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A NOTE ON THE HYERS-ULAM-RASSIAS STABILITY OF PEXIDER EQUATION

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ABSTRACT. In this paper we obtain the Hyers-Ulam-Rassias stability of the Pexider equation $f(x+y) = g(x) + h(y)$ in the spirit of Hyers, Ulam, Rassias and Găvruta.

1. Introduction

In 1940, S. M. Ulam [13] posed the following question concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by D. H. Hyers [2] under the assumption that G_1 and G_2 are Banach spaces.

Throughout this paper, we denote by X a Banach space. In 1978, Th. M. Rassias [11] gave a generalization of the Hyers' result in the following way:

Let V be a normed space and let $f : V \to X$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geq 0$ and $p \neq 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in V$(for all $x, y \in V \setminus \{0\}$ if $p < 0$). Then there exists a unique linear mapping $T : V \to X$ such that

$$\|T(x) - f(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p$$

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for all $x \in V$ (for all $x \in V \setminus \{0\}$ if $p < 0$). However, it was showed that a similar result for the case $p = 1$ does not hold (see [12]). Recently, Găvruta [1] also obtained a further generalization of the Hyers-Rassias theorem (see also [3,4,7,9]).

According to Theorem 6 in [10], a mapping $f : V \to X$ satisfying $f(0) = 0$ is a solution of the Jensen's functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$$

if and only if it satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$.

In this paper, using the idea from the papers of D. H. Hyers [2], Th. M. Rassias [11] and Găvruta [1], we obtain the Hyers-Ulam-Rassias stability of the Jensen equation and the Pexider equation:

$$f(x+y) = g(x) + h(y).$$

The following result follows from Lemma 2.1 and Lemma 3.1.

THEOREM 1.1. *Let V be a normed space and let $f : V \to X$ be a mapping. Assume that there exist $\theta \geq 0$ and $p \in [0, \infty) \setminus \{1\}$ such that*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in V$. Then there exists a unique additive mapping $T : V \to X$ such that

$$\|T(x) - f(x) + f(0)\| \leq \frac{2^p \theta}{|2 - 2^p|} \|x\|^p$$

for all $x \in V$.

We obtain the following theorem from Corollary 2.5 and Corollary 3.4.

THEOREM 1.2. *Let V be a normed space and let $f, g, h : V \to X$ be mappings. Assume that there exist $\theta \geq 0$ and $p \in [0, \infty) \setminus \{1\}$ such that*

$$\|f(x+y) - g(x) - h(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in V$. *Then there exists a unique additive mapping* $T : V \to X$ *such that*

$$\|T(x) - f(x) + f(0)\| \leq \frac{4\theta}{|2^p - 2|}\|x\|^p + \theta M$$
$$\|T(x) - g(x) + g(0)\| \leq \frac{(4 + 2^p)\theta}{|2^p - 2|}\|x\|^p + \theta M$$
$$\|T(x) - h(x) + h(0)\| \leq \frac{(4 + 2^p)\theta}{|2^p - 2|}\|x\|^p + \theta M$$

for all $x \in V$ *where* $M = \|f(0) - g(0) - h(0)\|$ (*if* $1 < p$ *then* $M = 0$).

2. Stability in the case $p < 1$

We denote by G an abelian group. We also denote by $\varphi : G \times G \to [0, \infty)$ a mapping such that

(1) $$\tilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. It is easy to show that $\tilde{\varphi}(0, 0) = 2\varphi(0, 0)$ and $\varphi(0, 0)$ can be replaced by an arbitrary nonnegative real number without the loss of property (1). The following lemma for the stability of Jensen's equation is well known (see [6,8]).

LEMMA 2.1. *Let* $f : G \to X$ *be a mapping such that*

(2) $$\left\|2f\left(\frac{x + y}{2}\right) - f(x) - f(y)\right\| \leq \varphi(x, y)$$

for all $x, y \in 2G$. *Then there exists a unique mapping* $T : G \to X$ *such that*

$$T(x + y) = T(x) + T(y) \quad \text{for all} \quad x, y \in G,$$

(3) $$\|T(x) - f(x) + f(0)\| \leq \frac{1}{2}\tilde{\varphi}(2x, 0) \quad \text{for all} \quad x \in G$$

and

$$T(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \quad \text{for} \quad x \in G.$$

Proof. Let $g(x) = f(x) - f(0)$. Then g satisfies (2) and φ satisfies (1) and (2) . From this we can assume that $f(0) = 0$ without the loss of generality.
Replacing x by $2^{n+1}x$ and y by 0 and dividing 2^{n+1} on the both sides in (2), we have

$$\|2^{-n}f(2^n x) - 2^{-n-1}f(2^{n+1}x)\| \leq 2^{-n-1}\varphi(2^{n+1}x, 0). \tag{4}$$

Hence

$$\begin{aligned} \|2^{-m}f(2^m x) - 2^{-n}f(2^n x)\| &\leq \sum_{i=m}^{n-1} \|2^{-i}f(2^i x) - 2^{-i-1}f(2^{i+1}x)\| \\ &\leq \sum_{i=m}^{n-1} 2^{-i-1}\varphi(2^{i+1}x, 0) \end{aligned} \tag{5}$$

for $n > m$. From (1) and (5), we obtain the sequence $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence. Because X is a Banach space, the sequence $\{2^{-n}f(2^n x)\}$ converges. Denote

$$T(x) = \lim_{n\to\infty} 2^{-n}f(2^n x)$$

for all x in G. By the definition of T and (2)

$$2T(x+y) = 2T\left(\frac{2x+2y}{2}\right) = T(2x) + T(2y) = 2(T(x) + T(y))$$

for all $x, y \in G$. This proves that

$$T(x+y) = T(x) + T(y) \text{ for all } x, y \in G.$$

From (4), we obtain

$$\begin{aligned} \|f(x) - 2^{-n}f(2^n x)\| &\leq \sum_{i=0}^{n-1} \|2^{-i}f(2^i x) - 2^{-i-1}f(2^{i+1}x)\| \\ &\leq \sum_{i=0}^{n-1} 2^{-i-1}\varphi(2^{i+1}x, 0) \\ &\leq 2^{-1}\tilde{\varphi}(2x, 0). \end{aligned}$$

Taking the limit in the above inequality, we obtain (3). Now we prove the uniqueness of T. Let $S : G \to X$ be another additive mapping satisfying (3). Then

$$\begin{aligned}\|S(x)-T(x)\| &\le \left\|\frac{S(2^n x)}{2^n}-\frac{f(2^n x)-f(0)}{2^n}\right\| \\ &+ \left\|\frac{f(2^n x)-f(0)}{2^n}-\frac{T(2^n x)}{2^n}\right\| \\ &\le \frac{\tilde{\varphi}(2^{n+1}x,0)}{2^n}.\end{aligned}$$

Taking the limit in the above inequality as $n \to \infty$, we obtain

$$S(x) = T(x).$$

□

From Lemma 2.1, we can modify the results of [5] in the following theorem.

THEOREM 2.2. *Let $f, g, h : G \to X$ be mappings such that*

$$\|f(x+y)-g(x)-h(y)\| \le \varphi(x,y) \tag{6}$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to X$ such that

$$\|T(x)-f(x)+f(0)\| \le \frac{1}{2}[\tilde{\varphi}(x,0)+\tilde{\varphi}(0,x)+\tilde{\varphi}(x,x)]+M, \tag{7}$$

$$\|T(x)-g(x)+g(0)\| \le \frac{1}{2}[\tilde{\varphi}(x,-x)+\tilde{\varphi}(x,0)+\tilde{\varphi}(2x,-x)]+M, \tag{8}$$

$$\|T(x)-h(x)+h(0)\| \le \frac{1}{2}[\tilde{\varphi}(-x,x)+\tilde{\varphi}(0,x)+\tilde{\varphi}(-x,2x)]+M \tag{9}$$

where $M = \|f(0)-g(0)-h(0)\|$ and

$$T(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^n} = \lim_{n\to\infty}\frac{g(2^n x)}{2^n} = \lim_{n\to\infty}\frac{h(2^n x)}{2^n} \quad \text{for } x \in G. \tag{10}$$

Proof. We can replace $\varphi(0,0)$ by $\|f(0)-g(0)-h(0)\|$ without the loss of property (1) and (6). From (6), we get

$$\begin{aligned}
&\Big\|2f\Big(\frac{x+y}{2}\Big)-f(x)-f(y)\Big\|\\
&\le \Big\|f\Big(\frac{x+y}{2}\Big)-g\Big(\frac{x}{2}\Big)-h\Big(\frac{y}{2}\Big)\Big\|+\Big\|f\Big(\frac{x+y}{2}\Big)-g\Big(\frac{y}{2}\Big)-h\Big(\frac{x}{2}\Big)\Big\|\\
&\qquad+\Big\|f(x)-g\Big(\frac{x}{2}\Big)-h\Big(\frac{x}{2}\Big)\Big\|+\Big\|f(y)-g\Big(\frac{y}{2}\Big)-h\Big(\frac{y}{2}\Big)\Big\|\\
&\le \varphi\Big(\frac{x}{2},\frac{y}{2}\Big)+\varphi\Big(\frac{y}{2},\frac{x}{2}\Big)+\varphi\Big(\frac{x}{2},\frac{x}{2}\Big)+\varphi\Big(\frac{y}{2},\frac{y}{2}\Big)
\end{aligned}$$

for all $x,y\in 2G$. Let

$$\varphi_1(x,y)=\varphi\Big(\frac{x}{2},\frac{y}{2}\Big)+\varphi\Big(\frac{y}{2},\frac{x}{2}\Big)+\varphi\Big(\frac{x}{2},\frac{x}{2}\Big)+\varphi\Big(\frac{y}{2},\frac{y}{2}\Big)$$

for all $x,y\in 2G$. Applying Lemma 2.1, there exists a unique mapping $T:G\to X$ satisfying (7) and

$$T(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}\quad\text{for}\quad x\in G. \tag{11}$$

From (6), we get

$$\begin{aligned}
&\Big\|2g\Big(\frac{x+y}{2}\Big)-g(x)-g(y)\Big\|\\
&\le \Big\|f\Big(\frac{y}{2}\Big)-g\Big(\frac{x+y}{2}\Big)-h\Big(\frac{-x}{2}\Big)\Big\|+\Big\|f\Big(\frac{x}{2}\Big)-g\Big(\frac{x+y}{2}\Big)-h\Big(\frac{-y}{2}\Big)\Big\|\\
&\qquad+\Big\|-f\Big(\frac{x}{2}\Big)+g(x)+h\Big(\frac{-x}{2}\Big)\Big\|+\Big\|-f\Big(\frac{y}{2}\Big)+g(y)+h\Big(-\frac{y}{2}\Big)\Big\|\\
&\le \varphi\Big(\frac{x+y}{2},-\frac{x}{2}\Big)+\varphi\Big(\frac{x+y}{2},-\frac{y}{2}\Big)+\varphi\Big(x,-\frac{x}{2}\Big)+\varphi\Big(y,-\frac{y}{2}\Big)
\end{aligned}$$

for all $x,y\in 2G$. Let

$$\varphi_2(x,y)=\varphi\Big(\frac{x+y}{2},-\frac{x}{2}\Big)+\varphi\Big(\frac{x+y}{2},-\frac{y}{2}\Big)+\varphi\Big(x,-\frac{x}{2}\Big)+\varphi\Big(y,-\frac{y}{2}\Big)$$

for all $x,y\in 2G$. Applying Lemma 2.1 again, there exists a unique mapping $T_1:G\to X$ satisfying (8) and

$$T_1(x)=\lim_{n\to\infty}\frac{g(2^n x)}{2^n}\quad\text{for}\quad x\in G. \tag{12}$$

From (6), we get

$$\begin{aligned}
&\left\|2h\left(\frac{x+y}{2}\right)-h(x)-h(y)\right\| \\
&\leq \left\|f\left(\frac{y}{2}\right)-g\left(\frac{-x}{2}\right)-h\left(\frac{x+y}{2}\right)\right\|+\left\|f\left(\frac{x}{2}\right)-g\left(\frac{-y}{2}\right)-h\left(\frac{x+y}{2}\right)\right\| \\
&\qquad+\left\|-f\left(\frac{x}{2}\right)+g\left(\frac{-x}{2}\right)+h(x)\right\|+\left\|-f\left(\frac{y}{2}\right)+g\left(-\frac{y}{2}\right)+h(y)\right\| \\
&\leq \varphi\left(-\frac{x}{2},\frac{x+y}{2}\right)+\varphi\left(-\frac{y}{2},\frac{x+y}{2}\right)+\varphi\left(-\frac{x}{2},x\right)+\varphi\left(-\frac{y}{2},y\right)
\end{aligned}$$

for all $x, y \in 2G$. Let

$$\varphi_3(x,y)=\varphi\left(-\frac{x}{2},\frac{x+y}{2}\right)+\varphi\left(-\frac{y}{2},\frac{x+y}{2}\right)+\varphi\left(-\frac{x}{2},x\right)+\varphi\left(-\frac{y}{2},y\right)$$

for all $x, y \in 2G$. Similarly, there exists a unique mapping $T_2 : G \to X$ satisfying (9) and

$$T_2(x)=\lim_{n\to\infty}\frac{h(2^n x)}{2^n} \quad \text{for} \quad x\in G. \tag{13}$$

Replacing x by $2^n x$ and y by 0 in (6), we get

$$\left\|\frac{f(2^n x)}{2^n}-\frac{g(2^n x)}{2^n}\right\|\leq\frac{1}{2^n}\varphi(2^n x,0). \tag{14}$$

Taking the limit in (14), we obtain

$$T(x)=T_1(x) \quad \text{for} \quad x\in G.$$

By the similar method we have $T = T_2$. From (11), (12) and (13), we obtain (10). □

COROLLARY 2.3. *Let V be a vector space. Let $f, g, h : V \to X$ be mappings such that*

$$\|f(x+y)-g(x)-h(y)\|\leq\varphi(x,y)$$

for all $x, y \in V$. Then there exists a unique additive mapping $T : V \to X$ satisfying (7), (8), (9) *and* (10).

The following corollary is a generalization of Theorem 1 in [4]

COROLLARY 2.4. *Let V be a normed space. Let $\psi : [0,\infty) \to R^+$ be a function such that*
(i) $\psi(ts) \le \psi(t)\psi(s)$ *for all* $t, s > 0$ *and*
(ii) $\psi(2)/2 < 1$.
Let $f, g, h : V \to X$ be mappings such that

$$\|f(x+y) - g(x) - h(y)\| \le \psi(\|x\|) + \psi(\|y\|) \text{ for } x \ne 0 \text{ or } y \ne 0.$$

Then there exists a unique additive mapping $T : V \to X$ such that

$$\|T(x) - f(x) + f(0)\| \le \frac{4\psi(\|x\|)}{2-\psi(2)} + 2\psi(0) + M$$

$$\|T(x) - g(x) + g(0)\| \le \frac{(4+\psi(2))\psi(\|x\|)}{2-\psi(2)} + \psi(0) + M \tag{15}$$

$$\|T(x) - h(x) + h(0)\| \le \frac{(4+\psi(2))\psi(\|x\|)}{2-\psi(2)} + \psi(0) + M$$

for all $x \in V$ where $M = \|f(0) - g(0) - h(0)\|$ and T satisfies (10).

Proof. Define $\varphi : V \times V \to [0,\infty)$ by

$$\varphi(x,y) = \begin{cases} \psi(\|x\|) + \psi(\|y\|)) & \text{if } x \ne 0 \text{ or } y \ne 0 \\ \|f(0) - g(0) - h(0)\| & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Then we get

$$\begin{aligned} \tilde{\varphi}(x,y) &= \sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) \\ &\le \sum_{n=0}^{\infty} (\psi(2)/2)^n(\psi(\|x\|) + \psi(\|y\|)) \\ &= \frac{\psi(\|x\|) + \psi(\|y\|)}{1 - \psi(2)/2} \end{aligned} \tag{16}$$

for all $x, y \in V\backslash\{0\}$ from (i) and (ii). By the similar method as (16), we obtain

$$\tilde{\varphi}(x,y) = \begin{cases} 2\psi(0) + \frac{2\psi(\|y\|)}{2-\psi(2)} & \text{if } x = 0 \text{ and } y \ne 0 \\ \frac{2\psi(\|x\|)}{2-\psi(2)} + 2\psi(0) & \text{if } x \ne 0 \text{ and } y = 0 \\ 2\|f(0) - g(0) - h(0)\| & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Applying Corollary 2.3, there exists a unique additive mapping $T : V \to X$ satisfying (15) for all $x \ne 0$. □

COROLLARY 2.5. *Let V be a normed space and let $f, g, h : V \to X$ be mappings. Assume that there exist $\theta > 0$ and $p \in [0, 1)$ such that*

$$\|f(x+y) - g(x) - h(y)\| \leq \theta(\|x\|^p + \|y\|^p) \text{ for } x \neq 0 \text{ or } y \neq 0.$$

Then there exists a unique additive mapping $T : V \to X$ such that

$$\|T(x) - f(x) + f(0)\| \leq \frac{4\theta}{2 - 2^p}\|x\|^p + \theta M$$
$$\|T(x) - g(x) + g(0)\| \leq \frac{(4 + 2^p)\theta}{2 - 2^p}\|x\|^p + \theta M$$
$$\|T(x) - h(x) + h(0)\| \leq \frac{(4 + 2^p)\theta}{2 - 2^p}\|x\|^p + \theta M$$

for all $x \in V$, where $M = \|f(0) - g(0) - h(0)\|$.

Proof. Define mappings $f_1, g_1, h_1 : V \to X$ by $f_1(x) = \frac{1}{\theta}f(x)$, $g_1(x) = \frac{1}{\theta}g(x)$, $h_1(x) = \frac{1}{\theta}h(x)$ for all $x \in V$. Define $\psi : [0, \infty) \to R^+$ by $\psi(t) = t^p$ and apply Corollary 2.4. □

It is easy to know that if $2f(\frac{x+y}{2}) - f(x) - f(y) = 0$, then $f(x) - f(0)$ is an additive mapping. The Pexider equation satisfying the similar result is shown in the following corollary.

COROLLARY 2.6. *Let V be a normed space. Let $f, g, h : V \to X$ be mappings such that*

$$f(x+y) - g(x) - h(y) = 0 \text{ for } x, y \in V.$$

Then $f(x) - f(0)$, $g(x) - g(0)$ and $h(x) - h(0)$ are additive mappings such that

$$f(x) - f(0) = g(x) - g(0) = h(x) - h(0)$$

for all $x \in V$.

Proof. Define $\varphi(x, y) : V \times V \to [0, \infty)$ by $\varphi(x, y) = 0$ for all $x, y \in V$, and apply Theorem 2.2. □

3. Stability in the case $p > 1$

Let $\phi : V \times V \to [0, \infty)$ be a mapping such that

$$(17) \qquad \tilde{\phi}(x, y) := \sum_{k=0}^{\infty} 2^k \phi(2^{-k}x, 2^{-k}y) < \infty.$$

It is easy to show that $\tilde{\phi}(0,0) = \phi(0,0) = 0$. Then we follow a similar approach as the above arguments and obtain the results from Lemma 3.1 to Corollary 3.4.

LEMMA 3.1. *Let V be a vector space. Let $f : V \to X$ be a mapping such that*

$$(18) \qquad \left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \phi(x, y)$$

for all $x, y \in V$. Then there exists a unique additive mapping $T : V \to X$ such that

$$(19) \qquad \|T(x) - f(x) + f(0)\| \leq \tilde{\phi}(x, 0) \quad \text{for all } \; x \in V$$

and

$$T(x) = \lim_{n \to \infty} 2^n (f(2^{-n}x) - f(0))$$

for all $x \in V$.

Proof. We can assume that $f(0) = 0$ without the loss of generality. We obtain the sequence $\{2^n f(2^{-n}x)\}$ is a Cauchy sequence. Denote

$$T(x) = \lim_{n \to \infty} 2^n f(2^{-n}x)$$

for all x in V. Then it is easy to show that T is a unique additive mapping satisfying (19). □

THEOREM 3.2. *Let V be a vector space. Let $f, g, h : V \to X$ be mappings such that*

$$(20) \qquad \|f(x + y) - g(x) - h(y)\| \leq \phi(x, y)$$

for all $x, y \in V$. Then there exists a unique additive mapping $T : V \to X$ such that

$$\|T(x) - f(x) + f(0)\| \le \tilde{\phi}\Big(\frac{x}{2}, 0\Big) + \tilde{\phi}\Big(0, \frac{x}{2}\Big) + \tilde{\phi}\Big(\frac{x}{2}, \frac{x}{2}\Big)$$
$$\|T(x) - g(x) + g(0)\| \le \tilde{\phi}\Big(\frac{x}{2}, -\frac{x}{2}\Big) + \tilde{\phi}\Big(\frac{x}{2}, 0\Big) + \tilde{\phi}\Big(x, -\frac{x}{2}\Big)$$
$$\|T(x) - h(x) + h(0)\| \le \tilde{\phi}\Big(-\frac{x}{2}, \frac{x}{2}\Big) + \tilde{\phi}\Big(0, \frac{x}{2}\Big) + \tilde{\phi}\Big(-\frac{x}{2}, x\Big)$$

for all $x \in V$ and

$$T(x) = \begin{cases} \lim_{n\to\infty} 2^n[f(2^{-n}x) - f(0)] \\ \lim_{n\to\infty} 2^n[g(2^{-n}x) - g(0)] \\ \lim_{n\to\infty} 2^n[h(2^{-n}x) - h(0)] \end{cases} \text{for all } x \in V. \tag{21}$$

Proof. As the result in the proof of Theorem 2.2, we get

$$\Big\|2f\Big(\frac{x+y}{2}\Big) - f(x) - f(y)\Big\| \le \phi\Big(\frac{x}{2}, \frac{y}{2}\Big) + \phi\Big(\frac{y}{2}, \frac{x}{2}\Big) + \phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \phi\Big(\frac{y}{2}, \frac{y}{2}\Big)$$
$$\Big\|2g\Big(\frac{x+y}{2}\Big) - g(x) - g(y)\Big\| \le \varphi\Big(\frac{x+y}{2}, -\frac{x}{2}\Big) + \varphi\Big(\frac{x+y}{2}, -\frac{y}{2}\Big) + \varphi\Big(x, -\frac{x}{2}\Big) + \varphi\Big(y, -\frac{y}{2}\Big)$$
$$\Big\|2h\Big(\frac{x+y}{2}\Big) - h(x) - h(y)\Big\| \le \varphi\Big(-\frac{x}{2}, \frac{x+y}{2}\Big) + \varphi\Big(-\frac{y}{2}, \frac{x+y}{2}\Big) + \varphi\Big(-\frac{x}{2}, x\Big) + \varphi\Big(-\frac{y}{2}, y\Big)$$

for all $x, y \in V$. From the above and Lemma 3.1, there exist T, T_1 and T_2 such that

$$\|T(x) - f(x) + f(0)\| \le \tilde{\phi}\Big(\frac{x}{2}, 0\Big) + \tilde{\phi}\Big(0, \frac{x}{2}\Big) + \tilde{\phi}\Big(\frac{x}{2}, \frac{x}{2}\Big)$$
$$\|T_1(x) - g(x) + g(0)\| \le \tilde{\phi}\Big(\frac{x}{2}, -\frac{x}{2}\Big) + \tilde{\phi}\Big(\frac{x}{2}, 0\Big) + \tilde{\phi}\Big(x, -\frac{x}{2}\Big)$$
$$\|T_2(x) - h(x) + h(0)\| \le \tilde{\phi}\Big(-\frac{x}{2}, \frac{x}{2}\Big) + \tilde{\phi}\Big(0, \frac{x}{2}\Big) + \tilde{\phi}\Big(-\frac{x}{2}, x\Big).$$

Replacing x by $2^{-n}x$ and y by 0 in (20) and multiplying 2^n on both sides of (20), we get

$$\begin{aligned}
&\|2^n[f(2^{-n}x) - f(0)] - 2^n[g(2^{-n}x) - g(0)]\| \\
&= 2^n\|f(2^{-n}x) - g(2^{-n}x) - h(0)\| \\
&\le 2^n\phi(2^{-n}x, 0).
\end{aligned} \tag{22}$$

Taking the limit in (22), we obtain

$$T(x) = T_1(x) \ \text{ for all } \ x \in V.$$

Similarly we have $T = T_2$. □

COROLLARY 3.3. *Let V be a normed space. Let a function $\psi : [0, a) \to [0, \infty)$ satisfy*
(i) $\psi(ts) \ge \psi(t)\psi(s) > 0$ *for all* $0 < t, s$,
(ii) $\psi(2)/2 > 1$ *and*
(iii) $\psi(0) = 0$.
Let $f, g, h : V \to X$ be mappings such that

$$\|f(x + y) - g(x) - h(y)\| \le \psi(\|x\|) + \psi(\|y\|)$$

for all $x, y \in V$. Then there exists a unique additive mapping $T : V \to X$ such that

$$\begin{aligned}
\|T(x) - f(x) + f(0)\| &\le \frac{4\psi(\|x\|)}{\psi(2) - 2} \\
\|T(x) - g(x) + g(0)\| &\le \frac{(4 + 2^p)\psi(\|x\|)}{\psi(2) - 2} \\
\|T(x) - h(x) + h(0)\| &\le \frac{(4 + 2^p)\psi(\|x\|)}{\psi(2) - 2}
\end{aligned}$$

for all $x \in V$ and T satisfies (21).

Proof. Let $\phi(x,y) = \psi(\|x\|) + \psi(\|y\|))$ for all $x, y \in V$. We get

$$\begin{aligned}
\tilde{\phi}(x,y) &= \sum_{n=0}^{\infty} 2^n \phi(2^{-n}x, 2^{-n}y) \\
&= \sum_{n=0}^{\infty} 2^n (\psi(\|2^{-n}x\|) + \psi(\|2^{-n}y\|)) \\
&\leq \sum_{n=0}^{\infty} (2/\psi(2))^n (\psi(\|x\|) + \psi(\|y\|)) \\
&= \frac{\psi(\|x\|) + \psi(\|y\|)}{1 - 2/\psi(2)} < \infty
\end{aligned}$$

from (i) , (ii) and (iii). Applying Theorem 3.2, the proof is completed.□

COROLLARY 3.4. *Let V be a normed space and let $f, g, h : V \rightarrow X$ be mappings. Assume that there exist $\theta > 0$ and $p > 1$ such that*

$$\|f(x+y) - g(x) - h(y)\| \leq \theta(\|x\|^p + \|y\|^p) \quad \text{for all } \; x, y \in V.$$

Then there exists a unique additive mapping $T : V \rightarrow X$ such that

$$\begin{aligned}
\|f(x) - T(x) - f(0)\| &\leq \frac{4\theta}{2^p - 2}\|x\|^p \\
\|g(x) - T(x) - g(0)\| &\leq \frac{(4 + 2^p)\theta}{2^p - 2}\|x\|^p \\
\|h(x) - T(x) - h(0)\| &\leq \frac{(4 + 2^p)\theta}{2^p - 2}\|x\|^p
\end{aligned}$$

for all $x \in V$ and T satisfies (21).

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