⊠연구논문

와이블 수명자료들에 대한 베이지안 가설검정

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Bayesian Hypotheses Testing for the Weibull Lifetime Data

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Abstract

In this paper, we address the Bayesian hypotheses testing for the comparison of Weibull distributions. In Bayesian testing problem, conventional Bayes factors can not typically accommodate the use of noninformative priors which are improper and are defined only up to arbitrary constants. To overcome such problem, we use the recently proposed hypotheses testing criterion called the intrinsic Bayes factor. We derive the arithmetic and median intrinsic Bayes factors for the comparison of Weibull lifetime model and we use these results to analyze real data sets.

1. Introduction

Lifetime studies are a precious source of information for product manufacturers. They allow one to compare lifetime distributions of competitors products. Comparisons of product designs, materials, suppliers and production periods are some of the issues addressed by these studies.

For lifetime studies, Weibull distribution is perhaps the most widely used lifetime model. Its application in connection with lifetimes of many types of manufactured items has been widely advocated, and it has been used as a model with diverse types of items such as vacuum tubes, ball bearings and electrical insulation.

In Bayesian testing problem, the Bayes

factor depend on rather strongly on the prior distributions, much more so than in, say, estimation. So, the Bayes factor under proper priors, have been very successful. However, elicitation of subjective prior distributions is impossible, because of time and cost limitations, or resistance or lack of training of clients. Also subjective elicitation can easily result in poor prior distribution and statistical analysis is often required to appear objective. So, the literature on noninformative priors has grown enormously over recent years. There have been several excellent books or review articles that have been concerned with discussing or comparing different approaches to developing noninformative priors (See Ghosh and Mukerjee, 1992).

But noninformative priors such as Jeffrey's (1961) priors or reference priors (Berger and Bernardo (1989, 1992)) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. So, Geisser and Eddy (1979), Spiegalhalter and Smith(1982), San Martini and Spezzaferri (1984) and O'Hagan (1995) have made efforts to compensate for that arbitrariness.

Berger and Pericchi (1996b) introduced a new model selection and hypotheses testing criterion, called the Intrinsic Bayes Factor (IBF) using a data-splitting idea, which would eliminate the arbitrariness of improper priors. These can be constructed in very general situation-nested, nonnested, and even irregular problems-and they seem to correspond to actual Bayes factors, at least asymptotically. This approach has shown to be quite useful (Berger and Pericchi (1996a), Varshavsky (1996) and Lingham and Sivaganesan (1997)).

In this paper, we use a Bayesian approach to the comparison of Weibull distributions using reference priors as improper priors. We derive intrinsic Bayes factors to solve our problem. Also, we give some numerical results with real data analysis to illustrate our results.

2. The Intrinsic Bayes Factor Methodology

In this Section, we firstly introduce the intrinsic Bayes factor in the general hypotheses testing. As a matter of convenience, we introduce the following notations.

$X=(X_1,\cdots,X_n)$: observation with density $f(x \mid \theta)$, where $\theta \in \Theta$ is a finite dimensional parameter and Θ is parameter space.

Θ_i : parameter space under ith hypothesis H_i, $i=1,2,\cdots,q$.

$f(x \mid \theta_i)$: the density under H_i, $i=1,2,\cdots,q$.

$\pi_i(\theta_i)$: the prior distribution under H_i, $i=1,2,\cdots,q$.

$m_i(x)$: the marginal density of X under H_i when use $\pi_i(\theta_i)$, $i=1,2,\cdots,q$.

p_i : the prior probability of H_i being true, $i=1,2,\cdots,q$.

$\pi_i^N(\theta_i)$: the improper prior distribution under H_i, $i=1,2,\cdots,q$.

$m_i^N(x)$: the marginal density of X under H_i when use $\pi_i^N(\theta_i)$, $i=1,2,\cdots,q$.

Then $\pi_i^N(\theta_i)$ is usually written as $\pi_i^N(\theta_i) \propto h_i(\theta_i)$, where h_i is a function whose integral over the Θ_i-space diverges. Formally, we can write $\pi_i^N(\theta_i) = c_i h_i(\theta_i)$, although the normalizing constant c_i does not exist, but treating it as an unspecified constant.

The posterior probability that H_i is true is given as

$$P(H_i \mid x) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} B_{ji} \right)^{-1}, \quad (1)$$

where B_{ji}, the Bayes factor of H_j to H_i, is defined by

$$B_{ji} = \frac{m_j(x)}{m_i(x)} = \frac{\int_{\Theta_j} f(x \mid \theta_j)\pi_j(\theta_j)d\theta_j}{\int_{\Theta_i} f(x \mid \theta_i)\pi_i(\theta_i)d\theta_i}. \quad (2)$$

The posterior probabilities in (1) are then used to select the most plausible hypothesis. If one were to use some noninformative priors, then (2) becomes

$$B_{ji}^N = \frac{m_j^N(x)}{m_i^N(x)} = \frac{\int_{\Theta_j} f(x \mid \theta_j)\pi_j^N(\theta_j)d\theta_j}{\int_{\Theta_i} f(x \mid \theta_i)\pi_i^N(\theta_i)d\theta_i}. \quad (3)$$

Hence, the corresponding Bayes factor, B_{ji}^N, is indeterminate. One solution to this indeterminacy problem is to use part of the data as a training sample. Let $x(l)$ denote the part of the data to be so used and let $x(-l)$ be the remainder of the data, such that

$$0 < m_i^N(x(l)) < \infty, \quad i = 1, \cdots, q. \quad (4)$$

In view (4), the posteriors $\pi_i^N(\theta_i \mid x(l))$ are well defined. Now, consider the Bayes factor, $B_{ji}(l)$, for the rest of the data $x(-l)$, using $\pi_i^N(\theta_i \mid x(l))$ as the priors:

$$B_{ji}(l) = \frac{\int_{\Theta_j} f(x(-l) \mid \theta_j, x(l))\pi_j^N(\theta_j \mid x(l))d\theta_j}{\int_{\Theta_i} f(x(-l) \mid \theta_i, x(l))\pi_i^N(\theta_i \mid x(l))d\theta_i}$$

$$= B_{ji}^N \times B_{ij}^N(x(l)) \quad (5)$$

where B_{ji}^N is given by (3) and

$$B_{ij}^N(x(l)) = \frac{m_i^N(x(l))}{m_j^N(x(l))}. \quad (6)$$

In (5), any arbitrary ratio, c_j/c_i say, that multiples B_{ji}^N would be cancelled by the ratio c_i/c_j forming the multiplicand in $B_{ij}^N(x(l))$. Also, while the expression (6) renders $B_{ji}(l)$ in terms of the simpler marginal densities of $x(l)$.

As training samples, arithmetic and median intrinsic Bayes factor play a fundamental role in our testing $H_i, i = 1, \cdots, q$, we introduce the following definitions.

Definition 1. (Berger and Pericchi (1996b)) A training sample $x(l)$, will be called *proper* if (4) holds and *minimal* if it is proper and none of its subsets is proper.

Definition 2. (Berger and Pericchi (1996b)) The Arithmetic Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \cdot \frac{1}{L} \sum_{l=1}^{L} B_{ij}^N(x(l)). \quad (7)$$

where L is the number of all possible minimal training samples.

Definition 3. (Berger and Pericchi (1998)) The Median Intrinsic Bayes factor of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^{N} \cdot ME[B_{ij}^{N}(\boldsymbol{x}(l))], \qquad (8)$$

where ME indicates the median, here to be taken over all the training sample Bayes factors.

We can also calculate the posterior probability of H_i using (1), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (7) and (8).

3. Intrinsic Bayes Factor

The Weibull distribution with parameters α, β is given by

$$f(x \mid \alpha, \beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right), \qquad (9)$$

where $x \geq 0$, and $\alpha > 0$ and $\beta > 0$ are parameters referred to as the scale and shape parameters of the distribution, respectively. Comparison of Weibull scale parameters is of interest mainly when the shape parameters of the distribution are equal. Also equality of Weibull shape parameters across different groups of individuals is an important. Consider two samples of sizes n_1, n_2 from Weibull distributions with parameters (α_1, β_1), (α_2, β_2), respectively. Then the observed sample consists of the failure times $\boldsymbol{x}_i = (x_{i1}, \cdots, x_{in_i})$, where $i = 1, 2$. Thus we want to test the hypotheses of (i) $H_1: \alpha_1 = \alpha_2$ vs. $H_2: \alpha_1 \neq \alpha_2$ under $\beta_1 = \beta_2 = \beta$, and (ii) $H_1: \beta_1 = \beta_2$ vs. $H_2: \beta_1 \neq \beta_2$.

3.1 Minimal Training Sample

The goal here is to determine the set of all possible minimal training sample(MTS) for the data $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ to test $H_1: \alpha_1 = \alpha_2$ vs. $H_2: \alpha_1 \neq \alpha_2$ when $\beta_1 = \beta_2 = \beta$. The reference priors for $H_1: \alpha_1 = \alpha_2$ vs. $H_2: \alpha_1 \neq \alpha_2$ are respectively given by

$$\pi_1^N(\alpha, \beta) = \frac{1}{\alpha\beta}, \qquad (10)$$

$$\pi_2^N(\alpha_1, \alpha_2, \beta) = \frac{1}{\alpha_1\alpha_2\beta}. \qquad (11)$$

To derive the marginals with respect to the reference priors given by (10) and (11), we first observe that the joint pdf of $\boldsymbol{X}_i$ is given by

$$f(\boldsymbol{x}_i \mid \alpha_i, \beta_i) = \left(\frac{\beta_i}{\alpha_i}\right)^{n}\left[\prod_{j=1}^{n_i}\left(\frac{x_j}{\alpha_i}\right)\right]^{\beta_i - 1} \exp\left(-\sum_{j=1}^{n_i}\left(\frac{x_i}{\alpha_i}\right)^{\beta_i}\right). \qquad (12)$$

Moreover, the joint pdf of (x_{ik}, x_{il}), $1 \leq k \neq l \leq n_i$, is given by

$$f(x_{ik}, x_{il} \mid \alpha_i, \beta_i) = \left(\frac{\beta_i}{\alpha_i}\right)^2\left(\frac{x_{ik}x_{il}}{\alpha_i^2}\right)^{\beta_i - 1} \exp\left[-\left(\left(\frac{x_{ik}}{\alpha_i}\right)^{\beta_i} + \left(\frac{x_{il}}{\alpha_i}\right)^{\beta_i}\right)\right]. \qquad (13)$$

In the following lemma, we give the

marginal densities for any two observations.

Lemma 1. We have the marginal density $m_h^N(x_{1i}, x_{1j}, x_{2k}, x_{2l})$ under $H_h, h=1,2$ as follows.

$$m_1^N(x_{1i}, x_{1j}, x_{2k}, x_{2l}) = \Gamma(4)A_1[x_{1i}x_{1j}x_{2k}x_{2l}]^{-1}, \tag{14}$$

$$m_2^N(x_{1i}, x_{1j}, x_{2k}, x_{2l}) = A_2[x_{1i}x_{1j}x_{2k}x_{2l}]^{-1}, \tag{15}$$

where $A_1 = \int_0^\infty \beta^2 \frac{(x_{1i}x_{1j}x_{2k}x_{2l})^\beta}{(x_{1i}^\beta + x_{1j}^\beta + x_{2k}^\beta + x_{2l}^\beta)^4} d\beta$,

$A_2 = \int_0^\infty \beta \frac{(x_{1i}x_{1j}x_{2k}x_{2l})^\beta}{(x_{1i}^\beta + x_{1j}^\beta)^2 (x_{2k}^\beta + x_{2l}^\beta)^2} d\beta$ and $1 \le i \ne j \le n_1$ and $1 \le k \ne l \le n_2$.

Since the marginal density of $(X_{1i}, X_{1j}, X_{2k}, X_{2l})$ is finite for all $1 \le i \ne j \le n_1$ and $1 \le k \ne l \le n_2$ under each hypothesis, we conclude that any training sample of size two is an MTS.

Nextly, we consider the test $H_1: \beta_1 = \beta_2$ vs. $H_2: \beta_1 \ne \beta_2$. The reference priors for $H_h, h=1,2$ are, respectively, given by

$$\pi_1^N(\alpha_1, \alpha_2, \beta) = \frac{1}{\alpha_1\alpha_2\beta}, \tag{16}$$

$$\pi_2^N(\alpha_1, \alpha_2, \beta_1, \beta_2) = \frac{1}{\alpha_1\beta_1} \frac{1}{\alpha_2\beta_2}. \tag{17}$$

In the following lemma, we now derive the marginals with respect to the reference priors given by (16) to (17).

Lemma 2. We have the marginal density $m_h^N(x_{1i}, x_{1j}, x_{2k}, x_{2l})$ under $H_h, h=1,2$ as follows.

$$m_1^N(x_{1i}, x_{1j}, x_{2k}, x_{2l}) = A_2[x_{1i}x_{1j}x_{2k}x_{2l}]^{-1}, \tag{18}$$

$$m_2^N(x_{1i}, x_{1j}, x_{2k}, x_{2l}) = \frac{1}{2x_{1i}x_{1j}|\log(x_{1j}/x_{1i})|} \frac{1}{2x_{2k}x_{2l}|\log(x_{2l}/x_{2k})|}, \tag{19}$$

where $A_2 = \int_0^\infty \beta \frac{(x_{1i}x_{1j}x_{2k}x_{2l})^\beta}{(x_{1i}^\beta + x_{1j}^\beta)^2 (x_{2k}^\beta + x_{2l}^\beta)^2} d\beta$, $1 \le i \ne j \le n_1$ and $1 \le k \ne l \le n_2$.

It is clear from the above that the marginal density of $(X_{1i}, X_{1j}, X_{2k}, X_{2l})$ is finite for all $1 \le i \ne j \le n_1$ and $1 \le k \ne l \le n_2$ under each hypothesis, and hence we conclude that any training sample of size two is an MTS.

3.2 Arithmetic and Median Bayes Factors

The marginal densities corresponding to the full data $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ for test $H_1: \alpha_1 = \alpha_2$ vs. $H_2: \alpha_1 \ne \alpha_2$ can also be expressed in the following lemma.

Lemma 3. For the full data, we have the marginal density $m_h^N(\boldsymbol{x}_1, \boldsymbol{x}_2)$ under $H_h, h=1,2$ as follows.

$$m_1^N(\boldsymbol{x}_1, \boldsymbol{x}_2) = \Gamma(n_1+n_2)C_1[\prod_{i=1}^{n_1} x_{1i}]^{-1}[\prod_{i=1}^{n_2} x_{2i}]^{-1}, \tag{20}$$

$$m_2^N(\boldsymbol{x}_1, \boldsymbol{x}_2) = \Gamma(n_1)\Gamma(n_2)C_2[\prod_{i=1}^{n_1} x_{1i}]^{-1} [\prod_{i=1}^{n_2} x_{2i}]^{-1}, \tag{21}$$

where

$$C_1 = \int_0^\infty \beta^{n_1+n_2-2}\left[\prod_{i=1}^{n_1} x_{1i}\right]^{\beta}\left[\prod_{i=1}^{n_2} x_{2i}\right]^{\beta} \left[\sum_{i=1}^{n_1} x_{1i}^{\beta}+\sum_{i=1}^{n_2} x_{2i}^{\beta}\right]^{-(n_1+n_2)} d\beta,$$

$$C_2 = \int_0^\infty \beta^{n_1+n_2-3}\left[\prod_{i=1}^{n_1} x_{1i}\right]^{\beta}\left[\prod_{i=1}^{n_2} x_{2i}\right]^{\beta} \left[\sum_{i=1}^{n_1} x_{1i}^{\beta}\right]^{-n_1}\left[\sum_{i=1}^{n_2} x_{2i}^{\beta}\right]^{-n_2} d\beta.$$

Nextly the marginal densities corresponding to the full data $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ for test $H_1: \beta_1=\beta_2$ vs. $H_2: \beta_1 \neq \beta_2$ can also be expressed in the following lemma.

Lemma 4. For the full data, we have the marginal density $m_h^N(\boldsymbol{x}_1, \boldsymbol{x}_2)$ under $H_h, h=1,2$ as follows.

$$m_1^N(\boldsymbol{x}_1, \boldsymbol{x}_2) = \Gamma(n_1)\Gamma(n_2)C_2\left[\prod_{i=1}^{n_1} x_{1i}\right]^{-1}\left[\prod_{i=1}^{n_2} x_{2i}\right]^{-1}, \quad (22)$$

$$m_2^N(\boldsymbol{x}_1, \boldsymbol{x}_2) = \Gamma(n_1)\Gamma(n_2)D_2\left[\prod_{i=1}^{n_1} x_{1i}\right]^{-1}\left[\prod_{i=1}^{n_2} x_{2i}\right]^{-1}, \quad (23)$$

where

$$C_2 = \int_0^\infty \beta^{n_1+n_2-3}\left[\prod_{i=1}^{n_1} x_{1i}\right]^{\beta}\left[\prod_{i=1}^{n_2} x_{2i}\right]^{\beta} \left[\sum_{i=1}^{n_1} x_{1i}^{\beta}\right]^{-n_1}\left[\sum_{i=1}^{n_2} x_{2i}^{\beta}\right]^{-n_2} d\beta,$$

$$D_2 = \int_0^\infty \beta_1^{n_1-2}\left[\prod_{i=1}^{n_1} x_{1i}\right]^{\beta_1}\left[\sum_{i=1}^{n_1} x_{1i}^{\beta_1}\right]^{-n_1} d\beta_1 \cdot \int_0^\infty \beta_2^{n_2-2}\left[\prod_{i=1}^{n_2} x_{2i}\right]^{\beta_2}\left[\sum_{i=1}^{n_2} x_{2i}^{\beta_2}\right]^{-n_2} d\beta_2 .$$

To test $H_1: \alpha_1=\alpha_2$ vs. $H_2: \alpha_1 \neq \alpha_2$, we get the following theorem from Lemmas 1 and 3.

Theorem 1. (i) The Bayes factor using the full data is given by

$$B_{21}^N = \frac{C_2\Gamma(n_1)\Gamma(n_2)}{C_1\Gamma(n_1+n_2)} . \quad (24)$$

(ii) The Bayes factor using the $\boldsymbol{x}(l) = (x_{1i}, x_{1j}, x_{2k}, x_{2l})$ is given by

$$B_{12}^N(\boldsymbol{x}(l)) = \Gamma(4)\frac{A_1}{A_2} . \quad (25)$$

From the Theorem 1, the arithmetic intrinsic Bayes factor B_{21}^{AI} to test $H_1: \alpha_1=\alpha_2$ vs. $H_2: \alpha_1 \neq \alpha_2$ is given by

$$B_{21}^{AI} = B_{21}^N \cdot \frac{1}{\binom{n_1}{2}\binom{n_2}{2}} \sum_l B_{12}^N(\boldsymbol{x}(l)). \quad (26)$$

Nextly we get the following theorem from Lemmas 2 and 4 to test $H_1: \beta_1=\beta_2$ vs. $H_2: \beta_1 \neq \beta_2$

Theorem 2. (i) The Bayes factor using the full data is given by

$$B_{21}^N = \frac{D_2}{C_2} . \quad (27)$$

(ii) The Bayes factor using the $\boldsymbol{x}(l) = (x_{1i}, x_{1j}, x_{2k}, x_{2l})$ is given by

$$B_{12}^N(\boldsymbol{x}(l)) = 4A_2\,|\log(x_{1j}/x_{1i})|\,|\log(x_{2l}/x_{2k})| . \quad (28)$$

From the Theorem 2, we can derive the

arithmetic intrinsic Bayes factor B_{21}^{AI} to test H_1 : $\beta_1=\beta_2$ vs. H_2 : $\beta_1\neq\beta_2$ as above (26).

Next we use the another intrinsic Bayes factor called median intrinsic Bayes factor (Berger and Pericchi (1998)). They showed that the median intrinsic Bayes factor seems to be a simple and very generally applicable intrinsic Bayes factor, which works well for nested or nonnested models, and even for small or moderate sample sizes.

From the Definition 3, Lemma 1, Lemma 3 and Theorem 1, we derive the median Bayes factors to test H_1 : $\alpha_1=\alpha_2$ vs. H_2 : $\alpha_1\neq\alpha_2$ as follow:

$$B_{12}^{MI}=B_{12}^{N}\cdot ME[B_{21}^{N}(x(l))], \qquad (30)$$

And also from the Definition 3, Lemma 2, Lemma 4 and Theorem 2, we can derive the median Bayes factors to test H_1 : $\beta_1=\beta_2$ vs. H_2 : $\beta_1\neq\beta_2$ as above.

4. Illustrative Examples

In this section, we present some examples to illustrate for our findings regarding the test (i) H_1 : $\alpha_1=\alpha_2$ vs. H_2 : $\alpha_1\neq\alpha_2$, and (ii) H_1 : $\beta_1=\beta_2$ vs. H_2 : $\beta_1\neq\beta_2$.

Example 1 : The data given below are the voltage levels at which failures occurred in two types of electrical cable insulation when specimens were subjected to an increasing voltage stress in a laboratory experiment (Lawless (1982)). The test involved 20 specimens of each type, an the failure voltages in kilovolts per millimeter were

Type I Insulation	32.0, 35.4, 36.2, 39.8, 41.2, 43.3, 45.5, 46.0, 46.2, 46.4, 46.5, 46.8, 47.3, 47.3, 47.6, 49.2, 50.4, 50.9, 52.4, 56.3
Type II Insulation	39.4, 45.3, 49.2, 49.4, 51.3, 52.0, 53.2, 53.2, 54.9, 55.5, 57.1, 57.2, 57.5, 59.2, 61.0, 62.4, 63.8, 64.3, 67.3, 67.7

For the type I insulation, the maximum likelihood estimates of α_1 and β_1 is $\widehat{\alpha}_1=47.781$ and $\widehat{\beta}_1=9.383$, respectively. And for type II insulation the maximum likelihood estimate of α_2 and β_2 are $\widehat{\alpha}_2=59.125$ and $\widehat{\beta}_2=9.141$, respectively. In table 1 and 2, we provide the P-value, Bayes factors and posterior probabilities for the test H_1 : $\alpha_1=\alpha_2$ vs. H_2 : $\alpha_1\neq\alpha_2$, and H_1 : $\beta_1=\beta_2$ vs. H_2 : $\beta_1\neq\beta_2$ for the failure voltages data. For the table 1, P-value computed by likelihood ratio statistic $\Lambda=-2\log L(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\beta})+2\log L(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\beta}_1, \widehat{\beta}_2)$ where $\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\beta}_1, \widehat{\beta}_2$ are the unrestricted MLE's of the parameters, and $\widetilde{\alpha}_1, \widetilde{\alpha}_2$ and $\widetilde{\beta}_1=\widetilde{\beta}_2=\widetilde{\beta}$ are the MLE's under H_1. And for the table 2, P-value computed by likelihood ratio statistic $\Lambda=-2\log L(\alpha^*, \alpha^*, \beta^*, \beta^*)+2\log L(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\beta}, \widetilde{\beta})$ where α^*, β^* are the MLE's of $\alpha_1=\alpha_2=\alpha$ and $\beta_1=\beta_2=\beta$ under H_1, and $\widetilde{\alpha}_1, \widetilde{\alpha}_2$ and $\widetilde{\beta}_1=\widetilde{\beta}_2=\widetilde{\beta}$ are the MLE's under H_2. The distribution of the above statistics, Λ, is approximately $\chi(1)$ under H_1 in large samples, respectively. From these tables, there is strong evidence for H_1 and H_2 in terms of the posterior probability and P-value, respectively.

Table 1 : P-value, Bayes factors and $P(H_1 \mid \boldsymbol{x})$ for testing $H_1 : \beta_1 = \beta_2$ vs. $H_2 : \beta_1 \neq \beta_2$ under the Voltage Data

$P-Value$	B_{21}^{AI}	B_{21}^{MI}	$P^{AI}(H_1 \mid \boldsymbol{x})$	$P^{MI}(H_1 \mid \boldsymbol{x})$
0.1697	0.1252	0.1844	0.8887	0.8443

Table 2 : P-value, Bayes factors and $P(H_1 \mid \boldsymbol{x})$ for testing $H_1 : \alpha_1 = \alpha_2$ vs. $H_2 : \alpha_1 \neq \alpha_2$ under the Voltage Data

$P-Value$	B_{21}^{AI}	B_{21}^{MI}	$P^{AI}(H_1 \mid \boldsymbol{x})$	$P^{MI}(H_1 \mid \boldsymbol{x})$
0.0000	14057.7413	5971.6329	0.0001	0.0002

Example 2 : The following data are time to breakdown of a type of electrical insulating fluid subject to a constant voltage stress (Nelson (1970)).

32 KV	0.27, 0.40, 0.69, 0.79, 2.75, 3.91, 9.88, 13.95, 15.93, 27.80, 53.24, 82.85, 89.29, 100.58, 215.10
38 KV	0.47, 0.73, 1.40, 0.74, 0.39, 1.13, 0.09, 2.38

For the 32KV, the maximum likelihood estimate of α_1 and β_1 is $\widehat{\alpha_1} = 28.94$ and $\widehat{\beta_1} = 0.561$, respectively. And for 38KV, the maximum likelihood estimate of α_2 and β_2 is $\widehat{\alpha_2} = 1.001$ and $\widehat{\beta_2} = 1.362$, respectively. In table 3, we provide the P-value, Bayes factors and posterior probabilities for the test $H_1 : \beta_1 = \beta_2$ vs. $H_2 : \beta_1 \neq \beta_2$ for the failures times for ball bearing data. From this table, there is evidence for H_2 in terms of the posterior probability and P-value. But in terms of the P-value, H_1 can accept for the significance level, α, is 0.01. It has been noticed that the generalized likelihood ratio test could be misleading, even when the sample size are large, (Berger, Brown and Wolpert, 1994). Now the Bayes factors are computed based on entire observations so that they give accurate interpretations.

Table 3 : P-value, Bayes factors and $P(H_1 \mid \boldsymbol{x})$ for testing $H_1 : \beta_1 = \beta_2$ vs. $H_2 : \beta_1 \neq \beta_2$ under the Voltage Breakdown Data

$P-Value$	B_{21}^{AI}	B_{21}^{MI}	$P^{AI}(H_1 \mid \boldsymbol{x})$	$P^{MI}(H_1 \mid \boldsymbol{x})$
0.0401	1.6536	1.7620	0.3768	0.3621

From the above examples, the arithmetic and median intrinsic Bayes factors are computed based on entire observations so that they give accurate interpretations and fairly steady answers.

5. Concluding Remark

We have suggested a Bayesian hypotheses testing criterion for the comparison of Weibull distributions via the intrinsic Bayes factor. We have derived the arithmetic and median intrinsic Bayes factors, and used these results to analyze real data sets. For the comparison of the classical test, P-values are computed by likelihood ratio statistic.

As we see from the numerical results, the arithmetic and median intrinsic Bayes factors are computed based on entire observations so that they give accurate interpretations and fairly steady answers.

In general, there has been a considerable amount of literature on the controversy between a P-value and a Bayes factor. It has been noticed that a P-value does not agree with the posterior probability that the null hypothesis is correct. Delampady and Berger (1990) have shown that the lower bounds of posterior probabilities in favor of null hypotheses are much larger than the corresponding P-values.

IBF methodology can be easily applied to nonnested as well as to nested problems. They can also be applied in general when the samples come from any distribution.

The study pertaining to applying the censored data to the Weibull lifetime model is left as a future study of interest.

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