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ON THE LANDSBERG SPACES OF DIMENSION TWO WITH A SPECIAL (α, β)-METRIC

HONG-SUH PARK AND IL-YONG LEE*

ABSTRACT. The present paper is devoted to studying the condition that a two-dimensional Finsler space with a special (α, β)-metric be a Landsberg space. It is proved that if a Finsler space with a special (α, β)-metric is a Landsberg space, then it is a Berwald space.

1. Introduction

We consider a Finsler space with the Cartan connection $C\Gamma$. If the covariant derivative $C_{hij|k}$ of the C-torsion tensor of $C\Gamma$ satisfies $C_{hij|k}y^k = 0$, then the Finsler space is called a Landsberg space. A Berwald space is characterized by $C_{hij|k} = 0$. Berwald spaces are specially interesting and important, because the connection is linear, and many examples of Berwald spaces have been known. On the other hand, if a Finsler space is a Landsberg space and satisfies some additional conditions, then it is merely a Berwald space ([3], [9]).

The purpose of the present paper is devoted to finding a Landsberg space in a two-dimensional Finsler space F^2 with a special (α, β)-metric $L(\alpha, \beta)$ satisfying $L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c_1, c_2, c_3 are non-zero constants. First we determine the difference vector and the main scalar of F^2 with $L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$. Next we derive the condition for F^2 with a special (α, β)-metric to be a Landsberg space. Finally we show that if F^2 with the above metric is a Landsberg space, then it is a Berwald space.

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2. Preliminaries

Let $F^n = (M^n,\ L(\alpha,\ \beta))$ be an n-dimensional Finsler space with an (α,β)-metric and $R^n = (M^n,\ \alpha)$ the associated Riemannian space, where $\alpha^2 = a_{ij}(x)y^iy^j$, $\beta = b_i(x)y^i$. In the following the Riemannian metric α is *not supposed to be positive-definite* and we shall restrict our discussions to a domain of (x,y), where β does not vanish. The covariant differentiation in the Levi-Civita connection $\gamma_j{}^i{}_k(x)$ of R^n is denoted by the semi-colon. Let us list the symbols here for the late use:

$$2r_{ij} = b_{i;j} + b_{j;i}, \quad 2s_{ij} = b_{i;j} - b_{j;i}, \quad r^i{}_j = a^{ir}r_{rj}, \quad s^i{}_j = a^{ir}s_{rj},$$
$$r_i = b_r r^r{}_i, \quad s_i = b_r s^r{}_i, \quad b^i = a^{ir}b_r, \quad b^2 = a^{rs}b_r b_s.$$

The Berwald connection $B\Gamma = (G_j{}^i{}_k,\ G^i{}_j)$ of F^n plays one of the leading roles in the present paper. Denote by $B_j{}^i{}_k$ the difference tensor ([8]) of $G_j{}^i{}_k$ from $\gamma_j{}^i{}_k$ as follows:

$$G_j{}^i{}_k(x,y) = \gamma_j{}^i{}_k(x) + B_j{}^i{}_k(x,y).$$

With the subscript 0, transvection by y^i, we have

$$G^i{}_j = \gamma_0{}^i{}_j + B^i{}_j, \quad 2G^i = \gamma_0{}^i{}_0 + 2B^i,$$

and then $B^i{}_j = \dot{\partial}_j B^i$ and $B_j{}^i{}_k = \dot{\partial}_k B^i{}_j$. It is noted that the Cartan connection also has the nonlinear connection $G^i{}_j$ common to $B\Gamma$. $B^i(x,y)$ is called the *difference vector* in the present paper.

Since $B\Gamma$ is L-metrical, $L(\alpha,\beta)$ satisfies

$$L_{|i} = \partial_i L - (\dot{\partial}_r L)G^r{}_i = 0 = L_1\alpha_{|i} + L_2\beta_{|i},$$

where $(L_1,\ L_2) = (\partial L/\partial\alpha,\ \partial L/\partial\beta)$, and so

$$\alpha_{|i} = -\frac{L_2}{L_1}\beta_{|i}. \tag{2.1}$$

It is observed that $\beta_{|i} = b_{s|i}y^s = (b_{s;i} - b_r B_s{}^r{}_i)y^s$, which implies

$$\beta_{|i}y^i = r_{00} - 2b_r B^r. \tag{2.2}$$

For the scalar b^2 we have $b^2_{|i}y^i = (\partial_i b^2)y^i = b^2_{;i}y^i = 2b^r(r_{ri}+s_{ri})y^i$, which shows

$$b^2_{|i}y^i = 2(r_0+s_0). \tag{2.3}$$

Next the quadratic form

$$\gamma^2 = b^2\alpha^2 - \beta^2 = (b^2a_{ij} - b_ib_j)y^iy^j,$$

plays a role in the following. From the equations above it is easy to show

$$\gamma^2_{|i}y^i = 2(r_0+s_0)\alpha^2 - 2\left(\frac{L_2}{L_1}b^2\alpha+\beta\right)(r_{00}-2b_rB^r). \tag{2.4}$$

The following Lemma has been shown as follows:

LEMMA 2.1. ([**2**], [**5**]) *If $\alpha^2 \equiv 0$ (mod. β), that is, $a_{ij}(x)y^iy^j$ contains $b_i(x)y^i$ as a factor, then the dimension n is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_ib^i = 2$.*

LEMMA 2.2. ([**5**]) *We consider the two-dimensional case.*

(1) *If $b^2 \neq 0$, then there exist a sign $\varepsilon = \pm 1$ and $\delta = d_i(x)y^i$ such that $\alpha^2 = \beta^2/b^2 + \varepsilon\delta^2$ and $d_ib^i = 0$.*

(2) *If $b^2 = 0$, then there exists $\delta = d_i(x)y^i$ such that $\alpha^2 = \beta\delta$ and $d_ib^i = 2$.*

If there are two functions $f(x)$ and $g(x)$ satisfying $f\alpha^2 + g\beta^2 = 0$, then $f = g = 0$ is obvious, because $f \neq 0$ implies a contradition $\alpha^2 = (-g/f)\beta^2$.

In the present paper we consider an n-dimensional Finsler space with a special (α,β)-metric $L(\alpha,\beta)$ satisfying

$$L^2(\alpha,\beta) = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2, \tag{2.5}$$

where c_1, c_2 and c_3 are non-zero constants. This metric was introduced and studied in [10] as a generalization of the Randers metric for the first time. If $c_1c_3 - c_2^2 = 0$, then the metric is a Randers metric. We shall deal with non-Randers space afterward. Therefore $c_1c_3 - c_2^2 \neq 0$ must be assumed. The following has been shown in [10]as follows:

PROPOSITION 2.3. *Let F^n be the Finsler space with a special (α,β)-metric $L(\alpha,\beta)$ satisfying* (2.5), *Then F^n is a Berwald space, if and only if $b_{i;j} = 0$ is satisfied.*

3. The Landsberg space with a special (α, β)-metric

Let $F^n = (M^n, L(\alpha, \beta))$ be an n-dimensional Finsler space with a special (α, β)-metric given by (2.5). By means of the method given in [8], the difference vector of F^n is given by

$$2B^i = \frac{1}{Z}(r_{00} - 2\alpha A s_0)(c_2 L^2 y^i + h\alpha^3 b^i) + 2\alpha A s_0^i. \tag{3.1}$$

where $Z = L^2(c_1\alpha + c_2\beta) + h\alpha\gamma^2$, $h = c_1c_3 - c_2^2$, $A = (c_2\alpha + c_3\beta)/(c_1\alpha + c_2\beta)$.

Before discussing our problem, we must consider the assumption $Z \neq 0$, because Z appears in the denominator in (3.1). If $Z = 0$, then we have

$$c_1^2\alpha^3 + 3c_1c_2\alpha^2\beta + 3c_2^2\alpha\beta^2 + c_2c_3\beta^3 + (c_1c_3 - c_2^2)b^2\alpha^3 = 0,$$

which is written in the form $P\alpha + Q = 0$, where

$$P = \{c_1^2 + (c_1c_3 - c_2^2)b^2\}\alpha^2 + 3c_2^2\beta^2, \quad Q = \beta(3c_1c_2\alpha^2 + c_2c_3\beta^2).$$

Since P and Q are rational polynomials of (y^i) and α is an irrational function of (y^i), we have $P = 0$ and $Q = 0$. These lead to $c_1 = c_2 = c_3 = 0$. This is a contradiction because c_1, c_2, c_3 are non zero constants. Hence $Z \neq 0$ is a proper assumption all through.

It follows from (3.1) that

$$r_{00} - 2b_rB^r = \frac{\alpha(c_1\alpha + c_2\beta)^2}{Z}(r_{00} - 2\alpha A s_0). \tag{3.2}$$

Now we deal with the condition for a two-dimensional Finsler space F^2 with (2.5) to be a Landsberg space. It is known that in the two-dimensional case, a general Finsler space is a Landsberg space, if and only if its main scalar $I(x, y)$ satisfies $I_{|i}y^i = 0$ ([7]).

Owing to [6], the main scalar of F^2 is obtained easily as follows:

$$\varepsilon I^2 = \frac{9c_2^2\gamma^2L^8}{4\alpha Z^3}. \tag{3.3}$$

Substituting (2.2) and (3.2) in the transvection of (2.1) by y^i, we have

$$-Z\gamma^2\alpha_{|i}y^i = \alpha A\gamma^2(c_1\alpha + c_2\beta)^2(r_{00} - 2\alpha A s_0). \tag{3.4}$$

Furthermore substitution of (2.5) and (3.2) in (2.4) leads to

$$\begin{aligned}\alpha Z\gamma^2_{|i}y^i =& 2\alpha^2\{(r_0 + s_0)\alpha Z \\ &- (Ab^2\alpha + \beta)(c_1\alpha + c_2\beta)^2(r_{00} - 2\alpha A s_0)\}.\end{aligned} \tag{3.5}$$

Making use of (2.1), (2.2), (2.4) and (3.2), we get

(3.6)

$$\begin{aligned}&- 3\alpha\gamma^2 Z_{|i}y^i \\ &= \frac{3h\alpha^2\gamma^2}{Z}\Big[\{\beta L^2 + (c_2\alpha + c_3\beta)\gamma^2 + 2\alpha(Ab^2\alpha + \beta)(c_1\alpha + c_2\beta)\} \\ &\qquad (c_1\alpha + c_2\beta)(r_{00} - 2\alpha A s_0) - 2\alpha^2(r_0 + s_0)Z\Big].\end{aligned}$$

The covariant differentiation of (3.3) leads to

$$4\alpha^3 Z^3\varepsilon I^2_{|i}y^i = \frac{9c_2^2L^8}{Z}(\alpha Z\gamma^2_{|i}y^i - Z\gamma^2\alpha_{|i}y^i - 3\alpha\gamma^2 Z_{|i}y^i). \tag{3.7}$$

Substituting (3.4), (3.5) and (3.6) in (3.7), we have

$$\begin{aligned}4\alpha^2 Z^3\varepsilon I^2_{|i}y^i =& \frac{9c_2^2L^8}{Z}\Big[2\alpha^2 Z\{(c_1\alpha + c_2\beta)L^2 - 2h\alpha\gamma^2\}(r_0 + s_0) \\ &-(c_1\alpha+c_2\beta)[Z(c_1\alpha+c_2\beta)\{A(b^2\alpha^2 + \beta^2)+ 2\alpha\beta\} \\ &- 3h\alpha\gamma^2\{\beta L^2 + (c_2\alpha + c_3\beta)(3b^2\alpha^2 - \beta^2) \\ &+ 2(c_1\alpha + c_2\beta)\alpha\beta\}](r_{00} - 2\alpha A s_0)\Big].\end{aligned}$$

Consequently, the two-dimensional Finsler space F^2 with (2.5) is a Landsberg space, if and only if

$$\begin{aligned}&(A_8\alpha^8 + A_7\alpha^7\beta + A_6\alpha^6\beta^2 + A_5\alpha^5\beta^3 + A_4\alpha^4\beta^4 + A_3\alpha^3\beta^5 \\ &+ A_2\alpha^2\beta^6)(r_0 + s_0) + (B_7\alpha^7 + B_6\alpha^6\beta + B_5\alpha^5\beta^2 + B_4\alpha^4\beta^3 \\ &+ B_3\alpha^3\beta^4 + B_2\alpha^2\beta^5 + B_1\alpha\beta^6 + B_0\beta^7)r_{00} + (C_8\alpha^8 + C_7\alpha^7\beta \\ &+ C_6\alpha^6\beta^2 + C_5\alpha^5\beta^3 + C_4\alpha^4\beta^4 + C_3\alpha^3\beta^5 + C_2\alpha^2\beta^6 \\ &+ C_1\alpha\beta^7)s_0 = 0,\end{aligned} \tag{3.8}$$

where

$A_8 = 2c_1^4 - 2c_1^2hb^2 - 4h^2b^4,\ \ A_7 = 12c_1^3c_2 - 6c_1c_2hb^2,$

$A_6 = 6c_1^2(c_1c_3 + 4c_2^2) + 6(c_1c_3 - 2c_2^2)hb^2,$

$A_5 = 2c_1c_2(11c_1c_3 + 9c_2^2) - 2c_2c_3hb^2,\ \ A_4 = 30c_1c_2^2c_3,$

$A_3 = 6c_2c_3(c_1c_3 + c_2^2),\ \ A_2 = 2c_2^2c_3^2,\ \ B_7 = -c_1^3c_2b^2 + 8c_1c_2hb^4,$

$B_6 = -3c_1^4 + c_1^2(6c_1c_3 - 11c_2^2)b^2 + 8(c_1c_3 + c_2^2)hb^4,$

$B_5 = -11c_1^3c_2 - 10c_1c_2^3b^2 + 8c_2c_3hb^4,$

$B_4 = -5c_1^2(2c_1c_3 + 3c_2^2) - 10c_1^2c_3^2b^2,$

$B_3 = -6c_1c_2(4c_1c_3 + c_2^2) - c_2c_3(11c_1c_3 - 6c_2^2)b^2,$

$B_2 = -20c_1c_2^2c_3 - c_2^2c_3^2b^2,\ \ B_1 = -c_2c_3(c_1c_3 + 6c_2^2),$

$B_0 = -c_2^2c_3^2,\ \ C_8 = -2c_1^2c_2^2b^2 - 16c_2^2hb^4,$

$C_7 = 4c_1^3c_2 - 10c_1c_2(c_1c_3 - 2c_2^2)b^2 - 32c_2c_3hb^4,$

$C_6 = 2c_1^2(c_1c_3 + 9c_2^2) - 4c_1c_3(3c_1c_3 - 8c_2^2)b^2 - 16c_3^2hb^4,$

$C_5 = 2c_1c_2(19c_1c_3 + 6c_2^2) + 4c_2c_3(8c_1c_3 - 3c_2^2)b^2,$

$C_4 = 20c_1c_3(c_1c_3 + 2c_2^2) + 10c_3^2(2c_1c_3 - c_2^2)b^2,$

$C_3 = 4c_2c_3(7c_1c_3 + 3c_2^2) + 2c_2c_3^2b^2,\ \ C_2 = 14c_2^2c_3^2,\ \ C_1 = 2c_2c_3^3.$

Seperating (3.8) in the rational and the irrational terms of (y^i), we have

$$\Big\{(A_8\alpha^8 + A_6\alpha^6\beta^2 + A_4\alpha^4\beta^4 + A_2\alpha^2\beta^6)(r_0 + s_0) + (B_6\alpha^6\beta + B_4\alpha^4\beta^3 + B_2\alpha^2\beta^5 + B_0\beta^7)r_{00} + (C_8\alpha^8 + C_6\alpha^6\beta^2 + C_4\alpha^4\beta^4 + C_2\alpha^2\beta^6)s_0\Big\}$$
$$+ \alpha\Big\{(A_7\alpha^6\beta + A_5\alpha^4\beta^3 + A_3\alpha^2\beta^5)(r_0 + s_0) + (B_7\alpha^6 + B_5\alpha^4\beta^2 + B_3\alpha^2\beta^4 + B_1\beta^6)r_{00} + (C_7\alpha^6\beta + C_5\alpha^4\beta^3 + C_3\alpha^2\beta^5 + C_1\beta^7)s_0\Big\}$$
$$= 0,$$

which yields two equations as follows:

$$\begin{aligned}(3.9)\qquad &(A_8\alpha^8 + A_6\alpha^6\beta^2 + A_4\alpha^4\beta^4 + A_2\alpha^2\beta^6)(r_0 + s_0)\\ &+ (B_6\alpha^6\beta + B_4\alpha^4\beta^3 + B_2\alpha^2\beta^5 + B_0\beta^7)r_{00}\\ &+ (C_8\alpha^8 + C_6\alpha^6\beta^2 + C_4\alpha^4\beta^4 + C_2\alpha^2\beta^6)s_0 = 0,\end{aligned}$$

$$(3.10)\qquad \begin{aligned}&(A_7\alpha^6\beta + A_5\alpha^4\beta^3 + A_3\alpha^2\beta^5)(r_0 + s_0)\\ &+ (B_7\alpha^6 + B_5\alpha^4\beta^2 + B_3\alpha^2\beta^4 + B_1\beta^6)r_{00}\\ &+ (C_7\alpha^6\beta + C_5\alpha^4\beta^3 + C_3\alpha^2\beta^5 + C_1\beta^7)s_0 = 0.\end{aligned}$$

From (3.9) and (3.10) we obtain respectively

$$(3.11)\qquad B_0\beta^7 r_{00} \equiv 0 \pmod{\alpha^2},$$

$$(3.12)\qquad B_1\beta^6 r_{00} + C_1\beta^7 s_0 \equiv 0 \pmod{\alpha^2}.$$

From $B_0 \neq 0$ (3.11) is reduced to

$$(3.11')\qquad \beta^7 r_{00} \equiv 0 \pmod{\alpha^2}.$$

In the following we shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity. For instance, $v_3 = v_{ijk}y^iy^jy^k$ is an $hp(3)$. Then (3.11′) is written as

$$\beta^7 r_{00} = \alpha^2 u_7,$$

where u_7 is an $hp(7)$. From $b^2 \neq 0$ it follows that $\alpha^2 \not\equiv 0 \pmod{\beta}$ and there must exist a function $f(x)$ such that $u_7 = \beta^7 f(x)$. Hence we have

$$(3.11'')\qquad r_{00} = \alpha^2 f(x)\ ;\ r_{ij} = a_{ij}f(x).$$

Then (3.12) is reduced to

$$(3.12')\qquad \beta^7 s_0 \equiv 0 \pmod{\alpha^2},$$

because of $C_1 \neq 0$. (3.12′) shows that there exists an $hp(6)$ u_6 satisfying $\beta^7 s_0 = \alpha^2 u_6$, which implies $u_6 = 0$, because $\alpha^2 u_6$ can not contain β^7 as a factor. Thus we have

$$(3.12'')\qquad s_0 = 0\ ;\ s_i = 0.$$

It is obvious that (3.11″) gives

$$(3.13)\qquad r_0 = \beta f(x)\ ;\ r_j = b_j f(x).$$

Therefore (3.11) and (3.12) are reduced to (3.11″), (3.12″) and (3.13), and (3.9), (3.10) are reduced respectively to

$$\begin{aligned} f(x)[(A_8+B_6)\alpha^6+(A_6+B_4)\alpha^4\beta^2+(A_4+B_2)\alpha^2\beta^4 \\ +(A_2+B_0)\beta^6]=0, \end{aligned} \tag{3.14}$$

$$\begin{aligned} f(x)[B_7\alpha^6+(A_7+B_5)\alpha^4\beta^2+(A_5+B_3)\alpha^2\beta^4 \\ +(A_3+B_1)\beta^6]=0. \end{aligned} \tag{3.15}$$

Let us assume $f(x)\neq 0$. Then (3.14) and (3.15) imply

$$(A_2+B_0)\beta^6=\alpha^2 v_4, \quad (A_3+B_1)\beta^6=\alpha^2 w_4,$$

where v_4, w_4 are $hp(4)$. Analogously to the above, these imply $v_4 = w_4 = 0$. We have, however,

$$A_2+B_0=(c_2c_3)^2\neq 0, \quad A_3+B_1=5c_1c_2c_3^2\neq 0.$$

Thus we arrive at a contradiction. Hence $f(x)=0$ must hold and we have $r_{00}=0$; $r_{ij}=0$ and $s=0$; $s_i=0$.

If $b^2=0$, then (3.9) and (3.10) are reduced to

$$\begin{aligned} &(D_8\alpha^8+D_6\alpha^6\beta^2+A_4\alpha^4\beta^4+A_2\alpha^2\beta^6)(r_0+s_0) \\ &+(E_6\alpha^6\beta+E_4\alpha^4\beta^3+E_2\alpha^2\beta^5+B_0\beta^7)r_{00} \\ &+(F_6\alpha^6\beta^2+F_4\alpha^4\beta^4+C_2\alpha^2\beta^6)s_0=0, \end{aligned} \tag{3.16}$$

$$\begin{aligned} &(D_7\alpha^6\beta+D_5\alpha^4\beta^3+A_3\alpha^2\beta^5)(r_0+s_0) \\ &+(E_5\alpha^4\beta^2+E_3\alpha^2\beta^4+B_1\beta^6)r_{00} \\ &+(F_7\alpha^6\beta+F_5\alpha^4\beta^3+F_3\alpha^2\beta^5+C_1\beta^7)s_0=0, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} &D_8=2c_1^4, \ D_7=12c_1^3c_2, \ D_6=6c_1^2(c_1c_3+4c_2^2), \\ &D_5=2c_1c_2(11c_1c_3+9c_2^2), \ E_6=-2c_1^4, \ E_5=-11c_1^3c_2, \\ &E_4=-5c_1^2(2c_1c_3+3c_2^2), \ E_3=-6c_1c_2(4c_1c_3+c_2^2), \\ &E_2=-20c_1c_2^2c_3, \ F_7=-4c_1^3c_2, \ F_6=2c_1^2(c_1c_3+9c_2^2), \\ &F_5=2c_1c_2(19c_1c_3+6c_2^2), \ F_4=20c_1c_3(c_1c_3+2c_2^2), \\ &F_3=4c_2c_3(7c_1c_3+3c_2^2). \end{aligned}$$

Making use of Lemma 2.1, (3.16) and (3.17) are reduced to

$$\begin{aligned}(3.18)\qquad &\beta\Big\{(D_6\delta^3 + A_4\delta^2\beta + A_2\delta\beta^2)(r_0 + s_0)\\ &+ (E_4\delta^2 + E_2\delta\beta + B_0\beta^2)r_{00} + (F_6\delta^3 + E_4\delta^2\beta + C_2\delta\beta^2)s_0\Big\}\\ &+ \Big\{D_8\delta^4(r_0 + s_0) + E_6\delta^3 r_{00}\Big\} = 0,\end{aligned}$$

$$\begin{aligned}(3.19)\qquad &\beta\Big\{(D_5\delta^2 + A_3\delta\beta)(r_0 + s_0) + (E_3\delta + B_1\beta)r_{00}\\ &+ (F_5\delta^2 + F_3\delta\beta + C_2\beta^2)s_0\Big\}\\ &+ \Big\{D_7\delta^3(r_0 + s_0) + E_5\delta^2 r_{00} + F_7\delta^3 s_0\Big\} = 0.\end{aligned}$$

From (3.18) and (3.19) we have

$$\delta^3\{D_8\delta(r_0 + s_0) + E_6 r_{00}\} \equiv 0 \quad (\text{mod. } \beta),$$
$$\delta^2\{D_7\delta(r_0 + s_0) + F_7\delta s_0 + E_5 r_{00}\} \equiv 0 \quad (\text{mod. } \beta).$$

Since $r_0 + s_0 = b^2_{;i}y^i/2$ vanishes because of $b^2 = 0$, the above equations are written as follows:

$$(3.20)\qquad E_6\delta^3 r_{00} \equiv 0 \quad (\text{mod. } \beta),$$

$$(3.21)\qquad \delta^2\{E_7\delta s_0 + E_5 r_{00}\} \equiv 0 \quad (\text{mod. } \beta).$$

Because of $E_6 \neq 0$, (3.20) is reduced to $\delta^3 r_{00} \equiv 0$ (mod. β). Then there exists an $hp(4)$ x_4 such that

$$\delta^3 r_{00} = \beta x_4.$$

Since $\delta^3 \equiv 0$ (mod. β), there exists an $hp(1)$ λ satisfying

$$(3.20')\qquad r_{00} = \lambda\beta; \quad r_{ij} = \frac{1}{2}(\lambda_i b_j + \lambda_j b_i).$$

Substituting (3.20′) in (3.21), there exists an $hp(3)$ w_3 such that

$$(3.21')\qquad \delta^2(F_7\delta s_0 + E_5\lambda\beta) = \beta w_3.$$

From $\delta^2 \not\equiv 0$ (mod. β) we have $w_3 = \mu\delta^2$ and $F_7\delta s_0 + E_5\lambda\beta = \mu\beta$, where μ is an $hp(1)$, that is, $F_7 s_0\delta = (\mu - E_5\lambda)\beta$. Therefore there exists a function $g(x)$ such that

$$(3.22) \qquad F_7 s_0 = g(x)\beta, \qquad \mu - E_5\lambda = g(x)\delta,$$

which implies $s_0 = f(x)\beta$, where $f(x) = g(x)/F_7$. Substituting (3.20′), $s_0 = f(x)\beta$ and $s_0 + r_0 = 0$ in (3.18) and (3.19), we get respectively

$$(3.23) \quad (E_4\delta^2 + E_2\delta\beta + B_0\beta^2)\lambda\beta + (F_6\delta^3\beta + C_2\delta\beta^2)f\beta + E_6\delta^3\lambda = 0,$$

$$(3.24) \quad (E_3\delta + B_1\beta)\lambda\beta + (F_5\delta^2 + F_3\delta\beta + C_1\beta^2)f\beta + E_5\delta^2\lambda + E_7\delta^3 f = 0.$$

The term $B_0\lambda\beta^3$ of (3.23) and the term $B_1\lambda\beta^2 + C_1 f\beta^3$ of (3.24) seemingly do not contain δ, and hence we must have $hp(3)$ X_3 and $hp(2)$ Y_2 satisfying

$$B_0\lambda\beta^3 = \delta X_3, \qquad B_1\lambda\beta^2 + C_1 f\beta^3 = \delta\beta Y_2$$

respectively. Eliminating λ from above the equations, we get

$$(3.25) \qquad C_1 B_0 f\beta^4 = \delta W_3,$$

where $W_3 = B_0\beta Y_2 - B_1 X_3$ is an $hp(3)$, and hence $W_3 = 0$, because δW_3 can not contain β^4 as a factor. Since $C_1 \neq 0$, $B_0 \neq 0$, we obtain $f = 0$. Substituting $f = 0$ in (3.24), we have

$$\lambda\{(E_3\delta + B_1\beta)\beta + E_5\delta^2\} = 0.$$

If $\lambda \neq 0$, then we have $B_1\beta^2 = -(E_3\beta + B_1\delta)\delta$, which implies $E_3\beta + E_5\delta = 0$, because $(E_3\beta + B_1\delta)\delta$ can not contain β^2 as a factor. Since (β, δ) are independant, we obtain $E_3 = E_5 = 0$. This is contradictory to $E_3 \neq 0$, $E_5 \neq 0$. Hence $\lambda = 0$. From (3.20′) and $s_0 = f\beta$ we have $r_{00} = 0$ and $s_0 = 0$ directly.

Summarizing up, we obtain $r_{00} = 0$ and $s_0 = 0$ in both cases of $b^2 \neq 0$ and $b^2 = 0$, that is,

$$(3.26) \qquad b_{i:j} + b_{j:i} = 0, \qquad b^r b_{r:i} = 0.$$

Consequently, we have the following

THEOREM 3.1. *The necessary and sufficient condition for a two-dimensional Finsler space F^2 with a special (α,β)-metric $L(\alpha,\beta)$ satisfying* (2.5) *to be a Landsberg space is that b_i is a Killing vector with constant length.*

Now we shall prove the following theorem.

THEOREM 3.2. *Let F^2 be a two-dimensional Finsler space with a special (α,β)-metric $L(\alpha,\beta)$ satisfying* (2.5). *If F^2 is a Landsberg space, then F^2 is a Berwald space.*

Proof. (3.26) of the two-dimensional case is written as

$$(3.26') \qquad b_{1;1}=0, \quad b_{2;2}=0, \quad b_{1;2}=-b_{2;1}.$$

$$b^1b_{1;1}+b^2b_{2;1}=b^2b_{2;1}=0, \quad b^1b_{1;2}+b^2b_{2;2}=-b^1b_{2;1}=0,$$

where (b^1,b^2) of (3.26′) is the contravariant component of (b_1,b_2). This is nothing but $b_{i;j}=0$, $i,j=1,2$, which coincides with the condition for the space to be a Berwald space from Proposition 2.3. Thus the proof is completed. □

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Hong-Suh Park
Department of Mathematics
Yeungnam University
Gyongsan 712-749, Korea
E-mail: phs1230@unitel.co.kr

Il-Yong Lee
Department of Mathematics
Kyungsung University
Pusan 608-736, Korea
E-mail: iylee@star.kyungsung.ac.kr