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ON p-ADIC q-BERNOULLI NUMBERS

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ABSTRACT. We give a proof of the distribution relation for q-Bernoulli polynomials $B_k(x:q)$ by using q-integral and evaluate the values of p-adic q-L-function.

1. Introduction

Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$.

Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log_p q)$ for $|x|_p \leq 1$.

In the p-adic case, Carlitz's q-Bernoulli number $\beta_k = \beta_k(q)$ are represented by a q-analogue form of Witt's formula and investigated some properties (see [4], [5]). In [6], Koblitz constructed a q-analogue of the p-adic L-functions which interpolated Carlitz's q-Bernoulli numbers $\beta_k(q)$. In the complex case (see [7]), Tsumura considered a q-analogue of the Dirichelet L-series which interpolated q-Bernoulli numbers $B_k(q)$.

Let d be a fixed integer and let p be a fixed prime number. We set

$$X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}),$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

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where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

For any positive integer N,

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]} = \frac{q^a}{[dp^N : q]}$$

is known to be a distribution on X (see [4], [5]).

This distribution yields an integral for each non-negative integer m:

$$\int_X [a]^m \, d\mu_q(a) = \beta_m(q) = I_q([a]^m),$$

where $\beta_m(q)$ are Carlitz's q-Bernoulli numbers (see [4], [5]).

In this paper, we show that q-Bernoulli numbers $B_k(q)$ can be represented as an integral by q-analogue μ_q of ordinary p-adic invariant measure and investigate some properties.

As an application, we give a proof of the distribution relation for p-adic q-Bernoulli polynomials $B_k(x; q)$ and construct p-adic q-Bernoulli measures to define p-adic q-L-series which interpolate q-Bernoulli numbers $B_k(q)$.

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2. q-Analogue of p-adic Bernoulli measures

In complex case [1], the Carlitz's numbers $\eta_k = \eta_k(q)$ are determined by

$$\eta_0 = 0, \quad (q\eta + 1)^k - \eta_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1. \end{cases}$$

These numbers η_k induce Carlitz's q-Bernoulli numbers $\beta_k(q) = \beta_k$ as

$$\beta_0 = 1, \quad q(q\beta + 1)^k - \beta_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1. \end{cases}$$

In [7], H. Tsumura modified the above numbers η_k, that is,

$$B_0(q) = \frac{q-1}{\log q}, \quad (qB(q) + 1)^k - B_k(q) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing $B_i(q)$ by $B^i(q)$.

We use the notation

$$[x] = [x : q] = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x : q] = x$. The q-Bernoulli numbers $B_k(q)$ satisfy the following relation

$$B_m(q) = \frac{1}{(q-1)^m}\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\frac{i}{[i]}.$$

This can be proved by the same method as [1: eq. 4.11].

In this paper, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. It is known (see [4], [5]) that

$$\int_{\mathbb{Z}_p} q^{-x}[x]^m d\mu_q(x) = \frac{q-1}{\log q}\int_{\mathbb{Z}_p}[x]^m d\mu_0(x),$$

where $\mu_0(x+p^N\mathbb{Z}_p) = \frac{1}{p^N}$.

In the p-adic case, the q-Bernoulli numbers $B_k(q)$ can be represented by

$$B_m(q) = \int_{\mathbb{Z}_p} q^{-x}[x]^m d\mu_q(x).$$

This is easily proved as in [4], [5].

Now, we define q-Bernoulli polynomials by

$$B_m(x : q) = \int_{\mathbb{Z}_p} q^{-t}[x+t]^m d\mu_q(t).$$

Then these can be rewritten as

$$(q^x B(q) + [x])^m = B_m(x : q),$$

for $m \geq 0$.

Indeed we have

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{-t}[x+t]^n d\mu_q(t) &= \int_{\mathbb{Z}_p} q^{-t}([x]+q^x[t])^n d\mu_q(t) \\ &= \sum_{k=0}^{n} \binom{n}{k} [x]^{n-k} q^{kx} \int_{\mathbb{Z}_p} q^{-t}[t]^k d\mu_q(t) \\ &= \sum_{k=0}^{n} \binom{n}{k} [x]^{n-k} q^{kx} B_k(q) = (q^x B(q) + [x])^n.\end{aligned}$$

As $q \to 1$, we have $B_k(q) \to B_k$ and $B_k(x:q) \to B_k(x)$.

LEMMA 1. *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} \chi(x) q^{-x}[x]^n d\mu_q(x) = \int_X \chi(x) q^{-x}[x]^n d\mu_q(x).$$

This can be easily proved as in [4], [5].

The following lemma is used to construct the p-adic q-Bernoulli measures. A simple proof in the p-adic case can be given by using I_q-integration of q-Bernoulli numbers.

LEMMA 2. *For any positive integer d, k we have*

$$[d]^{k-1} \sum_{i=0}^{d-1} B_k\left(\frac{x+i}{d} : q^d\right) = B_k(x:q).$$

Proof. From the definition of $B_k(x:q)$, we can write

$$\begin{aligned}B_k(x:q) &= \int_X q^{-t}[x+t]^k d\mu_q(t) = \lim_{\rho\to\infty} \frac{1}{[dp^\rho]} \sum_{n=0}^{dp^\rho-1} [x+n]^k \\ &= \lim_{\rho\to\infty} \frac{1}{[d]} \frac{1}{[p^\rho : q^d]} \sum_{i=0}^{d-1} \sum_{n=0}^{p^\rho-1} [x+i+dn]^k \\ &= \lim_{\rho\to\infty} \frac{1}{[d]} \frac{1}{[p^\rho : q^d]} \sum_{n=0}^{d-1} \sum_{n=0}^{p^\rho-1} \left(\left[\frac{x+i}{d} + n : q^d\right][d]\right)^k\end{aligned}$$

$$= \frac{1}{[d]} \sum_{i=0}^{d-1} [d]^k \int_{\mathbb{Z}_p} \left[\frac{x+i}{d} + t : q^d \right]^k d\mu_{q^d}(t)$$

$$= [d]^{k-1} \sum_{i=0}^{d-1} B_k \left(\frac{x+i}{d} : q^d \right). \qquad \square$$

THEOREM 1. *For any positive integer N, k and d, let $\mu_k^* = \mu_{k:q}^*$ be defined by*

$$\mu_k^*(a + dp^N) = [dp^N : q]^{k-1} B_k \left(\frac{a}{dp^N} : q^{dp^N} \right).$$

Then μ_k^* is measure on X.

Proof. It is suffices to check that

$$\sum_{i=0}^{p-1} \mu_k^*(a + idp^N + dp^{N+1}\mathbb{Z}_p) = \mu_k^*(a + dp^N \mathbb{Z}_p).$$

By definition of μ_k^*, we have

$$\begin{aligned}
&\sum_{i=0}^{p-1} \mu_k^*(a + idp^N + dp^{N+1}) \\
&= [dp^{N+1} : q]^{k-1} \sum_{i=0}^{p-1} B_k \left(\frac{a + idp^N}{dp^{N+1}} : q^{dp^{N+1}} \right) \\
&= [dp^N : q]^{k-1} [p : q^{dp^N}]^{k-1} \sum_{i=0}^{p-1} B_k \left(\frac{\frac{a}{dp^N} + i}{p} : (q^{dp^N})^p \right) \\
&= [dp^N : q]^{k-1} B_k \left(\frac{a}{dp^N} : q^{dp^N} \right) \\
&= \mu_k^*(a + dp^N \mathbb{Z}_p).
\end{aligned}$$

Thus we proved the above Theorem 1. $\square$

Let χ be a Dirichlet character with conductor f.

Now we define the generalized q-Bernoulli numbers as

$$B_{k,\chi}(q) = \int_{\mathbb{Z}_p} \chi(x) q^{-x} [x]^k d\mu_q(x).$$

From the above definition, we have

$$B_{k,\chi}(q) = [f]^{k-1} \sum_{a=0}^{f-1} \chi(a) B_k\left(\frac{a}{f} : q^f\right).$$

We can express a generalized q-Bernoulli number as an integral on X, by using the measure μ_k^*.

THEOREM 2. *For any positive integer k, we have*

$$\int_X \chi(x) d\mu_k^*(x) = B_{k,\chi}(q).$$

Proof. From the definition of μ_k^*, we see that

$$\begin{aligned}
&\int_X \chi(x) d\mu_k^* \\
&= \lim_{n\to\infty} \sum_{a=0}^{fp^n-1} \chi(a) \mu_k^*(a + fp^n\mathbb{Z}_p) \\
&= \lim_{n\to\infty} \sum_{a=0}^{fp^n-1} \chi(a) [fp^n : q]^{k-1} B_k\left(\frac{a}{fp^n} : q^{fp^n}\right) \\
&= \lim_{n\to\infty} [f]^{k-1} [p^n : q^f]^{k-1} \sum_{a=0}^{f-1} \sum_{b=0}^{p^n-1} \chi(a+fb) B_k\left(\frac{a+fb}{fp^n} : q^{fp^n}\right) \\
&= [f]^{k-1} \sum_{a=0}^{f-1} \chi(a) \lim_{n\to\infty} [p^n : q^f]^{k-1} \sum_{b=0}^{p^n-1} B_k\left(\frac{\frac{a}{f}+b}{p^n} : (q^f)^{p^n}\right).
\end{aligned}$$

By Lemma 2, we obtain that

$$\int_X \chi(x) d\mu_k^*(x) = [f]^{k-1} \sum_{a=0}^{f-1} \chi(a) B_k\left(\frac{a}{f} : q^f\right) = B_{k,\chi}(q).$$

Therefore the above Theorem 2 is proved. □

Next we give a relation between μ_k^* and μ_q.

THEOREM 3. *For any positive integer k, we have*

$$q^{-x}[x]^k d\mu_q(x) = d\mu_k^*(x).$$

Let p be a prime number and let χ be a Dirichelet character with values contained in the algebraic closure of $\mathbb{Q}_p$. We set $p^* = p$ for $p > 2$, and $p^* = 4$ for $p = 2$, and denote by $\bar{f} = (f, p^*)$ the least common multiple of conductor f of χ and p^*.

Let $B_{n,\chi}(q)$ denote the n-th generalized q-Bernoulli number belonging to the character χ.

Then we have q-analogue form of Witt's formula in the p-adic cyclotomic field $\mathbb{Q}_p(\chi)$ as follows:

$$B_{m,\chi}(q) = \lim_{\rho\to\infty} \frac{1}{[\bar{f}p^\rho]} \sum_{x=1}^{\bar{f}p^\rho} \chi(x)[x]^m$$

for all $n \geq 0$.

Herein as usual we set $\chi(x) = 0$ if x is not prime to the conductor f. In this section we shall give a few simple formulas of congruences for the generalized q-Bernoulli numbers. From the above formula for $B_{n,\chi}(q)$ we have

$$\begin{aligned} B_{n,\chi}(q) = \lim_{\rho\to\infty} \frac{1}{[\bar{f}p^\rho]} \sum_{1\leq x\leq \bar{f}p^\rho}^{*} \chi(x)[x]^n \\ + \lim_{\rho\to\infty} \frac{1}{[\bar{f}p^{\rho-1} : q^p][p]} \sum_{y=1}^{\bar{f}p^{\rho-1}} \chi(p)\chi(x)[p]^n[y : q^p]^n \end{aligned}$$

where $*$ means to takes sum over the rational integers prime to p in the given range.

Thus we have

$$\begin{aligned} B_{n,\chi}(q) = \lim_{\rho\to\infty} \frac{1}{[\bar{f}p^\rho]} \sum_{1\leq x\leq \bar{f}p^\rho}^{*} \chi(x)[x]^n \\ + [p]^{n-1}\chi(p)B_{n,\chi}(q^p), \end{aligned}$$

that is,

$$B_{n,\chi}(q)-[p]^{n-1}\chi(p)B_{n,\chi}(q^p)=\lim_{\rho\to\infty}\frac{1}{[\bar{f}p^\rho]}\sum_{1\le x\le \bar{f}p^\rho}^{*}\chi(x)[x]^n.$$

We choose a rational number $c\in\mathbb{Z}$ such that $(c,\bar{f})=1, c\neq\pm 1$.

Let x run over the range $1\le x\le \bar{f}p^\rho, (x,p)=1$, x_ρ run over the range $1\le x_\rho\le \bar{f}p^\rho, (x_\rho,p)=1$, and determine a number $r_\rho(x)\in\mathbb{Z}$ by $x_\rho=cx+r_\rho(x)\bar{f}p^\rho$.

Taking the n-th power of this equality and making sum with the character χ we obtain

$$\begin{aligned}&\frac{1}{[\bar{f}p^\rho]}\sum_{1\le x\le \bar{f}p^\rho}^{*}\chi(x_\rho)[x_\rho]^n\\ &\equiv\frac{1}{[\bar{f}p^\rho]}\sum_{1\le x\le \bar{f}p^\rho}^{*}\chi(cx)[cx]^n+n\sum_{1\le x\le \bar{f}p^\rho}^{*}\chi(cx)[cx]^{n-1}[r_\rho(x):q^{\bar{f}p^\rho}]\\ &\qquad(\bmod\ [\bar{f}p^\rho]).\end{aligned}$$

If $\rho\to\infty$, we get

$$\begin{aligned}B_{n,\chi}(q)-&[p]^{n-1}\chi(p)B_{n,\chi}(q^p)\\ &=\chi(c)[c]^n(B_{n,\chi}(q^c)-[p:q^c]^{n-1}\chi(p)B_{n,\chi}(q^{pc})\\ &\qquad+n\lim_{\rho\to\infty}\sum_{1\le x\le \bar{f}p^\rho}{}^{*}\chi(cx)[cx]^{n-1}[r_\rho(x):q^{\bar{f}p^\rho}].\end{aligned}$$

Thus we have

$$\begin{aligned}&\lim_{\rho\to\infty}\sum_{1\le x\le \bar{f}p^\rho}^{*}\chi(cx)[cx]^{n-1}[r_\rho(x):q^{\bar{f}p^\rho}]\\ &\qquad=\frac{1}{n}(B_{n,\chi}(q)-[p]^{n-1}\chi(p)B_{n,\chi}(q^p))\\ &\qquad\qquad-\frac{1}{n}\chi(c)[c]^n(B_{n,\chi}(q^c)-[p:q^c]^{n-1}\chi(p)B_{n,\chi}(q^{pc}).\end{aligned}$$

Let we define the operator $\chi^y=\chi^{y,k:q}$ on $f(q)$ by

$$\chi^y f(q)=[y]^{k-1}\chi(y)f(q^y),$$

$$\chi^x\chi^y=\chi^{x,k:q^y}\circ\chi^{y,k:q}.$$

Then we see that $\chi^x\chi^y=\chi^{xy}$.

Therefore we obtain the following

PROPOSITION 1. *For $n \geq 1$, we have*

$$\lim_{\rho\to\infty} \sum_{1\leq x\leq \bar{f}p^\rho}^{*} \chi(cx)[cx]^{n-1}[r_\rho(x) : q^{\bar{f}p^\rho}] = (1-\chi^p)(1-[c]\chi^c)\frac{B_{n,\chi}(q)}{n}.$$

Now we define

$$\mu_k^c = \frac{1}{k}(\mu_{k:q}^*(U) - [c]^k \mu_{k:q^c}^*(\frac{1}{c}U))$$

where $U \subset X$ is a compact open set. Note that μ_k^c is a measure on X. Here, we define $X^* = X - pX$. This measure yields an integral as follows.

$$\int_{X^*} \chi(x) d\mu_k^c(x) = (1-\chi^p)(1-[c]\chi^c)\frac{B_{k,\chi}(q)}{k}$$

Therefore we obtain the following

THEOREM 4. *For $k \geq 1$, we have*

$$\int_{X^*} \chi(x) d\mu_k^c(x) = \lim_{\rho\to\infty} \sum_{1\leq x\leq \bar{f}p^\rho}^{*} \chi(cx)[cx]^{k-1}\left[\left[-\frac{cx}{\bar{f}p^\rho}\right]_G : q^{\bar{f}p^\rho}\right],$$

where $[\cdot]_G$ is Gauss's symbol.

Let ω denote the Teichmüller character mod p^*. For $x \in X^*$, we set $\langle x\rangle = \langle x : q\rangle = [x]/\omega(x)$. Note that $\langle x\rangle^s$ is defined by $\exp(s\log\langle x\rangle)$, for $|s|_p \leq 1$.

For $r \in \mathbb{Z}$, if we define

$$\zeta_{p,q}(r) = \frac{1}{r-1}\lim_{k\to\infty}\frac{1}{[p^k]}\sum_{m=0}^{p^k-1}\frac{1}{[m]^{r-1}},$$

then we have

$$\zeta_{p,q}(1-k) = -\frac{B_k(q)}{k}.$$

Here, we can also define $L_{p,q}$ as follows.

$$\begin{aligned} L_{p,q}(r,\chi) &= \frac{1}{r-1}\lim_{k\to\infty}\frac{1}{[fp^k]}\sum_{1\leq n\leq fp^k}^{*}\frac{\chi(n)\omega^{r-1}(n)}{[n]^{r-1}} \\ &= \frac{1}{r-1}\lim_{k\to\infty}\frac{1}{[fp^k]}\sum_{1\leq n\leq fp^k}^{*}\chi(n)\langle n\rangle^{1-r}. \end{aligned}$$

Thus we find that

$$\begin{aligned} L_{p,q}(1-k,\chi\omega^k) &= -\frac{1}{k}\lim_{m\to\infty}\frac{1}{[fp^m]}\sum_{1\le n\le fp^m}^{*}\chi(n)[n]^k \\ &= -\frac{1}{k}\int_{X^*}\chi(x)[x]^k q^{-x}d\mu_q(x) \\ &= -\frac{1}{k}(B_{k,\chi}(q)-\chi(p)[p]^{k-1}B_{k,\chi}(q^p)). \end{aligned}$$

Therefore we obtain the following

PROPOSITION 2. (q-analogue of $L_p(1-k,\chi\omega^{-k})$) *For $k \geq 1$, we have*

$$L_{p,q}(1-k,\chi\omega^k) = -\frac{1}{k}(B_{k,\chi}(q)-\chi(p)[p]^{k-1}B_{k,\chi}(q^p)).$$

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