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MULTIPLICITY AND STABILITY OF SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS HAVING NOT NON-NEGATIVE MASS

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ABSTRACT. In this paper, the multiplicity, stability and the structure of classical solutions of semilinear elliptic equations of the form

$$\begin{cases} \Delta u + f(x,u) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

will be discussed. Here Ω is a smooth and bounded domain in $\mathbf{R}^n (n \geq 1)$, $f(x,u) = |u|^\alpha \text{sgn}(u) - h(x)$, $0 < \alpha < 1$, $(n \geq 1)$ and h is a τ- Hölder continuous function on $\bar{\Omega}$ for some $0 < \tau < 1$.

1. Introduction

Let Ω be a smooth and bounded domain in $\mathbf{R}^n (n \geq 1)$. In this paper, we are going to prove multiplicity, stability, and to understand a structure of classical solutions of the following semilinear elliptic boundary value problems having not non-negative mass:

$$(I) \qquad \begin{cases} \Delta u + f(x,u) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Here $f(x,u) = |u|^\alpha \text{sgn}(u) - h(x)$, $0 < \alpha < 1$, $(n \geq 1)$, h is a τ- Hölder continuous function on $\bar{\Omega}$ for some $0 < \tau < 1$, and $\text{sgn}(u) = 1$ if $u \geq 0$, $\text{sgn}(u) = -1$ if $u < 0$. In general, with $f(x,u) = g(u) + h(x)$, We say that the problem (I) has a positive mass (see [25]) if

$$\limsup_{u \to 0} \frac{g(u)}{u} < 0.$$

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Since the growth of the nonlinearity near the origin in our problem is the plus infinity, this fact motives us to describe our phenomenon by "non-negative mass" in contrast with the positive mass and we adapt it in our title.

In [20], J. Mawhin and K. Schmitt treated the solution structure of the following Landesman-Lazer type nonlinear problem when $f(x,u) = g(u) + h(x)$ with g is a nonlinearity which is allowed to grow at most a sublinear rate:

$$(II) \qquad \begin{cases} \Delta u + (\lambda_1 + \lambda)u + f(x,u) = 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Here λ_1 is the first eigenvalue for $-\Delta$ with the homogeneous Dirichlet data and λ is treated as a continuation parameter. They impose conditions on the behavior of g in a neighbourhood of ∞ of a kernel of linear problem to deduce results which say that there are multiple solutions of (II) for λ on one side of zero and guarantee the existence of at least one solution for $\lambda = 0$ and λ on the other side of zero. Those multiplicity results are obtained by Leray-Schauder degree theory and birfurcation from infinity technique.

To get multiplicity in our problem for any such h, first we choose sufficiently large number $c > 0$, and so we have a pair of negative subsolution $-c$ and positive supersolution c of (I). Then, by the sub and supersolution method, we may have a classical solution of (I) between them. From this fact, we can construct two pairs of subsolutions and supersolutions for (I), and then using fixed point index theory, we prove the existence of distinct three solutions of (I) under some condition h.

For stability in semilinear parabolic problems. In 1972, Sattinger [23] used the method of sub- and supersolutions to study the stability of solutions of the elliptic boundary value problem as equilibrium solutions of the parabolic problem

$$(III) \qquad \begin{cases} u_t = \Delta u + f(x,u) \quad \text{in} \quad \Omega \times (0,\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty) \end{cases}$$

Specifically, he showed that solutions of the elliptic problem which are obtained by monotone iteration from a sub- or supersolution have one-sided stability. If there is a unique solution between a sub-super solution pair, then it is stable. Several years later, Matano [18] established that an intermediate solution exists between two stable solutions. Existence of these intermediate solutions has also been established by others (see [2-7,10,14,17]), using degree theory, variational methods, infinite Morse index theory, or some combination thereof, especially in the case that either f is independent of x or

$$-\infty < \liminf_{u\to 0} \frac{f(x,u)}{u} \le \limsup_{u\to 0} \frac{f(x,u)}{u} < \infty$$

for all $x \in \bar{\Omega}$.

For more general theory of orbital stability for continuously differentiable increasing order-compact mapping, the existence of orbital stable solutions for quasilinear parabolic periodic-Neumann or periodic-regular oblique derivative boundary value problem, the multiplicity and orbital stability results for periodic-Dirichlet problem with jumping nonlinearity and with coercive growth nonlinearity for semilinear parabolic equations, we may find those results in [5], [6] and [11, 12, 15], respectively.

Since the function f in the problem (I) satisfies the following limits:

$$\lim_{u\to 0} \frac{u^{\alpha}\mathrm{sgn}(u)}{u} = \infty, \quad \lim_{u\to\infty} \frac{u^{\alpha}\mathrm{sgn}(u)}{u} = 0,$$

the above results do not imply the existence of distinct three solutions of (I). Also we understand the perturbation theory in variational methods (see [21, 25]) may not be applicable directly to prove some multiplicity of solutions. As we know, the stability for solutions of (I) is not fully understood.

The organization of this paper is as follows: In section 2, we solve a singular perturbation problem to construct two pairs of subsolutions and supersolutions of (I), and then by calculating the fixed point index (see [1], [8]), we show that if $\|h\| = \sup\{|h(x)| : x \in \bar{\Omega}\}$ is sufficiently small, (I) has three distinct classical solutions which contains a positive solution and a negative solution.

In section 3, using the limiting arguments (see [24], [25]) of the solutions depending on h, we prove stability by the uniqueness: when

h has one sign on Ω and $\|h\| \to \infty$, (I) has the unique solution. For the other case, by Green's identity, we show that if h is non-positive on Ω, (I) has the unique positive solution and if h is non-negative on Ω, the unique negative solution. Especially, we show that if h is identically zero, then (I) has the unique positive solution and the unique negative solution.

In section 4, using the calculation of the fixed point index, we find some structure of solutions varying h: There is a positive number $\|h^*\|$ such that (I) has at least three distinct solutions if $\|h\| < \|h^*\|$, (I) has two distinct solutions at $\|h^*\|$, and (I) has the unique solution if $\|h\| > \|h^*\|$. If h is constant, we prove the existence of $\mathcal{S}$-shaped curve (u, h) of solution structure on the range $-\infty < h < \infty$.

2. The Multiplicity

To get two pairs of subsolutions and supersolutions for (I), we first solve the following boundary value problem:

$$(I_{\lambda,\epsilon}) \qquad \begin{cases} \epsilon^2 \Delta u + |u|^\alpha \operatorname{sgn}(u) - \lambda u - h(x) = 0 & \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

Here $\lambda > 0$ and $\epsilon > 0$ are real constants.

We note that

$$(1) \qquad \lim_{|u| \to \infty} \frac{|u|^\alpha \operatorname{sgn}(u) - \lambda u - h(x)}{u} = -\lambda < 0.$$

That means that there exists $u_0 > 0$ such that $u_0 \leq u$ implies

$$|u|^\alpha \operatorname{sgn}(u) - h(x) - \lambda u < 0$$

and there exists $u_1 < 0$ such that $u \leq u_1$ implies

$$|u|^\alpha \operatorname{sgn}(u) - h(x) - \lambda u > 0.$$

We show that for any given $\lambda > 0$ with

$$(2) \qquad \left[1 - \frac{1}{\alpha}\right] \left(\frac{1}{\lambda}\right)^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} + h(x) < 0$$

for all $x \in \bar{\Omega}$, there is $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ the problem $(I_{\lambda,\epsilon})$ has a positive solution. Then we note that the solution is a positive subsolution of the problem $(I_{0,\epsilon})$.

LEMMA 2.1. *Let $\lambda > 0$ satisfy the inequality (2), let $x \in \bar{\Omega}$, and let*

$$f(x, u, \lambda) = u^\alpha - h(x) - \lambda u$$

for all $u > 0$. Then $f(x, \cdot, \lambda)$ has the positive maximum in u at $(\frac{\alpha}{\lambda})^{\frac{1}{1-\alpha}}$.

Proof. Let

$$w_x = \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}}.$$

Then

$$\frac{\partial}{\partial u} f(x, w_x, \lambda) = \alpha w_x^{\alpha-1} - \lambda = \alpha\left[\left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha-1} - \lambda = 0,$$

$$\frac{\partial}{\partial u}\left[\alpha w_x^{\alpha-1} - \lambda\right] = \alpha(\alpha - 1) w_x^{\alpha-2} < 0,$$

and

$$\begin{aligned} f(x, w_x, \lambda) &= \left[\left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha} - h(x) - \lambda\left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}} \\ &= \left(\frac{\alpha}{\lambda}\right)^{\frac{\alpha}{1-\alpha}} - h(x) - \alpha^{\frac{1}{1-\alpha}} \lambda^{\frac{\alpha}{\alpha-1}} \\ &= \lambda^{\frac{\alpha}{\alpha-1}} \alpha^{\frac{1}{1-\alpha}}\left[\frac{1}{\alpha} - 1\right] - h(x) \\ &> 0. \quad (\text{by } (2)) \end{aligned}$$

This completes the proof. □

From (1) and Lemma 2.1, there is $u_x > w_x$ so that

$$u_x^\alpha - h(x) - \lambda u_x = 0. \tag{3}$$

Furthermore,

$$\frac{\partial}{\partial u} f(x, u_x, \lambda) = \alpha u_x^{\alpha-1} - \lambda < \alpha \frac{\lambda}{\alpha} - \lambda = 0. \tag{4}$$

LEMMA 2.2. *Let $\lambda > 0$ satisfy* (2) *and u_x be the solution of the equations* (3) *and* (4). *Then*

$$\int_0^{u_x} \Big[u^\alpha - h(x) - \lambda u\Big] du > 0 \tag{5}$$

for all $x \in \partial\Omega$.

Proof. Since

$$\alpha < \frac{1}{\ln 2},$$

we get

$$\alpha \ln 2 - 1 < 0,$$

and hence

$$2^{1-\alpha}(\alpha \ln 2 - 1) < 0.$$

Consider the function $g(\alpha) = 2^{1-\alpha}\alpha$. Then $g(0) = 0$ and $g(1) = 1$. Now

$$\begin{aligned} \frac{d}{d\alpha} g(\alpha) &= 2^{1-\alpha} + \alpha(2^{1-\alpha})(-\ln 2) \\ &= 2^{1-\alpha}(1 - \alpha \ln 2) \\ &> 0. \end{aligned}$$

Hence,

$$2^{1-\alpha}\alpha < 1$$

for all $\alpha \in (0, 1)$. Taking ln on both sides of the above inequality, we then have

$$(1 - \alpha) \ln 2 + \ln \alpha < 0,$$

and so

$$\ln 2 < \ln 2^\alpha - \ln \alpha.$$

Therefore,

$$0 < \frac{1}{\alpha} 2^\alpha - 2$$

for all $\alpha \in (0, 1)$. Let

$$f(x, u, \lambda) = u^\alpha - h(x) - \lambda u$$

for all $(x,u) \in \bar{\Omega} \times [0,\infty)$ and let $w_x = (\frac{\alpha}{\lambda})^{\frac{1}{1-\alpha}}$. Then,

$$\begin{aligned} f(x,2w_x,\lambda) &= \left[2\left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha} - h(x) - \lambda\left[2\left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}}\right] \\ &= 2^{\alpha}\alpha^{\frac{\alpha}{1-\alpha}}\lambda^{\frac{\alpha}{\alpha-1}} - h(x) - 2\alpha^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{\alpha-1}} \\ &= \lambda^{\frac{\alpha}{\alpha-1}}\left[2^{\alpha}\alpha^{\frac{\alpha}{1-\alpha}} - 2\alpha^{\frac{1}{1-\alpha}}\right] - h(x) \\ &= \lambda^{\frac{\alpha}{\alpha-1}}\alpha^{\frac{1}{1-\alpha}}\left[2^{\alpha}\alpha^{\frac{\alpha-1}{1-\alpha}} - 2\right] - h(x) \\ &= \left(\frac{1}{\lambda}\right)^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}\left[2^{\alpha}\frac{1}{\alpha} - 2\right] - h(x). \end{aligned}$$

Therefore, if $\lambda \to 0^+$, then there is $\lambda > 0$ so that

$$f(x,2w_x,\lambda) > 0, \tag{6}$$

for all $x \in \bar{\Omega}$, and hence we can assume that

$$2w_x < u_x \tag{7}$$

for all $x \in \bar{\Omega}$. Then

$$\begin{aligned} \int_0^{u_x}[u^{\alpha} - h(x) - \lambda u]du &= \frac{u_x^{\alpha+1}}{\alpha+1} - h(x)u_x - \frac{\lambda}{2}u_x^2 \\ &= u_x\left[\frac{u_x^{\alpha}}{\alpha+1} - h(x) - \frac{\lambda}{2}u_x\right]. \end{aligned}$$

From (2), (3) and (7),

$$\begin{aligned} &\frac{u_x^{\alpha}}{\alpha+1} - h(x) - \frac{\lambda}{2}u_x \\ &= \frac{1}{\alpha+1}[\lambda u_x + h(x)] - h(x) - \frac{\lambda}{2}u_x \quad \text{(by (3))} \\ &= \frac{\lambda(1-\alpha)}{2(\alpha+1)}u_x - \frac{\alpha}{\alpha+1}h(x) \\ &> \frac{\lambda(1-\alpha)}{2(\alpha+1)}2\left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}} + \frac{\alpha}{\alpha+1}\left[1-\frac{1}{\alpha}\right]\left(\frac{1}{\lambda}\right)^{\frac{\alpha}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}} \quad \text{(by (2),(7))} \\ &= \frac{1-\alpha}{\alpha+1}\alpha^{\frac{1}{1-\alpha}}\lambda^{\frac{\alpha}{\alpha-1}} + \frac{\alpha}{\alpha+1}\left[1-\frac{1}{\alpha}\right]\lambda^{\frac{\alpha}{\alpha-1}}\alpha^{\frac{1}{1-\alpha}} \end{aligned}$$

$$= \alpha^{\frac{1}{1-\alpha}} \lambda^{\frac{\alpha}{\alpha-1}} \left[\frac{1-\alpha}{\alpha+1} + \frac{\alpha}{\alpha+1} \left[\frac{\alpha-1}{\alpha} \right] \right]$$
$$= 0.$$

Therefore,

$$\int_0^{u_x} [u^\alpha - h(x) - \lambda u] du > 0.$$

This completes the proof. □

Since the function $f(x, u, \lambda) = |u|^\alpha - h(x) - \lambda u$ satisfies the conditions (3), (4), and (5), by the construction (see [13,14,17]) of subsolutions of $(I_{\lambda,\epsilon})$, there is a small positive number ϵ_1, which also depends on λ and h, such that for any ϵ with $0 < \epsilon < \epsilon_1$ there is a solution $\underline{u}(\cdot;\epsilon)$ of $(I_{\lambda,\epsilon})$. Furthermore, $0 \leq \underline{u}(x;\epsilon) \leq u_x$ for all $x \in \bar{\Omega}$ and

$$\lim_{\epsilon \to 0} \underline{u}(x;\epsilon) = u_x$$

uniformly on every compact subset of Ω. Hence $\underline{u}(\cdot;\epsilon)$ is a subsolution of $(I_{0,\epsilon})$.

Secondly, for each $\epsilon > 0$, we find a supersolution of $(I_{0,\epsilon})$ which is larger than $\underline{u}(\cdot;\epsilon)$ on $\bar{\Omega}$. Let

$$M = \max\{|x|^2 : x \in \bar{\Omega}\}$$

and we choose $\beta > 0$ with

$$\epsilon^2 > \frac{(\frac{\beta}{2n})^\alpha (M - |x|^2 + 1)^\alpha - h(x)}{\beta}.$$

Let

$$\bar{u}(x;\epsilon) = \frac{\beta}{2n}(M - |x|^2 + 1).$$

Then

$$\begin{aligned}
&\epsilon^2 \Delta \bar{u}(x;\epsilon) + |\bar{u}(x;\epsilon)|^\alpha \operatorname{sgn}(\bar{u}(x;\epsilon)) - h(x) \\
&= -\epsilon^2 \beta + \left(\frac{\beta}{2n}\right)^\alpha (M - |x|^2 + 1)^\alpha - h(x) \\
&= -\beta \left[\epsilon^2 - \frac{(\frac{\beta}{2n})^\alpha (M - |x|^2 + 1)^\alpha - h(x)}{\beta} \right] \\
&< 0
\end{aligned}$$

if $\beta > 0$ is sufficiently large. Hence, if we choose $\beta > 0$ so that $\bar{u}(\cdot;\epsilon)$ is a supersolution of $(I_{0,\epsilon})$ and $u_x \leq \bar{u}(x;\epsilon)$ for all $x \in \bar{\Omega}$, then the problem $(I_{0,\epsilon})$ has a pair of positive subsolution $\underline{u}(\cdot;\epsilon)$ and positive supersolution $\bar{u}(\cdot;\epsilon)$ so that

$$\underline{u}(x;\epsilon) \leq u_x \leq \bar{u}(x;\epsilon)$$

for all $x \in \bar{\Omega}$. Then we have the following existence theorem of three solutions of $(I_{0,\epsilon})$.

THEOREM 2.1. *There is a positive number ϵ_0 such that for any ϵ with $0 < \epsilon < \epsilon_0$, the problem $(I_{0,\epsilon})$ has at least three solutions $u_1(x;\epsilon)$, $u_2(x;\epsilon)$, $u_3(x;\epsilon)$ such that*

$$u_2(x;\epsilon) \leq u_3(x;\epsilon) \leq u_1(x;\epsilon),$$

$$u_1(x;\epsilon) > 0, \quad u_2(x;\epsilon) < 0$$

for all $x \in \Omega$.

Proof. Let $u > 0$. We choose λ which satisfies the inequalities and the equality (2), (3), (4), and (5). Then from the above calculations, for any $0 < \epsilon < \epsilon_1$ we have already constructed a pair of subsolution $\underline{u}_1(x;\epsilon)$ and supersolution $\bar{u}_1(x;\epsilon)$ of $(I_{0,\epsilon})$ with the following properties;

$$0 \leq \underline{u}_1(x;\epsilon) \leq u_1(x) \leq \bar{u}_1(x;\epsilon)$$

for all $x \in \bar{\Omega}$, where $u_1(x)$ is the positive solution of the equations

$$t^\alpha - h(x) - \lambda t = 0$$

for all $x \in \bar{\Omega}$ and

$$u_1(x) > \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}}.$$

Furthermore,

$$\lim_{\epsilon \to 0} \underline{u}_1(x;\epsilon) = \underline{u}_1(x)$$

uniformly on every compact subset of Ω. And

$$\bar{u}_1(x;\epsilon) = \frac{\beta}{2n}(M - |x|^2 + 1)$$

for some sufficiently large $\beta > 0$.

We construct another pair of negative subsolution and negative supersolution of $(I_{0,\epsilon})$.

For any $u < 0$ and for any $\lambda > 0$ satisfying the inequality (2), let

$$f(x,u,\lambda) = -|u|^\alpha - h(x) - \lambda u.$$

Then, by the similar calculation to the previous paragraph, we get $f(x,\cdot,\lambda)$ has the negative minimum at

$$u = -\left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}}$$

on the interval $(-\infty, 0]$. Let $u_2(x)$ be the solution of the equations

$$-|t|^\alpha - h(x) - \lambda t = 0, \quad (t < 0).$$

Then

$$u_2(x) < -\left(\frac{\alpha}{\lambda}\right)^{\frac{1}{1-\alpha}},$$

$$\frac{\partial}{\partial u} f(x, u_2(x), \lambda) < 0$$

for all $x \in \bar{\Omega}$, and

$$\int_{u_2(x)}^t f(x,u,\lambda)du = \int_{u_2(x)}^t (-|u|^\alpha - h(x) - \lambda u)du < 0$$

for any $t \in (u_2(x), 0]$.

Again, by the construction ([13,14,17]) of supersolutions of $(I_{\lambda,\epsilon})$, there is a positive number ϵ_2 such that for any $0 < \epsilon < \epsilon_2$, $(I_{\lambda,\epsilon})$ has a negative solution $\bar{u}_2(x;\epsilon) \in C^2(\bar{\Omega})$ with the property

$$u_2(x) \le \bar{u}_2(x;\epsilon) \le 0$$

for all $x \in \bar{\Omega}$. Moreover,

$$\lim_{\epsilon \to 0} \bar{u}_2(x;\epsilon) = u_2(x)$$

uniformly on every compact subset of Ω. As before, we note that $\bar{u}_2(x;\epsilon)$ is a negative supersolution of $(I_{0,\epsilon})$. Let $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$. Then for any $0 < \epsilon < \epsilon_0$, we choose a positive number γ so that if we let

$$\underline{u}_2(x;\epsilon) = -\frac{\gamma}{2n}(M - |x|^2 + 1),$$

then $\underline{u}_2(x;\epsilon)$ is a subsolution of $(I_{0,\epsilon})$ and

$$\underline{u}_2(x;\epsilon) \le \bar{u}_2(x;\epsilon)$$

for all $x \in \bar{\Omega}$. □

To complete the proof, we introduce the following notations:

DEFINITION. (i) Let $v, w \in C(\bar{\Omega})$. $v < w$ means that $v(x) \le w(x)$ for all $x \in \bar{\Omega}$ but $v \ne w$. $v \le w$ means that either $v < w$ or $v = w$.

(ii) We denote an ordered interval

$$[v, w] = \{u \in C(\bar{\Omega}) \,:\, v \le u \le w\}.$$

With the above notation, if $0 < \epsilon < \epsilon_0$, then we get three ordered intervals $[\underline{u}_2, \bar{u}_1]$, $[\underline{u}_2, \bar{u}_2]$, $[\underline{u}_1, \bar{u}_1]$ where $\underline{u}_i = \underline{u}_i(x;\epsilon)$ and $\bar{u}_i = \bar{u}_i(x;\epsilon)$, $i = 1, 2$. Since $\bar{u}_2 < \underline{u}_1$, we know that

$$[\underline{u}_2, \bar{u}_2] \cap [\underline{u}_1, \bar{u}_1] = \emptyset.$$

To show the existence of three solutions of $(I_{0,\epsilon})$, we are going to calculate the fixed point index for some suitable nonlinear operator defined from $C(\bar{\Omega}) \cap [\underline{u}_2, \bar{u}_1]$ into $C(\bar{\Omega})$.

Consider the nonlinear operator $T : C^\tau(\bar{\Omega}) \cap [\underline{u}_2, \bar{u}_1] \to C(\bar{\Omega})$ defined as follows: for any $u \in C^\tau(\bar{\Omega}) \cap [\underline{u}_2, \bar{u}_1]$, $0 < \tau < 1$,

$$Tu = v$$

if v is the unique solution of the linear boundary value problem

$$\begin{cases} \epsilon^2 \Delta v - \lambda v + |u(x)|^\alpha \mathrm{sgn}(u(x)) - h(x) = 0 & \text{in} \quad \Omega \\ v = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Here, $\lambda > 0$ satisfies the inequality (2). In order that T is well-defined, we have to check the function $|u|^\alpha \mathrm{sgn}(u)$ is Hölder continuous in a suitable closed and bounded interval containing 0. In fact, it is α-Hölder continuous. We prove the following inequality:

$$||u|^\alpha \mathrm{sgn}(u) - |v|^\alpha \mathrm{sgn}(v)| \leq 2|u-v|^\alpha$$

for all ordered pairs (u, v) in $\mathbf{R}^2$. To show the inequality, we consider the function

$$f(u,v) = ||u|^\alpha \mathrm{sgn}(u) - |v|^\alpha \mathrm{sgn}(v)| - |u-v|^\alpha$$

on the set

$$D_K = \{(u,v) \,:\, 0 < u \leq K,\ 0 < v \leq K,\ v < u\}$$

for some positive constant K. Then

$$f(u,v) = u^\alpha - v^\alpha - (u-v)^\alpha$$

and

$$\frac{\partial f}{\partial u}(u,v) = \alpha u^{\alpha-1} - \alpha(u-v)^{\alpha-1}.$$

Hence f does not have the maximum in the interior of D_K. By simple calculation of f on the boundary ∂D_K of the domain, we can show

$$f(u,v) \leq 0$$

for all $(u,v) \in \bar{D}_K$. By the similar calculations, we can show that

$$f(u,v) \leq 0$$

for all (u, v) in the domain

$$\{(u,v) \,:\, 0 \leq u \leq K,\ 0 \leq v \leq K,\ u \leq v\}.$$

Likewise, we can prove that if $-K \leq u < 0$ and $-K \leq v < 0$, then

$$||u|^\alpha \mathrm{sgn}(u) - |v|^\alpha \mathrm{sgn}(v)| \leq |u-v|^\alpha.$$

Let $uv < 0$. Assume that $u > 0$ and $v < 0$. Then

$$\frac{||u|^\alpha \mathrm{sgn}(u) - |v|^\alpha \mathrm{sgn}(v)|}{|u-v|^\alpha} = \frac{u^\alpha + |v|^\alpha}{|u+|v||^\alpha} \leq 2.$$

Similarly, we can also show that the inequality for the case $u < 0$ and $v > 0$. Consequently,

$$||u|^\alpha \mathrm{sgn}(u) - |v|^\alpha \mathrm{sgn}(v)| \leq 2|u-v|^\alpha$$

for all (u, v) with $-K \leq u, v \leq K$. Since K was arbitrary, we are done.

The Continuation of the Proof of Theorem 2.1. From the above result, T is well-define, and by the maximum principle (see [21]), we can prove T is increasing if $\lambda > 0$ is sufficiently large. By the standard extension method ([17]) we can also show that T is a compact increasing operator define on $C(\bar{\Omega}) \cap [\underline{u}_2, \bar{u}_1]$ into itself. Moreover, we note that if u is a fixed point of the operator equation

$$Tu = u$$

in $C(\bar{\Omega})$, then u is a classical solution of $(I_{0,\epsilon})$ (see [23]). By the maximum principle, we have the following;

$$\underline{u}_i \leq T\underline{u}_i, \quad T\bar{u}_i \leq \bar{u}_i, \quad i = 1, 2.$$

Since T is increasing in the sense of the usual ordered Banach space $C(\bar{\Omega})$, we get the following results:

$$T([\underline{u}_2, \bar{u}_1]) \subset [\underline{u}_2, \bar{u}_1],$$
$$T([\underline{u}_i, \bar{u}_i]) \subset [\underline{u}_i, \bar{u}_i], \quad i = 1, 2,$$

and then we know that the following fixed point indices

$$i(T, [\underline{u}_2, \bar{u}_1], [\underline{u}_2, \bar{u}_1]) = 1,$$
$$i(T, [\underline{u}_i, \bar{u}_i], [\underline{u}_i, \bar{u}_i]) = 1 \quad \text{for} \quad i = 1, 2,$$

and, by the additivity of the fixed point index

$$i(T, [\underline{u}_2, \bar{u}_1] \setminus ([\underline{u}_1, \bar{u}_1] \cup [\underline{u}_2, \bar{u}_2]), [\underline{u}_2, \bar{u}_1] \setminus ([\underline{u}_1, \bar{u}_1] \cup [\underline{u}_2, \bar{u}_2])) = -1.$$

From the above calculation, $(I_{0,\epsilon})$ has three distinct solutions $u_1(x;\epsilon)$, $u_2(x;\epsilon)$, $u_3(x;\epsilon)$ so that $u_1 \in [\underline{u}_1, \bar{u}_1]$, $u_2 \in [\underline{u}_2, \bar{u}_2]$, and

$$u_3 \in [\underline{u}_2, \bar{u}_1] \setminus ([\underline{u}_2, \bar{u}_2] \cup [\underline{u}_1, \bar{u}_1]).$$

We note that, from the monotone property of T, there is a maximal solution $u_1(x;\epsilon) > 0$ in $[\underline{u}_1, \bar{u}_1]$ and a minimal solution $u_2(x;\epsilon) < 0$ in $[\underline{u}_2, \bar{u}_2]$. Hence, we have

$$u_2(x;\epsilon) < u_3(x;\epsilon) < u_1(x;\epsilon)$$

for all $x \in \Omega$. This completes the proof. □

To show the existence of three solutions of the problem (I) when $\|h\|$ is sufficiently small, we fix h and solve the following problem as $\eta \to 0^+$:

$$(I_\eta) \qquad \begin{cases} \Delta u + |u|^\alpha \mathrm{sgn}(u) = \eta h(x) & \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

THEOREM 2.2. *There is $\eta_0 > 0$ such that for all $0 \leq \eta < \eta_0$, (I_η) has at least three distinct solutions $u_1(\cdot;\eta)$, $u_2(\cdot;\eta)$, and $u_3(\cdot;\eta)$ such that $u_1 > 0$ and $u_2 < 0$.*

Proof. To prove the theorem, we construct two pairs of subsolutions and supersolutions of (I_η) when $\eta \geq 0$ is sufficiently small.

We choose the positive number p with $p > \frac{2}{1-\alpha}$. Let $h \neq 0$ and let $\eta = \eta(\epsilon) = \epsilon^{p-2}$ for all $0 < \epsilon < \epsilon_0$, where ϵ_0 is the constant in Theorem 2.1. Let u_ϵ be the positive solution of the problem $(I_{0,\epsilon})$ and let $\underline{u}_1(x;\eta) = \epsilon^p u_\epsilon$. Then

$$\begin{aligned}
&\Delta\underline{u}_1 + |\underline{u}_1|^\alpha \operatorname{sgn}(\underline{u}_1) - \eta h(x) \\
&= \epsilon^p \Delta u_\epsilon + |\epsilon^p u_\epsilon|^\alpha - \eta h(x) \\
&= \epsilon^{p-2}(-(u_\epsilon)^\alpha + h(x)) + \epsilon^{p\alpha} u_\epsilon^\alpha - \eta h(x) \\
&= (\epsilon^{p\alpha} - \epsilon^{p-2}) u_\epsilon^\alpha + \epsilon^{p-2} h(x) - \eta h(x) \\
&\geq 0
\end{aligned}$$

if $\epsilon > 0$ is sufficiently small. Hence, $\underline{u}_1$ is a positive subsolution of (I_η).

Let $\bar{u}_1 = \frac{\beta}{2n}(M - |x|^2 + 1)$. Then

$$\begin{aligned}
&\Delta\bar{u} + |\bar{u}|^\alpha \operatorname{sgn}(\bar{u}) - \eta h(x) \\
&= -\beta + \left(\frac{\beta}{2n}\right)^\alpha (M - |x|^2 + 1)^\alpha - \eta h(x) \\
&= -\beta\left[1 - \frac{\left(\frac{\beta}{2n}\right)^\alpha (M - |x|^2 + 1)^\alpha - \eta h(x)}{\beta}\right] \\
&< 0
\end{aligned}$$

if $\beta > 0$ is sufficiently large. Clearly, $\underline{u}_1(x) \leq \bar{u}_1(x)$ for all $x \in \bar{\Omega}$. Therefore, we have a pair of positive subsolution and positive supersolution of (I_η).

By the similar method, we can also construct another pair of negative subsolution and negative supersolution, $\underline{u}_2$ and $\bar{u}_2$, of (I_η) so that

$$\underline{u}_2 < \bar{u}_2 < 0 < \underline{u}_1 < \bar{u}_2.$$

Hence, the existence of three distinct solutions follows from the calculation of the fixed point index in the proof of Theorem 2.1.

The above method works for the case h is identically zero on $\bar{\Omega}$. This completes the proof. $\square$

The following theorem gives a limiting behavior of solutions of (I_η) as $\eta \to 0$.

THEOREM 2.3. *Let $h < 0$. Then there is $\eta_0 > 0$ such that for all $0 < \eta < \eta_0$, (I_η) has a positive solution u_η converging to a positive solution of (I_0) as $\eta \to 0$.*

Proof. We note, from Theorem 2.2, the existence of positive number η_0 such that for all $0 \le \eta < \eta_0$, the problem (I_η) has a positive solution and a negative solution.

For $0 < \eta < \eta_0$ and for any $x \in \bar{\Omega}$, we define

$$u_\eta^*(x) = \sup\{u(x) \,:\, u \text{ is a positive solution of } (I_\eta)\}.$$

We first claim that $\sup_{x\in\Omega} |u_\eta^*(x)| < \infty$. Suppose if not. Then there is a sequence $\{u_\eta^i\}_{i=1}^\infty$ of solutions of (I_η) such that $\|u_\eta^i\| \to \infty$ as $i \to \infty$. Divide the equation (I_η) by $\|u_\eta^i\|$, then

$$\Delta\left(\frac{u_\eta^i}{\|u_\eta^i\|}\right) + \frac{(u_\eta^i)^\alpha}{\|u_\eta^i\|} - \frac{\eta h}{\|u_\eta^i\|} = 0.$$

Hence, we have that $\Delta \frac{u_\eta^i}{\|u_\eta^i\|} \to 0$ as $i \to \infty$. Since $\left\{\frac{u_\eta^i}{\|u_\eta^i\|}\right\}$ is a bounded sequence, by the limiting arguments (see [24]), we can choose a convergent subsequence, and its limit u_0 satisfies that $\Delta u_0 = 0$ in Ω and $u = 0$ on $\partial\Omega$. Hence, $u_0 = 0$ from the maximum principle. But $\|\frac{u_\eta^i}{\|u_\eta^i\|}\| = 1$ for all i, and so $\|u_0\| = 1$. This leads to a contradiction.

We secondly claim that u_η^* is also a positive solution of (I_η). To show that, we choose a large supersolution $\bar{u}_\eta = \frac{\beta}{2n}(M - |x|^2 + 1)$ of (I_η) so that $u_\eta^* < \bar{u}_\eta$. By the monotone iteration method (see [23]), we have a maximal solution u so that $u_\eta^* \le u \le \bar{u}_\eta$. By the definition of u_η^*, $u_\eta^* = u$.

We note that if $\eta_2 < \eta_1$ and u_{η_1} is a solution of (I_{η_1}), then u_{η_1} is a subsolution of (I_{η_2}). Hence, $u^*_{\eta_1}$ is a subsolution of (I_{η_2}), and from the definition of $u^*_{\eta_2}$, we have the inequality $u^*_{\eta_1} < u^*_{\eta_2}$. By the mathematical induction, we have an increasing sequence $\{u^*_{\eta_m}\}$ of positive solutions of (I_{η_m}) as $\eta_m \to 0$ and $m \to \infty$. Let

$$u^*(x) = \sup\{u^*_{\eta_m}(x) \;:\; m = 1, 2, 3, \cdots\}$$

for any $x \in \bar{\Omega}$. From the similar argument as before, u^* is bounded above. By the limiting arguments and a regularity results (see [21]) u^* is a positive solution of (I_0). This completes the proof. □

COROLLARY. *Let $h > 0$. Then there is $\eta_0 > 0$ such that for all $0 < \eta < \eta_0$, (I_η) has a negative solution converging to a negative solution of (I_0) as $\eta \to 0$.*

3. Stability

To get the stability of the equilibrium solution of the parabolic problem (IlI), we want to prove the uniqueness of solutions of (I). Because we can easily construct a pair of subsolution and supersolution of (I). First, we show the uniqueness of the solution when $\|h\| \to \infty$. Let us consider the problems (I_η) as $\eta \to \infty$ and we fixed h which is not zero identically on Ω. We note that if $\eta > 0$, the problem (I_η) is equivalent to the following problem: Let $v = \frac{u}{\eta}$.

$$(I'_\eta) \qquad \begin{cases} \Delta v + \frac{1}{\eta^{1-\alpha}}|v|^\alpha \mathrm{sgn}(v) - h = 0 & \text{in} \quad \Omega \\ v = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

THEOREM 3.1. *Let $h < 0$. Then there is $\eta_0 > 0$ such that for any $\eta > \eta_0$, (I_η) has the unique solution. Hence, it is stable.*

Proof. Suppose that there is a sequence $\{\eta_m\}$ with $\eta_m \to \infty$ as $m \to \infty$ such that (I_{η_m}) has at least two distinct solutions u_m and u'_m. Let $v_m = \frac{u_m}{\eta_m}$ and $v'_m = \frac{u'_m}{\eta_m}$. Then v_m and v'_m are two distinct solutions of the problem (I'_{η_m}).

We claim that $\{v_m\}$ and $\{v'_m\}$ are bounded sequences in $C(\bar{\Omega})$. We only prove that $\{v_m\}$ is bounded. Suppose that $\|v_m\| \to \infty$ as $m \to \infty$. Let $w_m = \frac{v_m}{\|v_m\|}$. Then

$$\begin{aligned}\Delta w_m &= \frac{1}{\|v_m\|}\Delta v_m \\ &= \frac{1}{\|v_m\|}(h - \frac{1}{\eta_m^{1-\alpha}}|v_m|^\alpha \mathrm{sgn}(v_m)) \\ &= \frac{h}{\|v_m\|} - \frac{1}{\eta_m^{1-\alpha}\|v_m\|^{1-\alpha}}|w_m|^\alpha \mathrm{sgn}(w_m).\end{aligned}$$

Hence, $\Delta w_m \to 0$ as $m \to \infty$. By the limiting arguments and a regularity result, This leads to a contradiction.

Since $\{v_m\}$ and $\{v'_m\}$ are bounded sequences of solutions of elliptic boundary value problems, we can choose a convergent subsequence, we call it again, v_m and v'_m. Let $w = \lim_{m\to\infty} v_m$ and $w' = \lim_{m\to\infty} v'_m$. Then by the similar argument of the above, w and w' are classical solutions of the problem $\Delta u - h = 0$ in Ω and $u = 0$ on $\partial\Omega$. By the uniqueness of the linear problem $w = w'$ on $\bar{\Omega}$. Since $h < 0$, by the maximum principle, we note that $w(x) > 0$ for all $x \in \Omega$.

From the uniqueness of the solution w, we can assume that $u_m(x) \to +\infty$ and $u'_m(x) \to +\infty$ in Ω as $m \to \infty$. Then

$$\begin{aligned}&\left|\Delta\left(\frac{u_m(x) - u'_m(x)}{\|u_m - u'_m\|}\right)\right| \\ &= \frac{1}{\|u_m - u'_m\|}\left||u_m(x)|^\alpha \mathrm{sgn}(u_m(x)) - |u'_m(x)|^\alpha \mathrm{sgn}(u'_m(x))\right| \\ &= \frac{1}{\|u_m - u'_m\|}\frac{1}{\alpha|u^*_m(x)|^{1-\alpha}}|u_m(x) - u'_m(x)|\end{aligned}$$

in Ω and $u^*_m(x)$ lies between $u_m(x)$ and $u'_m(x)$. Since $u^*_m(x) \to +\infty$ almost everywhere in Ω, so

$$\lim_{m\to\infty} \Delta\left(\frac{u_m - u'_m}{\|u_m - u'_m\|}\right) = 0.$$

As the previous limiting arguments, this also leads to a contradiction. This completes the proof. □

COROLLARY. *Let $h > 0$. Then there is η_0 such that for any $\eta > \eta_0$, (I_η) has the unique solution. Hence, it is stable.*

The following uniqueness of the solutions of (I_η) depends on the sign of h, but they show stability of the equilibrium solution of the parabolic problem (II) if η is sufficiently large.

THEOREM 3.2. *Let $h < 0$. Then there is a unique positive solution of (I_η) for all $\eta > 0$. Hence it is stable.*

Proof. The existence of a positive solution is obvious, because u=0 is a subsolution of (I_η) and we can always choose a large positive supersolution of (I_η).

Suppose that there are two distinct positive solutions u and v of (I_η). Since $\max\{u(x), v(x)\}$ is a subsolution of (I_η), we have a solution which is larger than u and v. Hence, without loss of generality, we can assume that $u(x) > v(x)$ for all $x \in \Omega$. Then by Green's identity,

$$\begin{aligned} 0 &= \int_\Omega [v\Delta u - u\Delta v] \\ &= \int_\Omega [v(h - u^\alpha) - u(h - v^\alpha)] \\ &= \int_\Omega [h(v-u) + (uv)^\alpha(u^{1-\alpha} - v^{1-\alpha})] > 0. \end{aligned}$$

This leads to a contradiction.

The positive solution is stable. Because, the positive solution lies between a pair of subsolution and supersolution. □

COROLLARY 1. *Let $h > 0$. Then there is a unique negative solution of (I_η) for all $\eta > 0$. Hence it is stable.*

COROLLARY 2. *Let h be identically zero in Ω. The positive solution and the negative solution of (I) are stable.*

Proof. The existence of a positive solution and a negative solution of (I_0) comes from Theorem 2.2. The uniqueness follows from the method in the proof of Theorem 3.2. For the stability, as Section 2 we can easily construct a pair of positive subsolution and positive supersolution of (I_0) which produce the positive solution. By the similar way, we have a pair of negative subsolution and negative supersolution for the negative solution. □

4. Structure of Solutions

THEOREM 4.1. *Let $h > 0$ and have compact support in Ω. Then there is $\eta^* > 0$ such that*

(1) *(I_η) has three distinct solutions which contains a positive solution and a negative solution if $0 \leq \eta < \eta^*$,*
(2) *(I_{η^*}) has the unique positive solution and the unique negative solution,*
(3) *(I_η) has no positive solution if $\eta > \eta^*$, but only the negative solution.*

Proof. Let η^* be the largest positive real number so that $0 < \eta < \eta^*$ implies (I_η) has a positive solution. From Theorem 2.2, Corollary of Theorem 3.1, and Corollary 1 of Theorem 3.2, we note that $0 < \eta^* < \infty$.

For $0 < \eta < \eta^*$, we define a function u_η by

$$u_\eta(x) = \sup\{u(x) \, : \, u \, \text{is a positive solution of} \, (I_\eta)\}.$$

We note that u_η is a positive solution of (I_η) and that $0 < \eta < \eta' < \eta^*$ imply $u_{\eta'} < u_\eta$. By the limiting argument, we also note that u_η approaches a positive solution u_{η^*} of (I_{η^*}). For any $0 < \eta < \eta*$, the solution u_{η^*} is a subsolution of (I_η) and we can construct a supersolution of (I_η) which is larger than u_{η^*}, and then as the proof of Theorem 2.2, we have two pairs of subsolutions and supersolutions about (I_η), and hence using the fixed point index as before, we have distinct three solutions. This prove (1).

To show the uniqueness of u_{η^*}, we suppose that there are two positive solutions u and v of (I_{η^*}). Since we can choose a large supersolution of (I_{η^*}) so that it is larger than u and v, we can assume, without loss of generality, that $u > v > 0$. Then

$$\begin{aligned}&\Delta\left(\frac{u(x)+v(x)}{2}\right)+\left(\frac{u(x)+v(x)}{2}\right)^{\alpha}\\&>\frac{1}{2}\Delta(u(x)+v(x))+\frac{1}{2}((u(x))^{\alpha}+(v(x))^{\alpha})\\&=\eta^{*}h(x)\end{aligned}$$

for all $x \in \Omega$. Since h has compact support in Ω, there is a small positive number δ so that

$$\Delta\left(\frac{u(x)+v(x)}{2}\right)+\left(\frac{u(x)+v(x)}{2}\right)^{\alpha}\geq(\eta^{*}+\delta)h(x)$$

for all $x \in \Omega$. Hence $\frac{u+v}{2}$ is a subsolution of $(I_{\eta^*+\delta})$, and then $(I_{\eta^*+\delta})$ has a positive solution. Therefore, for any $\eta^* < \eta < \eta^* + \delta$, $u_{\eta^*+\delta}$ is a subsolution of (I_η), and hence (I_η) has a positive solution. This leads to a contradiction for the definition of η^*. The uniqueness about the negative solution has been proved in Corollary 1 of Theorem 3.2. This proves (2).

The nonexistence of positive solutions of (I_η) when $\eta > \eta^*$ is also trivial. Because if there is $\eta_0 > \eta^*$ so that (I_{η_0}) has a positive solution, by the same construction of a pair of subsolution and supersolution with a positive solution u_{η_0}, we have that, for all $0 < \eta < \eta_0$ (I_η), has a positive solution. This also leads to a contradiction. This completes the proof. □

We note that if $h \leq 0$, (I_η) has the unique positive solution for all $\eta \geq 0$.

COROLLARY. *Let $h < 0$ and have a compact support in Ω. Then there is $\eta^* > 0$ such that*

(1) *(I_η) has three distinct solutions which contains a positive solution and a negative solution if $0 \leq \eta < \eta^*$,*
(2) *(I_{η^*}) has the unique negative solution and the unique positive solution,*
(3) *(I_η) has no negative solution if $\eta > \eta^*$ but only the positive solution.*

We have from the following theorem a kind of structure of three solutions of (I_η).

THEOREM 4.2. *Let Ω be an open ball and let h be positive constant. Then there is a positive number η^* such that for all $0 < \eta < \eta^*$ (I_η) has two distinct positive solutions $u'_\eta < u_\eta$ so that*

(1) *(I_η) has three distinct solutions which contains two positive solutions and a negative solution if $0 < \eta < \eta^*$,*
(2) *(I_{η^*}) has a unique positive solution and a unique negative solution.*
(3) *u_η approaches the positive solution of (I_0) as $\eta \to 0$,*
(4) *if $\eta > \eta^*$, (I_η) has no positive solution but only the negative solution.*

Proof. Let η^* be the largest positive real number so that $0 < \eta < \eta^*$ implies (I_η) has a positive solution and let

$$(*) \qquad u_\eta(x) = \sup\{u(x) \,:\, u \text{ is a positive solution of } (I_\eta)\}.$$

By Corollary of Theorem 2.3 and Corollary of Theorem 3.2, u_η approaches a positive solution of (I_0) as $\eta \to 0$. This proves (3).

Using the limiting argument and maximum principles, we can easily prove the existence of a positive solution of (I_{η^*}) as $\eta \to \eta^*$. We will show the uniqueness for the solution in the end of this proof.

The nonexistence for $\eta > \eta^*$ of a positive solution of (I_η) are trivial from Corollary of Theorem 3.1 and Corollary 1 of Theorem 3.2.

To prove (1) we find another positive solution of (I_η) if $\eta < \eta^*$. Let u_0 be the large positive supersolution of (I_0) so that $u_\eta < u_0$ for all $0 < \eta < \eta^*$. We consider the order interval $[0, u_0]$ defined by

$$[0, u_0] = \{u \in C(\bar{\Omega}) \,:\, 0 \le u \le u_0\}$$

and we define the nonlinear operator T_η from $[0, u_0]$ into $C(\bar{\Omega})$ as follows: Give $0 < u < u_0$, $v = T_\eta u$ is the unique solution of the following problem:

$$\begin{cases} \Delta v + u^\alpha(x) - \eta h = 0 & \text{in} \quad \Omega \\ v = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Then T_η is trivially completely continuous and increasing. From Corollary of Theorem 3.1 and Corollary of Theorem 3.2, the fixed point index

$$i(T_{\eta_*}, [0, u_0], [0, u_0]) = 0 \quad \text{if} \quad \eta_* > \eta^*.$$

For the compact real interval $[\eta, \eta_*]$ we define the completely continuous map

$$H : [\eta, \eta_*] \times [0, u_0] \to C(\bar{\Omega})$$

by $H(\eta', u) = T_{\eta'} u$ for $\eta \le \eta' \le \eta_*$. From the symmetric property (see [9]) of positive solutions and maximum principles, we can prove that

$$H(\eta', u) \neq u \quad \text{for} \quad (\eta', u) \in [\eta, \eta_*] \times \partial[0, u_0],$$

where $\partial[0, u_0]$ is the boundary of the order interval $[0, u_0]$ in the ordered Banach space $C(\bar{\Omega})$ (see [1]). In fact, if there is u so that $T_{\eta'} u = u$ and $u \in \partial[0, u_0]$. Then u is a positive solution of $(I_{\eta'})$, and so u is radially symmetric, and hence $\frac{du}{dr} < 0$ with respect to the radius r (see [9]). Hence there is a point $x_0 \in \Omega$ such that $u(x_0) = u_0(x_0)$. This is impossible from maximum principles.

By the homotopy invariance of the fixed point index,

$$i(T_{\eta'}, [0, u_0], [0, u_0]) = 0$$

for all $\eta' \in [\eta, \eta_*]$. We note that if $\eta \le \eta' < \eta^*$ and if $u_{\eta'}$ is the positive solution defined by $(*)$ of $(I_{\eta'})$, then $u_{\eta'} < u_0$ and $u_{\eta'}$ is a subsolution of (I_η). Clearly $[u_{\eta'}, u_0] \subset [0, u_0]$ and

$$i(T_\eta, [u_{\eta'}, u_0], [u_{\eta'}, u_0]) = 1.$$

By the additivity of the fixed point index,

$$i(T_\eta, [0, u_0] \setminus [u_{\eta'}, u_0], [0, u_0] \setminus [u_{\eta'}, u_0]) = -1.$$

Therefore, there is another positive solution $u'_\eta \in [0, u_0] \setminus [u_{\eta'}, u_0]$ of (I_η). The existence of three distinct solutions can be obtained by the similar method in proof of Theorem 4.1.

We prove the uniqueness of the positive solution for (I_{η^*}). First, we choose a large positive supersolution u_1 of (I_0) so that $u_0 < u_1$.

Without loss of generality, we assume that the unique positive solution of (I_0) is less than u_0. Suppose that there are two distinct positive solutions v and w of (I_η). By the following inequality;

$$\Delta\left(\frac{v(x)+w(x)}{2}\right)+\left(\frac{v(x)+w(x)}{2}\right)^{\alpha}\geq \eta^* h$$

for all $x \in \bar{\Omega}$, we note that $\frac{v+w}{2}$ is a subsolution for all the problem (I_η) with $0 \leq \eta \leq \eta^*$. We can apply the homotopy invariance property in the order interval $[\frac{v+w}{2}, u_1]$ of the Leray-Schauder degree for the operator T_η as before. This means that we use the homotopy $H(\eta, u) = T_\eta u$ for $\eta \geq 0$. Then we have two distinct positive solutions of the problem (I_0). This leads to a contradiction for the uniqueness of the positive solution of (I_0). □

COROLLARY. *Let Ω be an open ball and let h be negative constant. Then there is a positive number η^* such that for all $0 < \eta < \eta^*$ (I_η) has two distinct negative solutions $u'_\eta < u_\eta$ so that*

(1) *(I_η) has three distinct solutions which contains two negative solutions and a positive solution if $0 < \eta < \eta^*$,*
(2) *(I_{η^*}) has a unique positive solution and a unique negative solution,*
(3) *u_η approaches the negative solution of (I_0) as $\eta \to 0$,*
(4) *if $\eta > \eta^*$, (I_η) has no negative solution but only the positive solution.*

REMARK. Assume h is a negative constant and Ω is a ball in $\mathbf{R}^n$. We define the ordered pair (u, η) by u is a solution of the problem (I_η). Let η be fixed and let $\mathcal{C}_\eta$ be the set of all possible pairs (u, η). By Theorem 4.2 and its Corollary, we note that a $\mathcal{S}$-shaped curve (u, η) is embedded in $\cup_{\eta \in \mathbf{R}^1} \mathcal{C}_\eta$.

References

[1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), no. 4, 620-709.

[2] K. Brown and H. Budin, *On the existence of positive solutions for a class of semilinear elliptic boundary value problems*, SIAM J. Math. Anal. **10** (1979), 875-883.

[3] K. Chang, *Infinite Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser, Boston, 1993.

[4] E. N. Dancer, *Multiple fixed points of positive mappings*, J. Reine Angew. Math. **371** (1986), 46-66.

[5] ———, *Upper and lower stability and index theory for positive mappings and applications*, Nonlinear Analysis T.M.A. **17** (1991), 205-217.

[6] E. Dancer and P. Hess, *On stable solutions of quasilinear periodic-parabolic problems*, preprint.

[7] D. DeFigueiredo, *On the existence of multiple ordered solutions of nonlinear eigenvalue problems*, Nonlinear Analysis **11** (1987), 481-492.

[8] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[9] B. Gidas, W. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209-243.

[10] P. Hess, *On multiple positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations **6** (1981), 951-961.

[11] N. Hirano and W. S. Kim, *Existence of stable and unstable solutions for semilinear parabolic problems with a jumping nonlinearity*, Nonlinear Analysis T.M.A. **26** (1996), no. 6, 1143-1160.

[12] ———, *Multiplicity and stability result for semilinear parabolic equations*, Discrete and Continuous dynamical Systems **2** (1996), no. 2, 271-280.

[13] F. Howes, *Singularly perturbed semilinear elliptic boundary value problems*, Comm. Partial Differential Equations **4** (1979), 1-39.

[14] W. Kelley and B. Ko, *Semilinear elliptic singular perturbation problems with nonuniform interior behavior*, J. Differential Equations **86** (1990), no. 1, 88-101.

[15] W. S. Kim, *Multiplicity result for periodic solutions of semilinear parabolic equations*, Comm. Appli. Anal. (to appear).

[16] ———, *Multiplicity result for semilinear dissipative hyperbolic equations*, J. Math. Anal. Appli. **231** (1999), 34-46.

[17] B. Ko, *The third solution of semilinear elliptic boundary value problems and applications to singular perturbation problems*, J. Differential Equations **101** (1993), no. 1, 1-14.

[18] H. Matano, *Asymptotic behavior and stability of solutions of semilinear diffusion equations*, Publ. Res. Inst. Math. Sci. **15** (1979), 401-454.

[19] J. Mawhin, *Topological degree method in nonlinear boundary value problems*, in "Regional Conference Ser. Math. N40", Amer. Math. Soc. Providence, 1977.

[20] J. Mawhin and K. Schmitt, *Landesman-Lazer type problems at an eigenvalue of odd multiplicity*, Results in Mathematics **14** (1988), 138-146.

[21] M. Protter and H. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, New Jersey, 1967.

[22] P. Rabinowitz, *Multiple critical points of perturbed symmetric functionals*, Trans. Amer. Math. Soc. **272** (1982), 753-770.

[23] D. Sattinger, *Monotone methods in nonlinear elliptic and parabolic equations*, Indiana Univ. Math. J. **21** (1972), 979-1000.

[24] K. Schmitt, *Boundary value problems for quasilinear second order elliptic equations*, Nonlinear Analysis **2** (1978), no. 3, 263-309.

[25] M. Struwe, *Variational Methods, applications to nonlinear partial differential equations and Hamiltonian systems*, Springer-Verlag, Berlin, 1990.

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