mu - r)F_t \, dt + \sigma \, F_t \, dW_t \, , \quad t \in [0,T] \tag{2.1}$$

where W is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{Q})$ and the coefficients μ and σ are positive constants. Equation (2.1) implies that the futures price follows a lognormal diffusion process with volatility parameter σ.

In this setting, Black (1976) developed a variation of his earlier Black-Scholes (1973) model to value European futures options. Let us denote the value functions of European call and put at time t by $c(F,\tau)$ and $p(F,\tau)$ defined on domain $\mathcal{D} \equiv \{(F,\tau); 0 < F < \infty, 0 < \tau \leq T\}$, where F is the futures price and $\tau = T - t$ is time to expiration. The formulas for the prices may be expressed as:

$$c(F,\tau) = e^{-r\tau} \left[F\aleph(d_1(F,\tau;K)) - K\aleph(d_2(F,\tau;K))\right] \tag{2.2}$$

$$p(F,\tau) = e^{-r\tau} \left[K\aleph(-d_2(F,\tau;K)) - F\aleph(-d_1(F,\tau;K))\right] \tag{2.3}$$

where $\aleph(\cdot)$ is the cumulative standard normal distribution function and

$$d_1(F,\tau;K) = \frac{\ln(F/K)+\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \tag{2.4}$$

$$d_2(F,\tau;K) = \frac{\ln(F/K)-\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \tag{2.5}$$

We will first present a useful identity between $d_1(\cdot,\cdot;\cdot)$ and $d_2(\cdot,\cdot;\cdot)$.

Lemma 2.1 *For all $x>0$, $y>0$, and $\tau > s \geq 0$,*

$$d_1(x,\tau-s;y) = -d_2\left(\frac{K^2}{x},\tau-s;\frac{K^2}{y}\right) \tag{2.6}$$

$$d_2(x,\tau-s;y) = -d_1\left(\frac{K^2}{x},\tau-s;\frac{K^2}{y}\right) \tag{2.7}$$

Proof of (2.6). Using equation (2.5), we see that

$$\begin{aligned} d_2\left(\frac{K^2}{x},\tau-s;\frac{K^2}{y}\right) &= \frac{\ln\left(\frac{K^2}{x}/\frac{K^2}{y}\right)-\frac{1}{2}\sigma^2(\tau-s)}{\sigma\sqrt{\tau-s}} \\ &= \frac{\ln(y/x)-\frac{1}{2}\sigma^2(\tau-s)}{\sigma\sqrt{\tau-s}} \\ &= -d_1(x,\tau-s;y) \end{aligned}$$

where the last equality follows from equation (2.4).

Proof of (2.7). The result is immediate from equation (2.6). □

We then give an identity between put and call value functions for European options on futures contracts.

PROPOSITION 2.1 *For all $x>0$ and $\tau>0$,*

$$c(x,\tau) = \frac{x}{K}p\left(\frac{K^2}{x},\tau\right) \tag{2.8}$$

where K is the exercise price.

Proof. Black's European put pricing formula (2.3) gives

$$p\left(\frac{K^2}{x},\tau\right) = e^{-r\tau}\left[K\aleph(-d_2(K^2/x,\tau;K))-\frac{K^2}{F}\aleph(-d_1(K^2/x,\tau;K))\right] \tag{2.9}$$

$$= e^{-r\tau}\left[K\aleph(d_1(x,\tau;K))-\frac{K^2}{F}\aleph(d_2(x,\tau;K))\right] \tag{2.10}$$

where we have used Lemma 2.1 with $y = K$. Multiplying both sides of equation (2.10) by x/K, we have

$$\begin{aligned} \frac{x}{K} p\left(\frac{K^2}{x}, \tau\right) &= e^{-r\tau}\left[x\aleph(d_1(F,\tau;K)) - K\aleph(d_2(x,\tau;K))\right] \\ &= c(x,\tau) \end{aligned}$$

where the last equality follows from Black's European call pricing formula (2.2).

3. American Options on Futures Contracts

Continuing with the setup and notation of the previous section, now consider the corresponding American call and put options written on the same futures contract. With American futures options, there is always the possibility of early exercise whether the underlying asset on which the futures contract is written pays dividends or not as long as the risk-free rate of interest is positive.[1] This implies that, for each time to maturity $\tau \in (0, T]$, there exists a critical futures price $G(\tau)(B(\tau))$ at or above (below) which the American call (put) should be exercised immediately. The optimal exercise boundary is defined as the time path of critical futures prices.

Let us denote the value functions of American call and put at time t by $C(F,\tau)$ and $P(F,\tau)$ defined on the domain $\mathcal{D}$. If the futures price is above (below) the optimal exercise boundary, the American call (put) is *dead* and its value is defined to be $C(F,\tau) = F - K$ $(P(F,\tau) = K - F)$. Although no one yet has found a closed-form solution for the values of *live* American futures options, the integral formulations express the value of a *live* American option as the sum of the corresponding European option value and the early-exercise premium. In summary, the American call and put futures option values are given by:

$$C(F,\tau) = \begin{cases} F - K & \text{if} \quad F \geq G(\tau) \\ c(F,\tau) + \int_0^\tau \phi(F, \tau - s; G(s))ds & \text{if} \quad F < G(\tau) \end{cases} \tag{3.1}$$

$$P(F,\tau) = \begin{cases} K - F & \text{if} \quad F \leq B(\tau) \\ p(F,\tau) + \int_0^\tau \psi(F, \tau - s; B(s))ds & \text{if} \quad F > B(\tau) \end{cases} \tag{3.2}$$

where $c(F,\tau)$ and $p(F,\tau)$ denote Black's formulas for European call and put futures options, respectively, and

$$\phi(F, \tau - s; G(s)) = re^{-r(\tau - s)}\left[F\aleph(d_1(F, \tau - s; G(s))) - K\aleph(d_2(F, \tau - s; G(s)))\right] \tag{3.3}$$

$$\psi(F, \tau - s; B(s)) = re^{-r(\tau - s)}\left[K\aleph(-d_2(F, \tau - s; B(s))) - F\aleph(-d_1(F, \tau - s; B(s)))\right] \tag{3.4}$$

We will give an identity between $\phi(\cdot,\cdot;\cdot)$ and $\psi(\cdot,\cdot;\cdot)$.

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[1] This fact has been well established in Ramaswamy and Sundaresan (1985), Brenner, Courtadon, and Subrahmanyam (1985), and Ball and Torous (1986).

Lemma 3.1 *For all* $x > 0$, $y > 0$, *and* $\tau > s \geq 0$,

$$\phi(x, \tau - s; y) = \frac{x}{K}\psi\left(\frac{K^2}{x}, \tau - s, \frac{K^2}{y}\right)$$

Proof. From equation (3.4), we have

$$\begin{aligned}
&\psi\left(\frac{K^2}{x}, \tau - s, \frac{K^2}{y}\right) \\
&= re^{-r(\tau-s)}\left[K\aleph\left(-d_2\left(\frac{K^2}{x}, \tau - s; \frac{K^2}{y}\right)\right) - \frac{K^2}{x}\aleph\left(-d_1\left(\frac{K^2}{x}, \tau - s; \frac{K^2}{y}\right)\right)\right] \\
&= re^{-r(\tau-s)}\left[K\aleph(d_1(x, \tau - s; y)) - \frac{K^2}{x}\aleph(d_2(x, \tau - s; y))\right] \\
&= \frac{K}{x}\phi(x, \tau - s; y)
\end{aligned}$$

where the second equality follows from Lemma 2.1 and the last equality follows from equation (3.3). □

By imposing an optimality condition on valuation formulas (3.1) and (3.2), the following nonlinear integral equations are obtained that implicitly define the optimal exercise boundaries.[2]

$$G(\tau) - K = c(G(\tau), \tau) + \int_0^\tau \phi(G(\tau), \tau - s; G(s))ds \tag{3.5}$$

$$K - B(\tau) = p(B(\tau), \tau) + \int_0^\tau \psi(B(\tau), \tau - s; B(s))ds \tag{3.6}$$

Jacka (1991) and van Moerbeke (1976) address the questions of uniqueness and regularity of the solution to integral equations (3.5) and (3.6). Although there is no explicit solution available for the optimal exercise boundary, we explicitly know the limiting behaviour of the boundary. It can be easily checked that $G(0)B(0) = G(\infty)B(\infty) = K^2$, where $G(0)$ and $B(0)$ represent the limits of $G(\tau)$ and $B(\tau)$ as τ tends to zero, and similarly $G(\infty)$ and $B(\infty)$ represent the limits of $G(\tau)$ and $B(\tau)$ as τ tends to infinity.[3] Note that $G(\infty)$ and $B(\infty)$ also stand for the critical futures prices for perpetual American call and put with otherwise similar terms. The next proposition uses integral equations (3.5) and (3.6) to generate an explicit expression for $B(\cdot)$ in terms of $G(\cdot)$.

PROPOSITION 3.1 *Assume that the integral equation (3.5) has a unique continuous solution* $G(\tau)$ *for* $\tau \in [0, T]$. *Then the integral equation (3.6) possesses a unique continuous solution and the solution is simply given by*

$$B(\tau) = \frac{K^2}{G(\tau)} \quad \text{for} \quad \tau \in [0, T] \tag{3.7}$$

[2] There are many other integral equations that define the unique optimal exercise boundary. See, for example, McKean (1965), Kim (1990), and Carr, Jarrow, and Myneni (1992)

[3] See Appendix for a proof of $G(\infty)B(\infty) = K^2$.

where K *is the exercise price.*

Proof. Let $G(\tau)$ be the unique continuous solution of integral equation (3.5). Consider then $\hat{B}(\tau)$ defined by

$$\hat{B}(\tau) = \frac{K^2}{G(\tau)} \quad \tau \in (0, T] \tag{3.8}$$

We will show that this $\hat{B}(\tau)$ satisfies integral equation (3.6). Now substitute $\hat{B}(\tau)$ into equation (3.6). Then

$$\begin{aligned} & p(\hat{B}(\tau), \tau) + \int_0^\tau \psi(\hat{B}(\tau), \tau - s; \hat{B}(s))ds \\ = \ & p(K^2/G(\tau), \tau) + \int_0^\tau \psi(K^2/G(\tau), \tau - s; K^2/G(s))ds \\ = \ & \frac{K}{G(\tau)} c(G(\tau), \tau) + \int_0^\tau \frac{K}{G(\tau)} \phi(G(\tau), \tau - s; G(s))ds \\ = \ & \frac{K}{G(\tau)}(G(\tau) - K) \\ = \ & K - \hat{B}(\tau) \end{aligned}$$

where the second equality follows from Proposition 2.1 and Lemma 3.1 with $x = G(\tau)$ and $y = G(s)$, the third equality follows from equation (3.5), and the last equality follows from equation (3.8). This proves that $\hat{B}(\tau)$ defined by (3.8) satisfies equation (3.6).

To show that $\hat{B}(\tau)$ is the only continuous solution, suppose there exists another continuous solution $\tilde{B}(\tau)$ of equation (3.6). Then equation (3.5) would have another continuous solution $\tilde{G}(\tau) \equiv K^2/\tilde{B}(\tau) \neq G(\tau)$, which contradicts the hypothesis that integral equation (3.5) has a unique continuous solution. Therefore we must have $\hat{B}(\tau) = \tilde{B}(\tau)$, $\tau \in (0, T]$, and there is only one continuous solution.[4] □

Proposition 3.1 says that the geometric mean of the optimal exercise boundaries $G(\tau)$ and $B(\tau)$ is equal to the exercise price; i.e., $\sqrt{G(\tau)B(\tau)} = K$, for $\tau \in [0, T]$. This relation gives a shortcut for computing the optimal exercise boundary of an American call/put given the optimal exercise boundary of its counterpart. By rearranging and taking logarithms on the relation (3.7) in Proposition 3.1, we obtain:

$$\ln\left(\frac{G(\tau)}{K}\right) + \ln\left(\frac{B(\tau)}{K}\right) \quad = \quad 0 \tag{3.9}$$

This says that the arithmetic mean of the logarithms of the optimal exercise boundaries devided by the exercise price is equal to zero as shown in Figure 1. Note the symmetry with respect to the time to maturity axis (τ-axis).

[4] Following the same lines of arguments, Proposition 3.1 can be generalized for American options on dividend-paying assets as follows: $G(\tau; r, \delta)B(\tau; \delta, r) = K^2$ where r is the riskfree rate and δ is the dividend yield.

Using Propositions 2.1 and 3.1, along with valuation formulas (3.1) and (3.2), the following identity between put and call value functions for American futures options, analogous to Proposition 2.1, obtains.

PROPOSITION 3.2 *For all $x > 0$ and $\tau \geq 0$,*

$$C(x,\tau) = \frac{x}{K} P\left(\frac{K^2}{x},\tau\right) \tag{3.10}$$

where K is the exercise price.[5]

Proof. American put valuation formula (3.2) gives

$$\begin{aligned} P\left(\frac{K^2}{x},\tau\right) &= p\left(\frac{K^2}{x},\tau\right) + \int_0^\tau \psi(K^2/x, \tau - s; B(s))ds \\ &= \frac{K}{x} c(x,\tau) + \int_0^\tau \psi(K^2/x, \tau - s; K^2/G(s))ds \\ &= \frac{K}{x} c(x,\tau) + \int_0^\tau \frac{K}{x} \phi(x, \tau - s; G(s))ds \\ &= \frac{K}{x} C(x,\tau) \end{aligned}$$

where the second equality follows from Propositions 2.1 and 3.1, the third equality follows from Lemma 3.1 with $y = G(s)$, and the last equality follows from American call valuation formula (3.1). □

Proposition 3.2 implies that if we know the value function of an American call/put futures option we can easily determine the value function of its counterpart.

Option traders are interested not only in price but also in the hedge parameters such as delta, gamma, and theta, which are used to evaluate and manage the risks of options. Differentiating both sides of equation (3.10) with respect to x or with respect to τ, the following identities for the hedge parameters can also be obtained.

PROPOSITION 3.3 *For all $x > 0$ and $\tau \geq 0$,*

$$C_F(x,\tau) = \frac{1}{K} P\left(\frac{K^2}{x},\tau\right) - \frac{K}{x} P_F\left(\frac{K^2}{x},\tau\right) \tag{3.11}$$

$$C_{FF}(x,\tau) = \left(\frac{K}{x}\right)^3 P_{FF}\left(\frac{K^2}{x},\tau\right) \tag{3.12}$$

$$C_\tau(x,\tau) = \frac{x}{K} P_\tau\left(\frac{K^2}{x},\tau\right) \tag{3.13}$$

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[5] Grabbe (1983) originally proposed this relationship in the context of American currency options pricing. McDonald and Schroder (1990) and Chesney and Gibson (1995) extend Grabbe's result to American options on dividend-paying assets.

where K is the exercise price.

It should be noted that Propositions 3.2 and 3.3 are valid for not only *live* but also *dead* options.

Proposition 3.2 shows a relationship between values of American call and put futures options at different futures levels. Now consider a relationship between values at the same futures level. Using equations (3.1), (3.2), Proposition 3.1 and put-call parity for European futures options, we obtain the following put-call parity for American futures options in terms of the optimal exercise boundary.

$$\begin{aligned} C + Ke^{-r\tau} &= P + Fe^{-r\tau} + \int_0^{\tau} re^{-r(\tau-s)} \left[F\left\{ \aleph(d_1) + \aleph(-d_1 - e) \right\} \right. \\ &\quad \left. - K\left\{ \aleph(d_2) + \aleph(-d_2 - e) \right\} \right] ds \end{aligned} \tag{3.14}$$

where C and P denote values for American call and put futures options at the same futures level F, respectively, and

$$\begin{aligned} d_1 &= d_1(F, \tau - s; G(s)) \\ d_2 &= d_2(F, \tau - s; G(s)) \\ e &= \frac{\ln\left(\frac{G(s)}{K}\right)}{\sigma\sqrt{\tau - s}} \geq 0 \end{aligned}$$

4. Conclusion

This paper presents a formula that relates the optimal exercise boundaries of American call and put options written on the same futures contract with the same exercise price. It is shown that the geometric mean of the put and call optimal exercise boundaries is equal to the exercise price which is a time invariant constant. In addition, this paper provides certain kinds of symmetrical relationships between put and call value functions available for both European and American futures options. The results are important not only for gaining insight into the qualitative behavior of the value functions and the optimal exercise boundaries, but are also useful in the design of effective numerical methods.

5. Appendix

Proof of $G(\infty)B(\infty) = K^2$. $G(\infty)$ and $B(\infty)$ are defined by

$$G(\infty) = \frac{\beta K}{\beta - 1} \quad , \quad \beta = \frac{\frac{1}{2}\sigma^2 + \sqrt{\frac{1}{4}\sigma^4 + 2\sigma^2 r}}{\sigma^2}$$

$$B(\infty) = \frac{\theta K}{\theta - 1} \quad , \quad \theta = \frac{\frac{1}{2}\sigma^2 - \sqrt{\frac{1}{4}\sigma^4 + 2\sigma^2 r}}{\sigma^2}$$

Straightforward multiplication yields

$$\begin{aligned} B(\infty)G(\infty) &= \frac{\beta\theta K^2}{(\beta-1)(\theta-1)} \\ &= \frac{\beta\theta K^2}{\beta\theta-(\beta+\theta)+1} \\ &= K^2 \end{aligned}$$

where we have used the fact that $\beta+\theta=1$. □

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