J. Korean Math. Soc. **37** (2000), No. 1, pp. 1–19

NON-CENTRAL LIMIT THEOREM FOR NON-LINEAR VECTOR FUNCTIONS OF GAUSSIAN VECTOR PROCESSES

TAE IL JEON

ABSTRACT. We formulate a non-central limit theorem for non-linear functionals of stationary Gaussian vector processes with dependence.

1. Introduction

Let $\mathbf{X}_t = (X_t^1, X_t^2)$ be a stationary Gaussian vector process such that $EX_t^1 = EX_t^2 = 0,\ E(X_t^1)^2 = E(X_t^2)^2 = 1$.
Let

$$\mathbf{G} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

be the corresponding random spectral matrix. Suppose that G_{11} and G_{22} are absolutely continuous. Then so are G_{12} and G_{21}. Let

$$g_{\alpha\beta}(\lambda) = \frac{d}{d\lambda} G_{\alpha\beta}(\lambda),$$

where $\alpha, \beta = 1$ or 2. Then we can write the relations between variances and spectral density functions

$$r_{\alpha\beta}(t) = EX_0^\alpha X_t^\beta = \int_{-\pi}^{\pi} e^{it\lambda} g_{\alpha\beta}(\lambda) d\lambda, \tag{1}$$

where $\alpha, \beta = 1$ or 2. Assume the random spectral matrix $d\mathbf{G}$ is strictly positive definite. Let $\mathbf{Z} = (Z_1, Z_2)$ be the corresponding random vector

Received September 1, 1997. Revised July 16, 1999.
1991 Mathematics Subject Classification: 60F05.
Key words and phrases: non-central limit theorem.
This paper was supported by the Basic Science Research Institute Program, Ministry of Education, Korea, 1996, Project No. BSRI-96-1439.

measure such that

$$X_t^1 = \int_{-\pi}^{\pi} e^{it\lambda} dZ_1(\lambda), \quad X_t^2 = \int_{-\pi}^{\pi} e^{it\lambda} dZ_2(\lambda).$$

Let $T = [-\pi, \pi]$. Let $\mathcal{B}(T)$ be its Borel subsets. Assume that the $\mathbb{R}^2$-valued noise $\mathbf{Z} = (Z_1, Z_2)$ in T satisfies the following condition: For any $A \in \mathcal{B}(T)$, $\mathbf{Z}(A) = (Z_1(A), Z_2(A))$ is defined on a probability space $(\Omega, \mathcal{F}, P)$ such that for any nonintersecting sets $A_1, \dots, A_n \in \mathcal{B}(T)$, $n \geq 1$, $\mathbf{Z}(A_1), \dots, \mathbf{Z}(A_n)$ are independent and $\mathbf{Z}(A_1) + \cdots + \mathbf{Z}(A_n) = \mathbf{Z}(\cup_{k=1}^n A_k)$. Introduce the Hilbert space $L^2(T)$ of functions $\mathbf{f} : T \to \mathbb{C}^2$ such that

$$\| \mathbf{f} \| = \left[\int_T (d\mathbf{G}(t)\mathbf{f}(t), \mathbf{f}(t)) \right]^{\frac{1}{2}} < \infty.$$

Since $d\mathbf{G}$ is strictly positive definite, $L^2(T)$ is complete if we identify as usual functions which are equal a.e. with respect to the Lebesgue measure. It can be shown that n-tuple tensor product $(\otimes L^2(T))^n$ can be identified with the Hilbert space $L^2(T^n)$ consisting of all functions

$$\mathbf{f} : T^n \longrightarrow (\otimes \mathbb{C}^2)^n, \quad \mathbf{f} = (f_{i_1, \dots, i_n})_{i_1, \dots, i_n = 1,2}$$

with finite norm

$$\| \mathbf{f} \|_n = \left[\int_{T^n} \sum_{\mathbf{j}^{(n)}, \mathbf{j}'^{(n)} = 1,2} f_{\mathbf{j}^{(n)}}(\mathbf{t}^{(n)}) \overline{f_{\mathbf{j}'^{(n)}}(\mathbf{t}^{(n)})} dG_{j_1 j_1'}(t_1) \cdots dG_{j_n j_n'}(t_n) \right]^{\frac{1}{2}} < \infty,$$

where

$$\mathbf{t}^{(n)} = t_1, \dots, t_n, \qquad \mathbf{j}^{(n)} = j_1, \dots, j_n,$$
$$\mathbf{j}'^{(n)} = j_1', \dots, j_n', \qquad \mathbf{j}^{(n)} = 1,2 \text{ means each } j_k = 1,2.$$

Symmetric tensor product $[\hat{\otimes} L^2(T)]^n$ can be identified with the subspace $\widehat{L^2(T^n)} \subset L^2(T^n)$ consisting of symmetric functions: $\mathbf{f} = sym\mathbf{f}$, where, for fixed $\mathbf{j}^{(n)}$

$$(sym\mathbf{f})_{\mathbf{j}^{(n)}}(\mathbf{t}^{(n)}) = \frac{1}{n!} \sum_{\sigma} f_{j_{\sigma(1)} \cdots j_{\sigma(n)}}(t_{\sigma(1)}, \cdots, t_{\sigma(n)})$$

and the sum is taken over all permutations σ on $\{1, \dots, n\}$.

Definition 1. A simple function $\mathbf{f}(t_1, \dots, t_n)$ is called special if $\mathbf{f}$ vanishes except for the case that $t_1, \dots, t_n$ are all different. We shall denote $L_s^2(T^n)$ the set of all special functions.

We state the following theorem without proof (see [6]).

THEOREM 1. *$L^2_s(T^n)$ is a dense linear subspace in $L^2(T^n)$.*

Define the multiple stochastic integral of the function $\mathbf{f} \in L^2(T^n)$ with respect to L^2 noise $\mathbf{Z} = (Z_1, Z_2)$ in T, following Surgailis [9]. See [9] for the proof of the following.

LEMMA 1. *If $\mathbf{f} \in L^2_s(T^n)$, then there exists r.v. $I^{(n)}(\mathbf{f})$ called multiple stochastic integral of $\mathbf{f}$ with respect to $\mathbf{Z}$ such that*

$$\begin{aligned} &\text{(i)} \quad I^{(n)}(\mathbf{f}) = I^{(n)}(sym\mathbf{f}) \in L^2(\Omega) \\ &\text{(ii)} \quad E[I^{(n)}(\mathbf{f})] = 0 \\ &\text{(iii)} \quad E[I^{(n)}(\mathbf{f})\overline{I^{(k)}(\mathbf{g})}] = \delta_{nk}\frac{1}{n!}\langle sym\mathbf{f}, \mathbf{g}\rangle \end{aligned}$$

for any $k \geq 1$ and $\mathbf{g} \in L^2_s(T^k)$, where δ_{nk} is the Kronecker delta and $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(T^n)$.

For arbitrary $\mathbf{f} \in L^2(T^n)$ let $I^{(n)}(\mathbf{f}) = \lim_{k\to\infty} I^{(n)}(\mathbf{f}^{(k)})$, where $\{\mathbf{f}^{(k)}\}$ is a sequence in $L^2_s(T^n)$ which converges to $\mathbf{f}$ in $L^2(T^n)$. By (iii) such a limit exists and the limit is independent of choice of $\{\mathbf{f}^{(k)}\}$. The limit also satisfies (i) - (iii). We also denote it by

$$I^{(n)}(\mathbf{f}) = \frac{1}{n!}\int_{T^n} \sum_{j_1,\cdots,j_n=1,2} f_{j_1,\cdots,j_n}(\mathbf{t}^{(n)}) Z_{j_1}(dt_1)\cdots Z_{j_n}(dt_n).$$

For simplicity of subscripts we use the following notation:

$$\begin{aligned} \mathbf{j}_k^{(n)} &= j_1, \cdots, j_{k-1}, j_{k+1}, \cdots, j_n \\ \mathbf{t}_k^{(n)} &= t_1, \cdots, t_{k-1}, t_{k+1}, \cdots, t_n \\ \mathbf{j}_{(k=i)}^{(n)} &= j_1, \cdots, j_{k-1}, i, j_{k+1}, \cdots, j_n. \end{aligned}$$

Define two functions generated from $\mathbf{f}(\mathbf{t}^{(n)}) = (f_{\mathbf{j}^{(n)}}(\mathbf{t}^{(n)}))_{\mathbf{j}^{(n)}=1,2} \in L^2(T^n)$ and $\mathbf{g}(t) = (g_1(t), g_2(t)) \in L^2(T)$:

$$(\mathbf{f} \times_{(k)} \mathbf{g})_{\mathbf{j}_k^{(n)}}(\mathbf{t}_k^{(n)}) = \int_T \sum_{i,j=1,2} f_{\mathbf{j}_{(k=i)}^{(n)}}(\mathbf{t}^{(n)}) g_j(t_k) dG_{ij}(t_k),$$

and

$$(\mathbf{f} \otimes \mathbf{g})_{\mathbf{j}^{(n-1)}}(\mathbf{t}^{(n+1)}) = f_{\mathbf{j}^{(n)}}(\mathbf{t}^{(n)}) g_{j_{n-1}}(t_{n+1}).$$

Then $(\mathbf{f} \otimes \mathbf{g})$ is an element in $L^2(T^{n+1})$ and its norm satisfies the inequality $\| (\mathbf{f} \otimes \mathbf{g}) \|_{n+1} \leq \| \mathbf{f} \|_n \cdot \| \mathbf{g} \|$. Let $\mathbf{f}^{(1)}, \cdots \mathbf{f}^{(n)}, \mathbf{g} \in L^2(T)$. Let

$\mathbf{f} = \otimes_{\ell=1}^{n} \mathbf{f}^{(\ell)}$. Then $\mathbf{f} \in L^2(T^n)$. Moreover $(\mathbf{f} \times_{(k)} \mathbf{g})$, $k = 1, 2, \ldots, n$, are in $L^2(T^{n-1})$ and $\| (\mathbf{f} \times_{(k)} \mathbf{g}) \|_{n-1} \leq \| \mathbf{f} \|_n \cdot \| \mathbf{g} \|$.

DEFINITION 2. $\mathbf{f}^{(1)} = (f_1^{(1)}, f_2^{(1)}), \cdots, \mathbf{f}^{(m)} = (f_1^{(m)}, f_2^{(m)}) \in L^2(T)$ are said to be orthonormal if, for any $i \neq j$, $\langle \mathbf{f}^{(i)}, \mathbf{f}^{(j)} \rangle = 0$, that is,

$$\int_T \sum_{\alpha,\beta=1,2} f_\alpha^{(i)}(t) \overline{f_\beta^{(j)}(t)} dG_{\alpha\beta}(t) = 0$$

and

$$\| \mathbf{f}^{(i)} \|^2 = \langle \mathbf{f}^{(i)}, \mathbf{f}^{(i)} \rangle = 1.$$

The following theorem is called Itô's formula (see [6] for the proof). Itô's formula for the case of one dimensional process X_t is developed in [8]. In this sense we may call the following 2-dimensional version of Itô's formula.

THEOREM 2. *Suppose* $\mathbf{f}^{(1)}, \cdots, \mathbf{f}^{(m)} \in L^2(T)$ *are orthonormal in* $L^2(T)$. *Then*

$$H_{n_1}[I^{(1)}(\mathbf{f}^{(1)})] H_{n_2}[I^{(1)}(\mathbf{f}^{(2)})] \cdots H_{n_m}[I^{(1)}(\mathbf{f}^{(m)})]$$
$$= (n_1 + \cdots + n_m)! \cdot I^{(n_1 + \cdots + n_m)}[(\otimes \mathbf{f}^{(1)})^{n_1} \otimes (\otimes \mathbf{f}^{(2)})^{n_2} \otimes \cdots \otimes (\otimes \mathbf{f}^{(m)})^{n_m}],$$

where $H_n(x)$ *is the Hermite polynomial of leading coefficient* 1 *defined by*

$$H_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \left\{ \exp\left(-\frac{x^2}{2}\right) \right\}.$$

Let $H(x, y) = H_k(x) H_\ell(y), k + \ell = m$. Consider a process

$$(2) \quad Y_H^N = \frac{1}{A_N} \sum_{t=0}^{N-1} H(X_t^1, X_t^2) = \frac{1}{A_N} \sum_{t=0}^{N-1} H_k(X_t^1) H_\ell(X_t^2), N = 1, 2, \ldots$$

with an appropriate norming factor A_N. Let $\mathbf{f}_t^{(1)}(\lambda) = (e^{it\lambda}, 0)$, $\mathbf{f}_t^{(2)}(\lambda) = (0, e^{it\lambda})$. Then $\langle \mathbf{f}_t^{(1)}, \mathbf{f}_t^{(2)} \rangle = 0$ in $L^2(T)$. Applying Itô's formula we have

$$(3) \qquad H_k(I^{(1)}(\mathbf{f}_t^{(1)})) H_\ell(I^{(1)}(\mathbf{f}_t^{(2)})) = m! I^{(m)}[(\otimes \mathbf{f}_t^{(1)})^k \otimes (\otimes \mathbf{f}_t^{(2)})^\ell].$$

Since

$$[(\otimes \mathbf{f}_t^{(1)})^k \otimes (\otimes \mathbf{f}_t^{(2)})^\ell]_{\mathbf{j}^{(m)}}(\lambda_1, \dots, \lambda_k, \lambda_{k+1}, \dots, \lambda_m)$$
$$(4) \qquad = \begin{cases} 0 & \text{if } \mathbf{j}^{(m)} \neq \underbrace{1, \dots, 1}_{k}, \underbrace{2, \dots, 2}_{\ell} \\ e^{it(\lambda_1 + \cdots + \lambda_k + \lambda_{k+1} + \cdots + \lambda_{(m)})} & \text{if } \mathbf{j}^{(m)} = \underbrace{1, \dots, 1}_{k}, \underbrace{2, \dots, 2}_{\ell} \end{cases}$$

and

$$H_k(I^{(1)}(\mathbf{f}_t^{(1)}))H_\ell(I^{(1)}(\mathbf{f}_t^{(2)}))$$
$$(5) \qquad = H_k\left(\int_{-\pi}^{\pi} e^{it\lambda} Z_1(d\lambda)\right) H_\ell\left(\int_{-\pi}^{\pi} e^{it\lambda} Z_2(d\lambda)\right)$$
$$= H_k(X_t^1)H_\ell(X_t^2),$$

we can rewrite (2) using (3), (4) and (5) as

$$(6) \quad Y_H^N = \frac{1}{A_N} \sum_{t=0}^{N-1} \int_{[-\pi,\pi]^m} e^{it(\lambda_1 + \ldots + \lambda_m)} Z_1(d\lambda_1) \cdots Z_1(d\lambda_k) Z_2(d\lambda_{k+1}) \cdots Z_2(d\lambda_m).$$

PROPOSITION 1. *Assume the stationary Gaussian vector process $\mathbf{X}_t$ satisfies the conditions stated at the beginning of the section and have the correlation functions*

$$(7) \qquad r_{11}(n) \sim |n|^{-\beta_1}, \quad r_{22}(n) \sim |n|^{-\beta_2} \quad \text{as} \quad |n| \to \infty$$

and

$$(8) \qquad r_{12}(n) \sim |n|^{-\beta_3}, \quad r_{21}(n) \sim |n|^{-\beta_4} \quad \text{as} \quad |n| \to \infty,$$

where $\beta_i > 0$ for $i = 1, 2, 3$ and 4. Since $|n|^{-\beta_4} \sim r_{21}(n) = r_{12}(-n) \sim |n|^{-\beta_3}$ we may assume $|n|^{-\beta_3} \sim |n|^{-\beta_4}$. The notation $\sim$ means that the two terms are asymptotically the same. Let $\mathbf{G}$ be the random spectral *matrix corresponding to* $\mathbf{X}_t$ *and define*

$$G_{11}^N(A) = N^{\beta_1} G_{11}\left(\frac{A}{N}\right),$$

$$G_{22}^N(A) = N^{\beta_2} G_{22}\left(\frac{A}{N}\right), \qquad N = 1, 2, \ldots$$

$$G_{12}^N(A) = N^{\beta_3} G_{12}\left(\frac{A}{N}\right), \qquad A \in \mathcal{B}([-\pi, \pi])$$

$$G_{21}^N(A) = N^{\beta_4} G_{21}\left(\frac{A}{N}\right).$$

Then there exist locally finite measures $G_{11}^0, G_{22}^0, G_{12}^0$, *and* G_{21}^0 *such that*

$$\lim_{N\to\infty} G_{\alpha\beta}^N = G_{\alpha\beta}^0, \qquad \alpha, \beta = 1 \text{ or } 2,$$

in the sense of locally weak convergence.

This result follows from [1] and [4]. Let $\mathbf{G}^N$ and $\mathbf{G}^0$ be the random spectral matrices with entries $G_{\alpha\beta}^N$ and $G_{\alpha\beta}^0, \alpha, \beta = 1, 2$. Now we state the main result.

THEOREM 3. *Suppose the stationary Gaussian vector process* $\mathbf{X}_t$ *satisfies the conditions stated at the beginning of the section and the conditions in* Proposition 1. *Assume* $\beta_1 < \beta_4, \beta_2 < \beta_3$ *and* $k\beta_4 + \ell\beta_3 < 1$. *With the choice of*

$$A_N = N^{1 - \frac{k\beta_1 + \ell\beta_2}{2}},$$

the distribution of the random variables defined in (2) *tends to that of random variable* Y_H^*, *given by*

$$Y_H^* = \int K_0(\mathbf{y}) Z_1^{G^0}(dy_1) \cdots Z_1^{G^0}(dy_k) Z_2^{G^0}(dy_{k+1}) \cdots Z_2^{G^0}(dy_m),$$

where $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$. *The last formula is a multiple stochastic integral with respect to the random spectral matrix determined by the measures* $G_{\alpha\beta}^0, \alpha, \beta = 1, 2$ *and*

$$K_0(\mathbf{y}) = \frac{e^{i(y_1 + \cdots + y_m)} - 1}{i(y_1 + \cdots + y_m)}.$$

Replacing A_N in (6) by the one in Theorem 3 and using the change of variables in multiple stochastic integrals, we can see that the random variables (6) have the same joint distribution for fixed N as the following one, which we identify with them for the sake of simplicity.

$$(9) \quad Y_H^N = \int_{[-\pi N,\pi N]^m} K_N(\mathbf{y}) \; Z_1^{G^N}(dy_1)\cdots Z_1^{G^N}(dy_k) Z_2^{G^N}(dy_{k+1})\cdots Z_2^{G^N}(dy_m),$$

where $\mathbf{Z}^{G^N} = (Z_1^{G^N}, Z_2^{G^N})$ is the random measure corresponding to the random spectral matrix $\mathbf{G}^N$ and

$$(10) \qquad K_N(\mathbf{y}) = \sum_{t=0}^{N-1} \frac{1}{N} e^{i\frac{1}{N}t(y_1+\cdots+y_m)} = \frac{1}{N}\frac{e^{i(y_1+\cdots+y_m)}-1}{e^{i\frac{1}{N}(y_1+\cdots+y_m)}-1}.$$

Note that the variance of Y_H^N is different from that of [1] because it involves cross-correlation functions. The following lemmas hold true under the assumptions of Theorem 3.

LEMMA 2.

$$E|Y_H^N|^2 = k!\ell! \int_{[-\pi N,\pi N]^m} \sum_{\mathbf{j}^{(m)}\in C} \frac{1}{N^b}|K_N(\mathbf{y})|^2 \; G_{j_1 1}^N(dy_1)\cdots G_{j_k 1}^N(dy_k) G_{j_{k+1} 2}^N(dy_{k+1})\cdots G_{j_m 2}^N(dy_m),$$

where

$$b = b(\mathbf{j}^{(m)}) = \sum_{i=1}^{k} \beta_{\delta(j_i)} + \sum_{i=k+1}^{m} \beta_{\gamma(j_i)} - k\beta_1 - \ell\beta_2,$$

$$\delta(j_i) = \begin{cases} 1 & \text{if} \quad j_i = 1 \\ 4 & \text{if} \quad j_i = 2, \end{cases} \qquad \gamma(j_i) = \begin{cases} 2 & \text{if} \quad j_i = 2 \\ 3 & \text{if} \quad j_i = 1, \end{cases}$$

and

$$(11) \quad C = \{\mathbf{j}^{(m)} = j_1, \dots j_m \mid k \text{ of } j_i's \text{ are 1 and } \ell \text{ of } j_i's \text{ are 2}\}.$$

Let us introduce the following piecewise constant modification of the Fourier transform:

$$(12) \quad \psi_N(\mathbf{x}) = k!\ell! \int_{[-\pi N,\pi N]^m} \sum_{\mathbf{j}^{(m)}\in C} \frac{1}{N^b} e^{i\frac{1}{N}(\nu_1 y_1+\cdots+\nu_m y_m)}|K_N(\mathbf{y})|^2 \; G_{j_1 1}^N(dy_1)\cdots G_{j_k 1}^N(dy_k) G_{j_{k+1} 2}^N(dy_{k+1})\cdots G_{j_m 2}^N(dy_m),$$

where $\mathbf{x} = (x_1, \dots, x_m)$, $\nu_p = [x_p N]$, $p = 1, 2, \dots, m$.

LEMMA 3. *Assume $\beta_1 < \beta_4, \beta_2 < \beta_3$ and $k\beta_4 < 1, \ell\beta_3 < 1$. For fixed $\mathbf{j}^{(m)} \in C$ let*

$$h_N(\mathbf{x}) = \int_{[-\pi N,\pi N]^m} e^{i\frac{1}{N}(\nu_1 y_1+\cdots+\nu_m y_m)}|K_N(\mathbf{y})|^2 G^N_{j_1 1}(dy_1)\cdots G^N_{j_k 1}(dy_k)G^N_{j_{k+1}2}(dy_{k+1})\cdots G^N_{j_m 2}(dy_m),$$

where $\nu_p = [x_p N]$. Then $h_N(\mathbf{x}) \to h(\mathbf{x})$ uniformly on every bounded region, where

$$h(\mathbf{x}) = \int_{[-1,1]} (1-|z|)\frac{1}{|x_1+z|^{\beta_{\delta(j_1)}}}\cdots\frac{1}{|x_k+z|^{\beta_{\delta(j_k)}}} \frac{1}{|x_{k+1}+z|^{\beta_{\gamma(j_{k+1})}}}\cdots\frac{1}{|x_m+z|^{\beta_{\gamma(j_m)}}}dz.$$

LEMMA 4. $\lim_{N\to\infty}\psi_N(\mathbf{x}) = g(\mathbf{x})$ *uniformly in every bounded region, where*

$$g(\mathbf{x}) = k!\ell!\int_{[-1,1]}(1-|z|) \frac{1}{|x_1+z|^{\beta_1}}\cdots\frac{1}{|x_k+z|^{\beta_1}}\frac{1}{|x_{k+1}+z|^{\beta_2}}\cdots\frac{1}{|x_m+z|^{\beta_2}}dz.$$

LEMMA 5. *Let $\mu_1, \mu_2, \cdots$ be a sequence of finite measures on $\mathbb{R}^m$ such that*

$$\mu_N(\mathbb{R}^m - [-\pi N, \pi N]^m) = 0.$$

Let

$$\phi_N(\mathbf{x}) = \int_{\mathbb{R}^m} e^{i\frac{1}{N}(\nu_1 y_1+\cdots+\nu_m y_m)}\mu_N(d\mathbf{y}),$$

where $\nu_p = [x_p N]$, $p = 1,\dots,m$. If, for every $\mathbf{x} = (x_1,\dots,x_m) \in \mathbb{R}^m$, the sequence $\phi_N(\mathbf{x})$ tends to $\phi(\mathbf{x})$, which is continuous at the origin, then μ_N converges weakly to a finite measure μ_0, where $\phi(\mathbf{x})$ is the Fourier transform of μ_0.

LEMMA 6. *Suppose $G^N_{\alpha\beta}$ converges locally weakly to $G^0_{\alpha\beta}$ for each $\alpha, \beta = 1, 2$ and $K_N(\mathbf{y})$ converges to a continuous function $K_0(\mathbf{y})$ uniformly in any $[-A, A]^m$. Moreover, let the functions K_N satisfy the relation*

$$(13)\quad \lim_{A\to\infty} \int_{\mathbb{R}^m - [-A,A]^m} \sum_{\mathbf{j}^{(m)}\in C} \frac{1}{N^b} |K_N(\mathbf{y})|^2 G^N_{j_1 1}(dy_1) \cdots G^N_{j_m 2}(dy_m) = 0$$

uniformly for $N = 1, 2, \dots$, and K_0 satisfies the relation

$$(14)\quad \lim_{A\to\infty} \int_{\mathbb{R}^m - [-A,A]^m} \sum_{\mathbf{j}^{(m)}\in C} |K_0(\mathbf{y})|^2 G^0_{j_1 1}(dy_1) \cdots G^0_{j_m 2}(dy_m) = 0.$$

Then the multiple stochastic integral

$$(15)\qquad \int K_0(\mathbf{y}) Z_1^{G^0}(dy_1) \cdots Z_1^{G^0}(dy_k) Z_2^{G^0}(dy_{k+1}) \cdots Z_2^{G^0}(dy_m)$$

exists and the sequence of multiple stochastic integrals

$$\int_{[-\pi N, \pi N]^m} K_N(\mathbf{y}) Z_1^{G^N}(dy_1) \cdots Z_1^{G^N}(dy_k) Z_2^{G^N}(dy_{k+1}) \cdots Z_2^{G^N}(dy_m)$$

tends in distribution to the integral (15) *as $N \to \infty$.*

We will apply Lemma 6 to show the distribution of the random variable Y_H^N tends to that of the random variable Y_H^*. We have to check the validity of the conditions of Lemma 6. The convergences of $G^N_{\alpha\beta} \to G^0_{\alpha\beta}$ is stated in Proposition 1. The uniform convergence of $K_N \to K_0$ in $[-A, A]^m$ is easy to show. Let

$$\mu_N(A) = \int_A \sum_{\mathbf{j}^{(m)}\in C} \frac{1}{N^b} |K_N(\mathbf{y})|^2 \\ G^N_{j_1 1}(dy_1) \cdots G^N_{j_k 1}(dy_k) G^N_{j_{k+1} 2}(dy_{k+1}) \cdots G^N_{j_m 2}(dy_m),$$

where $N = 1, 2, \dots$ and

$$\mu_0(A) = \int_A \sum_{\mathbf{j}^{(m)}\in C} |K_0(\mathbf{y})|^2 \\ G^0_{j_1 1}(dy_1) \cdots G^0_{j_k 1}(dy_k) G^0_{j_{k+1} 2}(dy_{k+1}) \cdots G^0_{j_m 2}(dy_m),$$

where $A \in \mathcal{B}(\mathbb{R}^m)$. The measures $\mu_1, \mu_2, \ldots$ are finite and concentrated on the rectangle $[-\pi N, \pi N]^m$. Then we have

$$\psi_N(\mathbf{x}) = k!\ell! \int_{\mathbb{R}^m} e^{i\frac{1}{N}(\nu_1 y_1 + \cdots + \nu_m y_m)} \mu_N(d\mathbf{y}),$$

and therefore $\psi_N \to g$ by Lemma 4. Since g is continuous at the origin, $\mu_N \to \mu_0$ weakly by Lemma 5, where g is the Fourier transform of μ_0. Weakly convergence of $\mu_N \to \mu_0$ is equivalent to $\mu_N \to \mu_0$ locally weakly and $\lim_{A\to\infty} \sup_A \mu_N(|\mathbf{x}| > A) = 0$. Therefore condition (13) of Lemma 6 is satisfied. Thus Lemma 6 implies Theorem 3.

2. Proof of the Lemmas

Proof of Lemma 2. The integrations in the rest of this paper are over the range $[-\pi N, \pi N]^m$ unless we specify it. By (9)

$$E|Y_H^N|^2 = E\left|\int_{[-\pi N,\pi N]^m} K_N(\mathbf{y}) Z_1^{G_N}(dy_1)\cdots Z_1^{G_N}(dy_k) \right.$$
$$\left. Z_2^{G_N}(dy_{k+1})\cdots Z_2^{G_N}(dy_m)\right|^2.$$

Note that

$$K_N(\mathbf{y}) = \left(\frac{1}{N}\sum_{t=0}^{N-1}[(\otimes \mathbf{f}_t^{(1)})^k \otimes (\otimes \mathbf{f}_t^{(2)})^\ell]\right)_{1,\ldots,1,2,\ldots,2}(\mathbf{y}) \tag{16}$$

with $\mathbf{f}_t^{(1)}(y) = (e^{i\frac{1}{N}ty}, 0)$ and $\mathbf{f}_t^{(2)}(y) = (0, e^{i\frac{1}{N}ty})$ in $L^2([-\pi N, \pi N])$. The subscript in (16) is $\underbrace{1,\ldots,1}_{k},\underbrace{2,\ldots,2}_{\ell}$. For simplicity, let

$$(\mathbf{F}_N)_{\mathbf{j}^{(m)}}(\mathbf{y}) = \begin{cases} K_N(\mathbf{y}) & \text{if } \mathbf{j}^{(m)} = \underbrace{1,\ldots,1}_{k},\underbrace{2,\ldots,2}_{\ell}. \\ 0 & \text{otherwise.} \end{cases}$$

To avoid complicated subscript we use $\mathbf{F}$ instead of $\mathbf{F}_N$ for the time being. If we need the precise notation in any context we will specify it. By (iii) in Lemma 1,

$$E|Y_H^N|^2 = (m!)^2 E[I^{(m)}(\mathbf{F})\overline{I^{(m)}(\mathbf{F})}] = m!\langle sym\mathbf{F}, \mathbf{F}\rangle_{L^2([-\pi N,\pi N]^m)}.$$

Now compute $sym\mathbf{F}$. For fixed $\mathbf{j}^{(m)}$, by the definition of $sym\mathbf{F}$,

$$(sym\mathbf{F})_{\mathbf{j}^{(m)}}(\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{j}^{(m)} \notin C \\ \dfrac{1}{m!}\sum_{\sigma} \mathbf{F}_{j_{\sigma(1)}\cdots j_{\sigma(m)}}(y_{\sigma(1)}, \cdots, y_{\sigma(m)}) & \text{if } \mathbf{j}^{(m)} \in C, \end{cases}$$

where C is the set in (11) and σ is a permutation on $\{1, 2, \dots, m\}$. Examining $\mathbf{F}$, for $\mathbf{j}^{(m)} \in C$, we have

$$(sym\mathbf{F})_{\mathbf{j}^{(m)}}(\mathbf{y}) = \frac{k!\ell!}{m!}\mathbf{F}_{1,\dots,1,2,\dots,2}(\mathbf{y}) = \frac{k!\ell!}{m!}K_N(\mathbf{y}). \tag{17}$$

Thus

$$E|Y_H^N|^2 = m! \int_{[-\pi N,\pi N]^m} \sum_{\mathbf{j}^{(m)},\mathbf{j}'^{(m)}=1,2} \frac{1}{N^\tau}(sym\mathbf{F})_{\mathbf{j}^{(m)}}(\mathbf{y})\overline{\mathbf{F}_{\mathbf{j}'^{(m)}}}(\mathbf{y})$$
$$G^N_{j_1 j'_1}(dy_1)\cdots G^N_{j_m j'_m}(dy_m),$$

where $\tau = \tau(\mathbf{j}^{(m)}, \mathbf{j}'^{(m)}) = \sum_{i=1}^{m} \rho(j_i, j'_i) - k\beta_1 - \ell\beta_2$,

$$\rho(j_i, j'_i) = \begin{cases} \beta_1 & \text{if } j_i = j'_i = 1 \\ \beta_2 & \text{if } j_i = j'_i = 2 \\ \beta_3 & \text{if } j_i = 1 \text{ and } j'_i = 2 \\ \beta_4 & if\ j_i = 2 \text{ and } j'_i = 1. \end{cases}$$

Using (17) we have

$$E|Y_H^N|^2 = k!\ell! \int_{[-\pi N,\pi N]^m} \sum_{\mathbf{j}^{(m)}\in C} \frac{1}{N^b}|K_N(\mathbf{y})|^2$$
$$G^N_{j_1 1}(dy_1)\cdots G^N_{j_k 1}(dy_k)G^N_{j_{k+1} 2}(dy_{k+1})\cdots G^N_{j_m 2}(dy_m).$$

This completes the proof of Lemma 2.

Proofs of Lemmas 3, 4 *and* 5. First note that

$$|K_N(\mathbf{y})|^2 = \frac{1}{N^2}\sum_{u=-N+1}^{N-1}(N-|u|)e^{i\frac{1}{N}u(y_1+\cdots+y_m)}.$$

Therefore

$$\begin{aligned} h_N(\mathbf{x}) &= \int_{[-\pi N,\pi N]^m} \frac{1}{N^2} \sum_{u=-N+1}^{N-1} (N-|u|) e^{i\frac{1}{N}\{(u+\nu_1)y_1+\cdots(u+\nu_m)y_m\}} \\ &\qquad G^N_{j_1 1}(dy_1)\cdots G^N_{j_k 1}(dy_k) G^N_{j_{k+1}2}(dy_{k+1})\cdots G^N_{j_m 2}(dy_m) \\ &= \int_{[-\pi,\pi]^m} \frac{1}{N} \sum_{u=-N+1}^{N-1} (1-\frac{|u|}{N}) e^{i\prod_{q=1}^m (u+\nu_q)\lambda_q} N^{\sum_{p=1}^k \beta_{\delta(j_p)}+\sum_{p=k+1}^m \beta_{\gamma(j_p)}} \\ &\qquad G_{j_1 1}(d\lambda_1)\cdots G_{j_k 1}(d\lambda_k) G_{j_{k+1}2}(d\lambda_{k+1})\cdots G_{j_m 2}(d\lambda_m) \\ &= \frac{1}{N} \sum_{-N+1}^{N-1} (1-\frac{|u|}{N}) N^{\sum_{p=1}^k \beta_{\delta(j_p)}+\sum_{p=k+1}^m \beta_{\gamma(j_p)}} \\ &\qquad \prod_{p=1}^k \int_{-\pi}^{\pi} e^{i(u+\nu_p)} G_{j_p 1}(d\lambda_p) \prod_{p=k+1}^m \int_{-\pi}^{\pi} e^{i(u+\nu_p)} G_{j_p 2}(d\lambda_p) \end{aligned}$$

By (1) and assumptions of Proposition 1

$$\begin{aligned} h_N(\mathbf{x}) &\sim \frac{1}{N} \sum_{-N+1}^{N-1} (1-\frac{|u|}{N}) N^{\sum_{p=1}^k \beta_{\delta(j_p)}+\sum_{p=k+1}^m \beta_{\gamma(j_p)}} \\ &\qquad \cdot \prod_{p=1}^k |u+\nu_p|^{-\beta_{\delta(j_p)}} \prod_{p=k+1}^m |u+\nu_p|^{-\beta_{\gamma(j_p)}} \\ &= \frac{1}{N} \sum_{-N+1}^{N-1} (1-\frac{|u|}{N}) \prod_{p=1}^k [\frac{|u+\nu_p|}{N}]^{-\beta_{\delta(j_p)}} \prod_{p=k+1}^m [\frac{|u+\nu_p|}{N}]^{-\beta_{\gamma(j_p)}}. \end{aligned}$$

Let

$$\begin{aligned} &f_N(x_1,\dots,x_m,z) \\ &\qquad = (1-\frac{|[zN]|}{N}) \prod_{p=1}^k \frac{r_{j_p 1}([zN]+\nu_p)}{N^{-\beta_{\delta(j_p)}}} \prod_{p=k+1}^m \frac{r_{j_p 2}([zN]+\nu_p)}{N^{-\beta_{\gamma(j_p)}}} \end{aligned}$$

and

$$f(x_1,\dots,x_m,z) = (1-|z|) \prod_{p=1}^k \frac{1}{|x_p+z|^{\beta_{\delta(j_p)}}} \prod_{p=k+1}^m \frac{1}{|x_p+z|^{\beta_{\gamma(j_p)}}},$$

where $z \in [-1,1]$. We can show that

$$h_N(x_1,\dots,x_m)=\int_{-1}^{1} f_N(x_1,\dots,x_m,z)dz.$$

Let

$$A_\varepsilon(\mathbf{x})=\{z\in[-1,1] \mid |x_p+z|<\varepsilon \text{ for some } p=1,\dots,m\},$$

and

$$A_\varepsilon(x_i)=\{z\in[-1,1] \mid |x_i+z|<\varepsilon\}.$$

Then

$$\begin{aligned}
|h_N-h| &= \left|\int_{-1}^{1}(f_N-f)dz\right| \le \int_{-1}^{1}|f_N-f|dz\\
&\le \int_{[-1,1]-A_\varepsilon(\mathbf{x})}|f_N-f|dz+\int_{A_\varepsilon(\mathbf{x})}|f_N|dz+\int_{A_\varepsilon(\mathbf{x})}|f|dz\\
&\le \int_{[-1,1]-A_\varepsilon(\mathbf{x})}|f_N-f|dz+\sum_{i=1}^{m}\int_{A_\varepsilon(x_i)}|f_N|dz+\sum_{i=1}^{m}\int_{A_\varepsilon(x_i)}|f|dz
\end{aligned}$$

Let $|x_p|\le K,\ \ p=1,\dots,m$, and $\varepsilon>0$. Then it is not hard to show

$$\lim_{N\to\infty}\sup_{\substack{|x_p|\le K,\\ p=1,\dots,m}}\int_{[-1,1]-A_\varepsilon(\mathbf{x})}|f_N(\mathbf{x},z)-f(\mathbf{x},z)|dz=0.$$

In order to complete the proof of Lemma 3 it is sufficient to show that

$$\int_{A_\varepsilon(x_q)}|f_N(\mathbf{x},z)|dz \ < \ M(\varepsilon) \tag{18}$$

and

$$\int_{A_\varepsilon(x_q)}|f(\mathbf{x},z)|dz \ < \ M(\varepsilon), \tag{19}$$

for every $q=1,\dots,m$, if $|x_p|<K, p=1,\dots,m$, where $M(\varepsilon)\to 0$ as $\varepsilon\to 0$. Let us show (19) first. By *Hölder's* inequality

$$\int_{A_\varepsilon(x_q)}|f(\mathbf{x},z)|dz \le M_0\prod_{p=1}^{k}\left\{\int_{-x_p-\varepsilon}^{-x_p+\varepsilon}\frac{1}{|x_p+z|^{k\beta_{\delta(j_p)}}}dz\right\}^{\frac{1}{k}}
\prod_{p=k+1}^{m}\left\{\int_{-x_p-\varepsilon}^{-x_p+\varepsilon}\frac{1}{|x_p+z|^{\ell\beta_{\gamma(j_p)}}}dz\right\}^{\frac{1}{\ell}}.$$

For a fixed p

$$\int_{-x_p-\varepsilon}^{-x_p+\varepsilon} \frac{1}{|x_p+z|^{k\beta_{\delta(j_p)}}}dz \leq K_p\varepsilon^{1-k\beta_{\delta(j_p)}},$$

$$\int_{-x_p-\varepsilon}^{-x_p+\varepsilon} \frac{1}{|x_p+z|^{k\beta_{\gamma(j_p)}}}dz \leq K_p\varepsilon^{1-k\beta_{\gamma(j_p)}}.$$

Therefore

$$\begin{aligned}\int_{A_\varepsilon(x_q)} |f(\mathbf{x},z)|dz &\leq M_0 \prod_{p=1}^{k} \{M_p\varepsilon^{1-k\beta_{\delta(j_p)}}\}^{\frac{1}{k}} \prod_{p=k+1}^{m} \{M_p\varepsilon^{1-\ell\beta_{\delta(j_p)}}\}^{\frac{1}{\ell}} \\ &\leq M\varepsilon^{2-\sum_{p=1}^{k}\beta_{\delta(j_p)}-\sum_{p=k+1}^{m}\beta_{\gamma(j_p)}} \\ &\leq M\varepsilon^{2-k\beta_4-\ell\beta_3}.\end{aligned}$$

This completes the proof of (19). Therefore, for large enough N,

$$\int_{A_\varepsilon(x_p)} |f_N(\mathbf{x},z)|dz$$

$$\sim \int_{A_\varepsilon(x_p)} (1-|z|) \prod_{p=1}^{k} \frac{1}{|z+x_p|^{\beta_{\delta(j_p)}}} \prod_{p=k+1}^{m} \frac{1}{|z+x_p|^{\beta_{\gamma(j_p)}}} dz.$$

An estimation similar to the argument above implies (18) and it completes the proof of Lemma 3.

The following argument and Lemma 3 imply Lemma 4. Examining (12) we can interchange the order of sum and integration because each term has finite integral. Therefore

$$\psi_N(\mathbf{x}) = k!\ell! \sum_{\mathbf{j}^{(m)}\in C} \frac{1}{N^b} \int_{[-\pi N,\pi N]^m} e^{i\frac{1}{N}(\nu_1 y_1+\cdots+\nu_m y_m)} |K_N(\mathbf{y})|^2$$
$$G^N_{j_1 1}(dy_1)\cdots G^N_{j_k 1}(dy_k) G^N_{j_{k+1} 2}(dy_{k+1})\cdots G^N_{j_m 2}(dy_m).$$

For $\mathbf{j}^{(m)} \in C$ whose $b = b(\mathbf{j}^{(m)})$ is positive, the term converges to 0 in any bounded region by Lemma 3. The only term which converges to nonzero is the one with $\mathbf{j}^{(m)} = \underbrace{1,\ldots,1}_{k}\underbrace{2,\ldots,2}_{\ell}$. Indeed this subscript yields $b(1,\ldots,1,2,\ldots,2) = 0$. Therefore the proof of Lemma 4 is completed. Lemma 5 is an analogue of the theorem about the equivalence of the weak convergence of measures and the convergence of their Fourier transforms. See [1] for the proof.

Proof of Lemma 6. If $\mathbf{h} \in L_s^2([-\pi N, \pi N]^m)$, then

$$\lim_{N\to\infty} I^{(m)}_{\mathbf{Z}^{G^N}}(\mathbf{h}) = I^{(m)}_{\mathbf{Z}^{G^0}}(\mathbf{h}), \tag{20}$$

in the sense of convergence in distribution, where $\mathbf{Z}^{G^N} = (Z_1^{G^N}, Z_2^{G^N})$ is the random measure corresponding to the random spectral matrix $\mathbf{G}^N$ and $\mathbf{Z}^{G^0} = (Z_1^{G^0}, Z_2^{G^0})$ is the random measure corresponding to the random spectral matrix $\mathbf{G}^0$. Indeed the multiple stochastic integral of (20) is a sum of terms like the constant times of the following forms:

$$Z_1^{G^N}(\Delta_1)\cdots Z_1^{G^N}(\Delta_k) Z_2^{G^N}(\Delta_{k+1})\cdots Z_2^{G^N}(\Delta_m), \tag{21}$$

where Δ_i is in a regular system of rectangles $\Delta(M)$, for some positive integer M. Since the joint distribution of the variables in (21) tends to the joint distribution of the variables

$$Z_1^{G^0}(\Delta_1)\cdots Z_1^{G^0}(\Delta_k) Z_2^{G^0}(\Delta_{k+1})\cdots Z_2^{G^0}(\Delta_m),$$

(20) is true. Let $\varepsilon > 0$. Then by (13), there exists A_0 such that

$$\int_{\mathbb{R}^m - [-A_0, A_0]^m} \sum_{\mathbf{j}^{(m)}\in C} \frac{1}{N^b} |K_N(\mathbf{y})|^2 G^N_{j_1 1}(dy_1)\cdots G^N_{j_m 2}(dy_m) < \varepsilon$$

uniformly for $N = 1, 2, \ldots$. Therefore, for sufficiently large $N > A_0/\pi$, we have

$$\int_{\mathbb{R}^m} \sum_{\mathbf{j}^{(m)}\in C} \frac{1}{N^b} |K_N(\mathbf{y})|^2 G^N_{j_1 1}(dy_1)\cdots G^N_{j_m 2}(dy_m)$$
$$\leq \int_{[-\pi N, \pi N]^m} \sum_{\mathbf{j}^{(m)}\in C} \frac{1}{N^b} |K_N(\mathbf{y})|^2 G^N_{j_1 1}(dy_1)\cdots G^N_{j_m 2}(dy_m) + \varepsilon.$$

Since K_N converges to K_0 uniformly on any bounded region, we have

$$\int_{[-\pi N, \pi N]^m} \sum_{\mathbf{j}^{(m)}\in C} \frac{1}{N^b} |K_N(\mathbf{y}) - K_0(\mathbf{y})|^2 G^N_{j_1 1}(dy_1)\cdots G^N_{j_m 2}(dy_m) < \varepsilon$$

for sufficiently large N, which implies the following

$$E\Big| \int_{[-\pi N, \pi N]^m} (K_N(\mathbf{y}) - K_0(\mathbf{y})) \\ Z_1^{G^N}(dy_1)\cdots Z_1^{G^N}(dy_k) Z_2^{G^N}(dy_{k+1})\cdots Z_2^{G^N}(dy_m)\Big|^2 < \varepsilon \tag{22}$$

for large N. Since K_0 can be approximated by $\mathbf{h} \in L_s^2[-\pi N, \pi N]^m$, by (20), we have the following

(23)

$$E\Big|\int_{[-\pi N,\pi N]^m} K_0(\mathbf{y}) Z_1^{G^N}(dy_1)\cdots Z_1^{G^N}(dy_k) Z_2^{G^N}(dy_{k+1})\cdots Z_2^{G^N}(dy_m)$$
$$-\int_{[-\pi N,\pi N]^m} K_0(\mathbf{y}) Z_1^{G^0}(dy_1)\cdots Z_1^{G^0}(dy_k) Z_2^{G^0}(dy_{k+1})\cdots Z_2^{G^0}(dy_m)\Big|^2 < \varepsilon$$

for large N. By (14) we can show that, for large N,

$$(24)\quad E\Big|\int_{\mathbb{R}^m-[-\pi N,\pi N]^m} K_0(\mathbf{y})\, Z_1^{G^0}(dy_1)\cdots Z_1^{G^0}(dy_k) Z_2^{G^0}(dy_{k+1})\cdots Z_2^{G^0}(dy_m)\Big|^2 < \varepsilon.$$

Therefore, by (23) and (24), we have

$$(25)\quad \int_{[-\pi N,\pi N]^m} K_0(\mathbf{y}) Z_1^{G^N}(dy_1)\cdots Z_1^{G^N}(dy_k) Z_2^{G^N}(dy_{k+1})\cdots Z_2^{G^N}(dy_m)$$
$$\xrightarrow{d} \int K_0(\mathbf{y}) Z_1^{G^0}(dy_1)\cdots Z_1^{G^0}(dy_k) Z_2^{G^0}(dy_{k+1})\cdots Z_2^{G^0}(dy_m)$$

Consequently (22) and (25) complete the proof of Lemma 6.

Suppose that

$$H(x,y) = \sum_{k+\ell=m} c_{(k,\ell)} H_k(x) H_\ell(y).$$

Then we have

$$(26)\qquad Y_H^N = \frac{1}{A_N}\sum_{t=0}^{N-1}\sum_{k+\ell=m} c_{(k,\ell)} H_k(X_t^1) H_\ell(X_t^2).$$

Using similar technique we can have

$$Y_H^N = \sum_{k+\ell=m} c_{(k,\ell)} \int_{[-\pi N,\pi N]^m} K_N(\mathbf{y})\, Z_1^{G^N}(dy_1)\cdots Z_1^{G^N}(dy_k) Z_2^{G^N}(dy_{k+1})\cdots Z_2^{G^N}(dy_m),$$

where $K_N(\mathbf{y})$ is the same to (10). Considering (26) we can formulate the more general result. The norming constant A_N should be uniform for all terms in (26).

THEOREM 4. *Suppose the stationary Gaussian vector process* $\mathbf{X}_t$ *satisfies the conditions on section* 1 *and the conditions in* Proposition 1. *Assume* $\beta_1 < \beta_4, \beta_2 < \beta_3$ *and* $\xi_m = \min\{k\beta_1 + \ell\beta_2 | k + \ell = m\} < 1$. *With the choice of*

$$A_N = N^{1-\frac{\xi_m}{2}}$$

the distribution of the random variables defined in (26) *tends to that of the random variable* Y_H^*, *given by the formula*

$$Y_H^* = \sum_{(k,\ell)\in D_m} c_{(k,\ell)} \int K_0(\mathbf{y})$$
$$Z_1^{G^0}(dy_1)\cdots Z_1^{G^0}(dy_k) Z_2^{G^0}(dy_{k+1})\cdots Z_2^{G^0}(dy_m),$$

where $D_m = \{(k,\ell) | k+\ell = m, k\beta_1 + \ell\beta_2 = \xi_m\}$ *and* $\mathbf{y} = (y_1, \cdots, y_m) \in \mathbf{R}^m$. *The last formula is a multiple stochastic integral with respect to the random spectral matrix determined by the measures* $G^0_{\alpha\beta}, \alpha\beta = 1,2$ *and*

$$K_0(\mathbf{y}) = \frac{e^{i(y_1+\cdots+y_m)} - 1}{i(y_1 + \cdots + y_m)}.$$

Since the proof of Theorem 4 is basically the same to that of Theorem 3 we will skip it. Consider the case of general $H(x,y)$ which has the Hermite expansion $H(x,y) = \sum_{j=m}^{\infty} \sum_{|\mathbf{m}|=j} c_{\mathbf{m}} H_{m_1}(x) H_{m_2}(y)$ with

$$\sum_{j=m}^{\infty} \left(\sum_{|\mathbf{m}|=j} |c_{\mathbf{m}}| \right)^2 j! < \infty, \tag{27}$$

where, $\mathbf{m} = (m_1, m_2)$ and $|\mathbf{m}| = m_1 + m_2$. Define

$$Z_H^N = \sum_{t=0}^{N-1} \sum_{j=m+1}^{\infty} \sum_{|\mathbf{m}|=j} c_{\mathbf{m}} H_{m_1}(X_t^1) H_{m_2}(X_t^2) \qquad N = 1, 2, \ldots .$$

Then we have (see [7] for the detail)

$$E(Z_H^N)^2$$
$$= \sum_{j=m+1}^{\infty} \sum_{|\mathbf{m}|=|\mathbf{l}|=j} c_{\mathbf{m}} c_{\mathbf{l}} \sum_{s=max(0,m_1-l_2)}^{min(m_1,l_1)} \left[\frac{m_1! m_2! l_1! l_2!}{s!(l_1-s)!(m_1-s)!(l_2-m_1+s)!} \right]$$
$$\cdot \left[\frac{1}{N} \sum_{t_1=0}^{N-1} \sum_{t_2=0}^{N-1} r_{11}^{s}(t_1-t_2) r_{12}^{m_1-s}(t_1-t_2) r_{22}^{l_2-m_1+s}(t_1-t_2) r_{21}^{l_1-s}(t_1-t_2) \right].$$

It is easy to check with the help of (27), (7) and (8) that

$$E(Z_H^N)^2 = O(N^{2-\xi_{m+1}}) + O(N) \quad as \quad N \to \infty.$$

Therefore $A_N^{-1} Z_H^N \to 0$ in probability as $N \to \infty$ and this implies that $H(x,y)$ can be replaced with $\sum_{|\mathbf{m}|=m} c_{\mathbf{m}} H_{m_1}(x) H_{m_2}(y)$ in Theorem 4.

References

[1] Dobrushin, R. L. and Major, P., *Non-central limit theorems for non-linear functionals of Gaussian fields*, Z. Wahrsch. Verw. Gebiete **50** (1979), 27-52.

[2] Fox, R., and Taqqu, M., *Multiple stochastic integrals with dependent integrators*, J. Multivariate Analysis **21** (1987), 105-127.

[3] Giraitis, L. and Surgailis, D., *CLT and other limit theorems for functionals of Gaussian processes*, Z. Wahrsch. Verw. Gebiete **70** (1985), 191-212.

[4] Ho, H. C., and Sun, T. C., *Limiting distributions of non-linear vector functions of stationary Gaussian process*, Ann. Probability **18** (1990), No. 3, 1159-1173.

[5] Hsiao, C. T., *Central limit theorem for stationary random vectors*, unpublished note (1983).

[6] Jeon, T. I., *L^2 multiple stochastic integral and Itô formula*, J. Chungcheong Math. Soc. **6** (1989), 71-87.

[7] Jeon, T. I. and Sun, T. C., *A Central limit theorem for non-linear vector Gaussian processes*, Proc. 1990 Taipei Symposium in Stat. Inst. of Stat. Sci., Academia Sinica (1990).

[8] Major, P., *Multiple Wiener-Itô Integrals*, Lecture notes in Math. 849, Springer-Verlag, New York/Berlin (1981).

[9] Surgailis, D., *On L^2 and non-L^2 multiple stochastic integration.* In Stochastic Differential Systems. Lecture Notes in Control and Information Sciences, Vol. 36, Springer-Verlag, New York, 1981, pp. 212-226.

[10] Taqqu, M., *Laws of the iterated logarithm for sums of non-linear functions of Gaussian variables that exhibit a long-range dependence*, Z. Wahrsch. Verw. Gebiete **40** (1977), 103-238.

[11] Zygmund, A., *Trigonometric Series*, Cambridge: Cambridge Univ. Press, 1959.

Department of Mathematics
Taejon University
Taejon, 300-716, Korea
E-mail: tijeon@dragon.taejon.ac.kr