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# STABILITY OF ISOMETRIES ON RESTRICTED DOMAINS

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Abstract. In the present paper, the classical results of the stability of isometries obtained by some authors will be generalized; More precisely, the stability of isometries on restricted (unbounded or bounded) domains will be investigated.

## 0. Introduction

Let $E$ and $F$ be real Banach spaces. A function $I : E \to F$ is called an isometry if $I$ satisfies the equality

$$\|I(x) - I(y)\| = \|x - y\|$$

for all $x, y \in E$.

Following D. H. Hyers and S. M. Ulam [8], a function $f : E \to F$ is called a $\delta$-isometry if $f$ satisfies the inequality

$$(*) \qquad \big|\, \|f(x) - f(y)\| - \|x - y\| \,\big| \le \delta$$

for all $x, y \in E$. If in this case there exist an isometry $I : E \to F$ and a constant $k \ge 0$ such that $\|f(x) - I(x)\| \le k\delta$ for all $x$ in $E$, then we say that the isometry from $E$ into $F$ is stable (in the sense of Hyers and Ulam).

Hyers and Ulam proved in the same paper the stability of isometries between real Hilbert spaces. More precisely, they proved that if a surjective function $f : E \to E$, where $E$ is a real Hilbert space, satisfies $f(0) = 0$ as well as the inequality $(*)$ for some $\delta \ge 0$ and for all

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$x, y \in E$, then there exists a surjective isometry $I : E \to E$ such that $\|f(x) - I(x)\| \leq 10\delta$ for any $x$ in $E$.

This result of Hyers and Ulam was the first one concerning the stability of isometries and was further generalized by D. G. Bourgin [1]. Indeed, Bourgin proved the following: Assume that $E$ is a Banach space and that $F$ belongs to a class of uniformly convex spaces which includes the spaces $L_p(0,1)$ for $1 < p < \infty$. If a function $f : E \to F$ satisfies $f(0) = 0$ as well as the inequality $(*)$ for some $\delta \geq 0$ and for all $x, y$ in $E$, then there exists a linear isometry $I : E \to F$ such that $\|f(x) - I(x)\| \leq 12\delta$ for each $x$ in $E$.

Subsequently, D. H. Hyers and S. M. Ulam [9] studied the stability problem for spaces of continuous functions: Let $S_1$ and $S_2$ be compact metric spaces and $C(S_i)$ denote the space of real-valued continuous functions on $S_i$ equipped with the metric topology with $\|\cdot\|_\infty$. If a homeomorphism $T : C(S_1) \to C(S_2)$ satisfies the inequality

$$(**) \qquad \big|\, \|T(f) - T(g)\|_\infty - \|f - g\|_\infty \,\big| \leq \delta$$

for some $\delta \geq 0$ and for all $f, g \in C(S_1)$, then there exists an isometry $U : C(S_1) \to C(S_2)$ such that $\|T(f) - U(f)\|_\infty \leq 21\delta$ for every $f \in C(S_1)$.

This result of Hyers and Ulam was significantly generalized by D. G. Bourgin again (see [2]): Let $S_1$ and $S_2$ be completely regular Hausdorff spaces and let $T : C(S_1) \to C(S_2)$ be a surjective function satisfying the inequality $(**)$ for some $\delta \geq 0$ and for all $f, g \in C(S_1)$. Then there exists a linear isometry $U : C(S_1) \to C(S_2)$ such that $\|T(f) - U(f)\|_\infty \leq 10\delta$ for any $f \in C(S_1)$.

The study of stability problems for isometries on finite dimensional Banach spaces was continued by R. D. Bourgin [3].

In 1978, P. M. Gruber [6] obtained an elegant result as follows: Let $E$ and $F$ be real normed spaces. Suppose that $f : E \to F$ is a surjective function and it satisfies the inequality $(*)$ for some $\delta \geq 0$ and for all $x, y \in E$. Furthermore, assume that $I : E \to F$ is an isometry with $f(p) = I(p)$ for some $p \in E$. If $\|f(x) - I(x)\| = o(\|x\|)$ as $\|x\| \to \infty$ uniformly, then $I$ is a surjective linear isometry and $\|f(x) - I(x)\| \leq 5\delta$ for all $x \in E$. If in addition $f$ is continuous, then $\|f(x) - I(x)\| \leq 3\delta$ for all $x \in E$.

J. Gevirtz [5] established the stability of isometries between arbitrary Banach spaces: Given real Banach spaces $E$ and $F$, let $f : E \to F$ be a surjective function satisfying the inequality $(*)$ for some $\delta \geq 0$ and for all $x, y \in E$. Then there exists a surjective isometry $I : E \longrightarrow F$ such that $\|f(x) - I(x)\| \leq 5\delta$ for each $x$ in $E$.

On the other hand, R. L. Swain [14] considered the stability of isometries on bounded metric spaces and proved the following result: Let $M$ be a subset of a compact metric space $(E, d)$ and let $\varepsilon > 0$ be given. Then there exists a $\delta > 0$ such that if $f : M \longrightarrow E$ satisfies the inequality $(*)$ for all $x, y \in M$, then there exists an isometry $I : M \longrightarrow E$ with $d(f(x), I(x)) \leq \varepsilon$ for any $x \in M$.

The stability problem of isometries on bounded subsets of $\mathbf{R}^n$ was studied by J. W. Fickett [4]: For $t \geq 0$, let us define $K_0(t) = K_1(t) = t$, $K_2(t) = 3\sqrt{3t}$, $K_i(t) = 27t^{m(i)}$, where $m(i) = 2^{1-i}$ for $i \geq 3$. Let $S$ be a bounded subset of $\mathbf{R}^n$ with diameter $d(S)$, and suppose that $3K_n(\delta/d(S)) \leq 1$ for some $\delta \geq 0$. If a function $f : S \longrightarrow \mathbf{R}^n$ satisfies the inequality $(*)$ for all $x, y \in S$, then there exists an isometry $I : S \longrightarrow \mathbf{R}^n$ such that $|f(x) - I(x)| \leq d(S)K_{n+1}(\delta/d(S))$ for each $x \in S$.

For more detailed information on the stability of isometries and related topics, one can refer to [11] and [12] (see also [10] and [13]).

In this paper, the results mentioned above will be generalized by studying the stability problems of isometries on restricted (bounded or unbounded) domains.

## 1. Stability on Unbounded Domains

In the following theorem, we will prove the stability of isometries (in the sense of Hyers, Ulam and Rassias) on restricted (unbounded) domains by applying the direct method. The direct method was first devised by D. H. Hyers to construct a true additive function from an approximate additive function (see [7]).

THEOREM 1. *Let $E$ be a real Hilbert space with the associated inner product $\langle \cdot, \cdot \rangle$. Let $d \geq 1$, $\delta \geq 0$, $0 \leq \theta \leq 16$ and $0 < p < 1$ be constants with $2d - 2\theta d^p - \delta \geq 0$. Denote by $B$ the punctured sphere defined by $B = \{x \in E : 0 < \|x\| \leq d\}$. If a function $f : E \longrightarrow E$ satisfies the functional inequality*

$$\big|\, \|f(x) - f(y)\| - \|x - y\| \,\big| \leq \delta + \theta\, \|x - y\|^p$$

*for all* $x, y \in E \backslash B$*, then there exists a unique linear isometry* $I : E \to E$ *such that*

$$\|f(x) - I(x) - f(0)\| \leq 2\delta + \frac{\sqrt{6\delta + 8\theta}}{\sqrt{2} - \sqrt{2^p}} \|x\|^{(1+p)/2} \tag{1}$$

*for all* $x \in E \backslash B$*.*

*Proof.* If we define a function $g : E \to E$ by $g(x) = f(x) - f(0)$, then we have

$$| \, \|g(x) - g(y)\| - \|x - y\| \, | \leq \delta + \theta \, \|x - y\|^p \tag{2}$$

for any $x, y \in E \backslash B$.

Let $x \in E \backslash (B \cup \{0\})$ be given. Putting $y = 0$ and $y = 2x$ separately in (2), we get

$$\begin{aligned} | \, \|g(x)\| - \|x\| \, | &\leq \delta + \theta \, \|x\|^p, \\ | \, \|g(x) - g(2x)\| - \|x\| \, | &\leq \delta + \theta \, \|x\|^p. \end{aligned} \tag{3}$$

Furthermore, replacing $x$ and $y$ by $2x$ and 0 in (2), respectively, yields

$$| \, \|g(2x)\| - 2\|x\| \, | \leq \delta + 2\theta \, \|x\|^p. \tag{4}$$

We will now prove that there is a constant $C > 0$ such that

$$\|g(x) - (1/2)g(2x)\| \leq \delta + C \, \|x\|^{(1+p)/2}. \tag{5}$$

It follows from (3) that

$$\|g(x)\|^2 \leq (\|x\| + \theta \, \|x\|^p + \delta)^2 \tag{6}$$

and

$$\begin{aligned} \|g(x) - g(2x)\|^2 &= \|g(x)\|^2 + \|g(2x)\|^2 - 2\langle g(x), g(2x)\rangle \\ &\leq (\|x\| + \theta \, \|x\|^p + \delta)^2 . \end{aligned} \tag{7}$$

The inequalities (6), (7) and (4), together with the assumption for $d$, $\delta$, $\theta$ and $p$, yield

$$\begin{aligned} & 2 \, \|g(x) - (1/2)g(2x)\|^2 \\ &= 2 \, \|g(x)\|^2 + (1/2)\|g(2x)\|^2 - 2 \, \langle g(x), g(2x)\rangle \\ &= \|g(x)\|^2 + \|g(x)\|^2 + \|g(2x)\|^2 - 2 \, \langle g(x), g(2x)\rangle - \ (1/2) \, \|g(2x)\|^2 \\ &\leq (\|x\| + \theta \, \|x\|^p + \delta)^2 + (\|x\| + \theta \, \|x\|^p + \delta)^2 \\ &\quad - (1/2) \, (2\|x\| - 2\theta \, \|x\|^p - \delta)^2 \end{aligned}$$

$$\leq (3/2)\delta^2 + 8\theta \|x\|^{1+p} + 2\delta\theta \|x\|^p + 6\delta \|x\|$$
$$\leq 2\left(\delta + \sqrt{4\theta + 3\delta}\, \|x\|^{(1+p)/2}\right)^2.$$

Hence, if we put $C := \sqrt{4\theta + 3\delta}$ in the last inequality, we get the inequality (5).

Next, we use induction to prove

$$\|g(x) - 2^{-n}g(2^n x)\| \leq \delta \sum_{i=0}^{n-1} 2^{-i} + C\,\|x\|^{(1+p)/2} \sum_{i=0}^{n-1} 2^{-(1-p)i/2} \tag{8}$$

for all $x \in E\backslash(B \cup \{0\})$ and $n \in \mathbf{N}$. The inequality (5) implies the validity of (8) for $n = 1$. Assume that the inequality (8) holds for some $n \geq 2$. If we replace $x$ by $2x$ in (8) and then divide by 2 the resulting inequality, we can conclude by considering (5) that (8) holds for $n + 1$, which completes the proof of (8).

If we substitute $2^m x$ and $n - m$ ($n > m$) for $x$ and $n$ in (8), respectively, and then divide the resulting inequality by $2^m$, we get

$$\|2^{-m}g(2^m x) - 2^{-n}g(2^n x)\| \leq \delta \sum_{i=m}^{n-1} 2^{-i} + C\,\|x\|^{(1+p)/2} \sum_{i=m}^{n-1} 2^{-(1-p)i/2},$$

which implies that the sequence $\{2^{-n}g(2^n x)\}$ is a Cauchy sequence for each $x \in E\backslash B$. Since $E$ is assumed to be complete, we can define a function $I : E \to E$ by

$$I(x) := 2^{-n(x)} \lim_{n\to\infty} 2^{-n}g(2^{n+n(x)}x), \tag{9}$$

where $n(x) = \min\{n \in \mathbf{N}_0 : 2^n x \notin B\}$. Because of the fact that $n(x) = 0$ for $x \in E\backslash B$, the definition (9) reduces to

$$I(x) = \lim_{n\to\infty} 2^{-n}g(2^n x) \tag{10}$$

for each $x \in E\backslash B$. Since if $x \in B$ then $2^{n(x)}x \in E\backslash B$, the function $I : E \to E$ is well defined.

Let $x, y \in E\backslash B$ be given. Then $2^n x, 2^n y$ also belong to $E\backslash B$ for all $n \in \mathbf{N}$. Hence, by (2) we obtain

$$\big|\, \|2^{-n}g(2^n x) - 2^{-n}g(2^n y)\| - \|x - y\| \,\big| \leq 2^{-n}\delta + 2^{-(1-p)n}\theta\,\|x - y\|^p$$

for every $n \in \mathbf{N}$. Taking the limit as $n \to \infty$ in the last inequality and considering (10), we see that $I|_{E\backslash B}$ is an isometry.

Assume that $x \in B$ and $y \in E\backslash B$. Since $2^{n(x)}x$ and $2^{n(x)}y$ belong to $E\backslash B$, it follows from (9) and (10) that

$$2^{n(x)}\|I(x) - I(y)\| = \|I(2^{n(x)}x) - I(2^{n(x)}y)\| = 2^{n(x)}\|x - y\|.$$

Assume that both $x$ and $y$ belong to $B$. By (9) and (10) again, we have

$$\begin{aligned} 2^{n(x)+n(y)}\|I(x) - I(y)\| &= \|I(2^{n(x)+n(y)}x) - I(2^{n(x)+n(y)}y)\| \\ &= 2^{n(x)+n(y)}\|x - y\|. \end{aligned}$$

Altogether, we conclude that $I : E \to E$ is an isometry.

We will now prove the uniqueness of $I$. According to Lemma 6.2 in [12], every inner product preserving function between Hilbert spaces is linear. We can easily see that the isometry $I$ is inner product preserving because of the fact $I(0) = 0$. Therefore, $I$ is a linear isometry. Assume that there is another linear isometry $I^* : E \to E$ satisfying the inequality (1) for all $x$ in $E\backslash B$ instead of $I$. Since $I$ and $I^*$ are linear, we obtain

$$\begin{aligned} \|I(x) - I^*(x)\| &= 2^{-n}\,\|I(2^n x) - I^*(2^n x)\| \\ &\le 2^{-(n-2)}\delta + 2^{1-(1-p)n/2}\frac{\sqrt{6\delta + 8\theta}}{\sqrt{2} - \sqrt{2^p}}\,\|x\|^{(1+p)/2} \\ &\to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

for every $x$ in $E$.

Inequality (8), together with (10), yields the validity of the inequality (1). □

In the case when $\theta = 0$ in the inequality given in Theorem 1, we will improve the modulus in the inequality (1) by using the method presented in the paper [8] of Hyers and Ulam.

First, we will prove in the following lemma that if $x$ and $u$ are orthogonal, then the inner product of $f(x)$ and $I(u)$ is not so large.

LEMMA 2. *Let $E$, $B$, $d$ and $\delta$ be those stated in* Theorem 1, *let a function $f : E \to E$ satisfy the inequality*

$$(11) \qquad \big|\,\|f(x) - f(y)\| - \|x - y\|\,\big| \le \delta$$

*for all $x, y \in E\backslash B$, and let $I$ be the isometry given in* Theorem 1. *Assume that $x \in E\backslash(B \cup \{0\})$ and that $u \in E$ satisfies $\|u\| = 1$. It then holds*

$$(12) \qquad |\,\langle f(x) - f(0), I(u)\rangle\,| \le 3\delta$$

*whenever* $\langle x, u\rangle = 0$.

*Proof.* As was in the proof of Theorem 1, if we define a function $g : E \to E$ by $g(x) = f(x) - f(0)$, then $g$ itself satisfies the inequality (11) instead of $f$.

Assume that $x$ and $u$ are orthogonal. We can choose an integer $n \geq n(u)$ such that

$$y = x + ru \notin B \cup \{0\}, \tag{13}$$

where $r = 2^n - \sqrt{2^{2n} - \|x\|^2}$. Put $z = 2^n u$. Then

$$\|y - z\|^2 = \langle y, y\rangle - 2\langle y, z\rangle + \langle z, z\rangle = \|z\|^2. \tag{14}$$

Since $g$ satisfies the inequality (11) for all $x, y \in E \backslash B$, it follows from (11) and (14) that

(15)
$$\|g(y) - g(z)\| = \zeta(y, z) + \|y - z\| = \zeta(y, z) + \|z\| = \eta(y, z) + \|g(z)\|,$$

where $\zeta$ and $\eta$ are appropriate correction factors with $|\zeta(y, z)| \leq \delta$ and $|\eta(y, z)| \leq 2\delta$.

Squaring both sides of (15), we get

$$2\langle g(y), g(z)\rangle = \langle g(y), g(y)\rangle - 2\eta(y, z)\|g(z)\| - \eta(y, z)^2;$$

and dividing by $2^{n+1}$ both sides of the last equality allows us to obtain

(16)
$$\langle g(y), 2^{-n} g(2^n u)\rangle = 2^{-(n+1)}(\langle g(y), g(y)\rangle - \eta(y, z)^2) - \eta(y, z)\|2^{-n} g(2^n u)\|.$$

The relation (13) yields

$$\|y - x\| = r \to 0 \quad \text{as} \quad n \to \infty. \tag{17}$$

Furthermore, we see by (9) that

$$\lim_{n\to\infty} 2^{-n} g(2^n u) = 2^{-n(u)} \lim_{n\to\infty} 2^{-(n-n(u))} g(2^{n-n(u)} 2^{n(u)} u) = I(u). \tag{18}$$

*Since $I$ is an isometry with $I(0) = 0$, $\|u\| = 1$ implies $\|I(u)\| = 1$.*

In view of (16), (17) and (18), we can choose a sufficiently large integer $n$ such that the following inequalities hold for an arbitrary $\beta > 0$:

$$\begin{aligned} |\langle g(x), I(u)\rangle| &\le |\langle g(x), I(u) - 2^{-n}g(2^n u)\rangle| + |\langle g(y), 2^{-n}g(2^n u)\rangle| \\ &\quad + |\langle g(x) - g(y), 2^{-n}g(2^n u)\rangle| \\ &\le \|g(x)\|\,\|I(u) - 2^{-n}g(2^n u)\| + \beta + 2\delta\,\|2^{-n}g(2^n u)\| \\ &\quad + \|g(x) - g(y)\|\,\|2^{-n}g(2^n u)\| \\ &\le \beta + \beta + 2\delta(1+\beta) + (\delta+\beta)(1+\beta). \end{aligned}$$

Hence, taking the limit as $\beta \to 0$, we see that the inequality (12) follows. □

In the following theorem, we will improve the result of Theorem 1 in the case when $\theta = 0$ in the relevant inequality.

THEOREM 3. *Let $E$ be a real Hilbert space with the associated inner product $\langle \cdot, \cdot \rangle$. Let $d \ge 1$ and $0 \le \delta \le (1/20)d$ be fixed, and by $B$ denote the punctured sphere defined in* Theorem 1. *If a function $f : E \to E$ satisfies the inequality* (11) *for all $x, y \in E\backslash B$, then there exists a unique linear isometry $I : E \to E$ such that*

$$\|f(x) - I(x) - f(0)\| \le 5\delta \tag{19}$$

*for all $x \in E\backslash B$.*

*Proof.* Let us define a function $g : E \to E$ by $g(x) = f(x) - f(0)$. Then, $g$ satisfies the inequality (11) for all $x, y \in E\backslash B$. Furthermore, define the isometry $I : E \to E$ by (9). By Lemma 6.2 in [12], $I$ is a linear isometry.

Assume that $x \in E\backslash(B \cup \{0\})$ is given and that $M$ is the (linear) subspace orthogonal to $x$. Then, $I(M)$ is the subspace orthogonal to $I(x)$. Let $w$ be the projection of $g(x)$ on $I(M)$. Define

$$t = \begin{cases} 0 & \text{for } w = 0, \\ w/\|w\| & \text{for } w \ne 0. \end{cases}$$

From Lemma 2 it follows that

$$|\langle g(x), t\rangle| \le 3\delta, \tag{20}$$

since we can choose a $u \in M$ such that $\langle x, u\rangle = 0$, $\|u\| = 1$ and $I(u) = t$ (for $t \ne 0$). Put $v = I(x)/\|x\|$. Then, $v$ is a unit vector orthogonal to $t$

and is coplanar with $g(x)$ and $t$. By the Pythagorean theorem, we have

$$\|g(x) - I(x)\|^2 = \langle g(x), t\rangle^2 + (\|x\| - \langle g(x), v\rangle)^2 . \tag{21}$$

Let $z_n = 2^n x$ for a non-negative integer $n$. By $w_n$ denote the projection of $g(z_n)$ on $I(M)$ and define

$$t_n = \begin{cases} 0 & \text{for } w_n = 0, \\ w_n/\|w_n\| & \text{for } w_n \neq 0. \end{cases}$$

We then obtain $\langle t_n, v\rangle = 0$ and by Lemma 2 we have

$$|\langle g(z_n), t_n\rangle| \leq 3\delta, \tag{22}$$

since there is a $u_n \in M$ such that $\langle z_n, u_n\rangle = 0$, $\|u_n\| = 1$ and $I(u_n) = t_n$ (for $t_n \neq 0$). By (22), we can obtain

$$\begin{aligned} 0 \;\leq\; \|g(z_n)\| - |\langle g(z_n), v\rangle| \;&=\; \|g(z_n)\| - \left(\|g(z_n)\|^2 - \langle g(z_n), t_n\rangle^2\right)^{1/2} \\ \leq\; \|g(z_n)\| \left(1 - \left(1 - \frac{\langle g(z_n), t_n\rangle^2}{\|g(z_n)\|^2}\right)\right) \;&\leq\; (1/2)\delta, \end{aligned}$$

since $\|g(z_n)\| \geq \|z_n\| - \delta > d - \delta \geq 19\delta$. The fact $\|z_n\| \leq \|g(z_n)\| + \delta$, together with the last inequality, implies that

$$\big|\, \|z_n\| - |\langle g(z_n), v\rangle| \,\big| \leq (3/2)\delta. \tag{23}$$

In the case of $\langle g(x), v\rangle \geq 0$, we put $n = 0$ in (23) and use (21) and (20) to obtain

$$\|g(x) - I(x)\| \leq (3/2)\sqrt{5}\,\delta.$$

In the case of $\langle g(x), v\rangle < 0$, there exists an integer $m \geq 0$ such that $\langle g(z_m), v\rangle < 0$ and $\langle g(z_{m+1}), v\rangle \geq 0$ because $\langle I(x), v\rangle$ is positive and $I(x) = \lim_{n\to\infty} 2^{-n} g(z_n)$. Hence, (23) yields

$$\|g(z_{m+1}) - g(z_m)\| \geq \langle g(z_{m+1}), v\rangle - \langle g(z_m), v\rangle \geq 3\,\|z_m\| - 3\delta.$$

Since $\|g(z_{m+1}) - g(z_m)\| \leq \|z_m\| + \delta$, we get from the last inequality that $\|x\| \leq \|z_m\| \leq 2\delta$, and hence

$$\|g(x) - I(x)\| \leq \|g(x)\| + \|I(x)\| \leq \|x\| + \delta + \|x\| \leq 5\delta.$$

The uniqueness of the linear isometry $I$ can be easily proved in the same way as in the proof of Theorem 1. □

By a slight modification of the proof of Theorem 3 we may easily prove the following corollary. Hence, we omit the proof.

COROLLARY 4. *Let $E$, $B$, $d$ and $f$ be those defined in* Theorem 3. *For each $\varepsilon > 0$ there exists a sufficiently small $\delta > 0$ such that if a function $f : E \to E$ satisfies the inequality* (11) *for all $x, y \in E\backslash B$, then there exists a unique linear isometry $I : E \to E$ with*

$$\|f(x) - I(x) - f(0)\| \le 4\delta + \varepsilon$$

*for any $x \in E\backslash B$.*

## 2. Stability on Bounded Domains

In the previous section, we have investigated the stability problems in connection with the unbounded domains. We will now prove the stability of isometries (in the sense of Hyers, Ulam and Rassias) on bounded domains by using the direct method.

THEOREM 5. *Let $E$ be a real Hilbert space with the associated inner product $\langle \cdot, \cdot \rangle$. Given $0 < d \le 1$, denote by $D$ the sphere of radius $d$ and center at 0, i.e., $D = \{x \in E : \|x\| \le d\}$. If a function $f : E \to E$ satisfies the inequality*

$$\big|\, \|f(x) - f(y)\| - \|x - y\| \,\big| \le \theta\, \|x - y\|^p$$

*for some $0 \le \theta \le 1$, some $p > 1$ and for all $x, y \in D$, then there exists a unique linear isometry $I : E \to E$ such that*

$$\|f(x) - I(x) - f(0)\| \le \frac{2^{(1+p)/2}}{2^{(p-1)/2} - 1} \sqrt{\theta}\, \|x\|^{(1+p)/2} \tag{24}$$

*for every $x \in D$.*

*Proof.* If we define a function $g : E \to E$ by $g(x) = f(x) - f(0)$, then $g$ satisfies

$$\big|\, \|g(x) - g(y)\| - \|x - y\| \,\big| \le \theta\, \|x - y\|^p \tag{25}$$

for all $x, y \in D$.

Let $x \in D$ be given. Putting $y = 0$ in (25) we get

$$\big|\, \|g(x)\| - \|x\| \,\big| \le \theta\, \|x\|^p. \tag{26}$$

Replacing $y$ by $(1/2)x$ in (25) again yields

$$\begin{aligned} \|g(x) - g((1/2)x)\|^2 &= \|g(x)\|^2 + \|g((1/2)x)\|^2 - 2\langle g(x), g((1/2)x)\rangle \\ &\le \left((1/2)\|x\| + 2^{-p}\theta\, \|x\|^p\right)^2. \end{aligned} \tag{27}$$

Using (26) and (27) we obtain

$$\begin{aligned}&(1/2)\|g(x) - 2g((1/2)x)\|^2 \\ &= (1/2)\|g(x)\|^2 + 2\,\|g((1/2)x)\|^2 - 2\langle g(x), g((1/2)x)\rangle \\ &\le -\,(1/2)\,(\|x\| - \theta\,\|x\|^p)^2 + ((1/2)\|x\| + 2^{-p}\theta\,\|x\|^p)^2 \\ &\quad +\ ((1/2)\|x\| + 2^{-p}\theta\,\|x\|^p)^2 \\ &\le 2\theta\,\|x\|^{1+p}.\end{aligned}$$

Hence, we have

$$\|g(x) - 2g((1/2)x)\| \le 2\sqrt{\theta}\,\|x\|^{(1+p)/2}. \tag{28}$$

We will now use induction on $n$ to prove

$$\|g(x) - 2^n g(2^{-n}x)\| \le 2\sqrt{\theta}\,\|x\|^{(1+p)/2}\sum_{i=0}^{n-1} 2^{-(p-1)i/2} \tag{29}$$

for all $x \in D$ and for every $n \in \mathbf{N}$. The inequality (28) implies the validity of (29) for $n = 1$. Assume that the inequality (29) holds for some $n \ge 2$. If we replace $x$ by $(1/2)x$ in (29) and then multiply by 2 the resulting inequality, we may conclude by considering (28) that (29) holds for $n + 1$.

If we substitute $2^{-m}x$ and $n - m$ $(n > m)$ for $x$ and $n$ in (29), respectively, and then multiply the resulting inequality by $2^m$, we obtain

$$\|2^m g(2^{-m}x) - 2^n g(2^{-n}x)\| \le 2\sqrt{\theta}\,\|x\|^{(1+p)/2}\sum_{i=m}^{n-1} 2^{-(p-1)i/2},$$

which implies that the sequence $\{2^n g(2^{-n}x)\}$ is a Cauchy sequence. Hence, we can define a function $I : E \to E$ by

$$I(x) = \lim_{n\to\infty} 2^{n+m(x)} g(2^{-n-m(x)}x)$$

for each $x \in E$, where we set

$$m(x) = \min\{n \in \mathbf{N}_0 : 2^{-n}x \in D\}.$$

It is not difficult to prove that $I$ is an isometry (cf. proof of Theorem 1).

The inequality (24) immediately follows from the inequality (29) because $m(x) = 0$ holds for each $x$ in $D$.

Due to Lemma 6.2 in [12], the isometry $I$ is linear (cf. the proof of Theorem 1 or 3). If $I^* : E \longrightarrow E$ is another linear isometry satisfying the inequality (24), then

$$\begin{aligned} \|I(x) - I^*(x)\| &= 2^n \|I(2^{-n}x) - I^*(2^{-n}x)\| \\ &\le \frac{2^{(3+p)/2}}{2^{(p-1)/2} - 1}\sqrt{\theta}\, 2^{-(1+p)n/2} \|x\|^{(1+p)/2} \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

for every $x$ in $E$. □

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