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CONDITION FOR SMOOTHNESS OF HIGH ACCURACY WAVELET BASIS

Soon-Geol Kwon

Abstract. High accuracy wavelet basis β is constructed in [2]. We derive a condition for smoothness of the basis function $\beta(x)$.

1. Introduction

Let ϕ be the scaling function of an orthogonal multiresolution approximation ([1]). The wavelet approximation of a function in a Hilbert space $\mathcal{H}$ onto the subspace V_k at the resolution $h = 2^{-k}$ is the projection

$$\mathcal{P}_k f(x) = \sum_{j=-\infty}^{\infty} \langle f, \phi_j^k \rangle \phi_j^k(x), \tag{1}$$

where V_k is spanned by

$$\phi_j^k(x) = 2^{k/2}\phi(2^k x - j), \qquad \text{for } j \in \mathbf{Z}. \tag{2}$$

The accuracy of the wavelet approximation at the resolution $h = 2^{-k}$ is

$$||f(x) - \mathcal{P}_k f(x)|| = \mathcal{O}(h^M),$$

where M is the vanishing moments of the wavelet ψ ([4]). In some applications, such as wavelet-Galerkin methods, we may know the wavelet coefficients of a solution function to high accuracy.

We would like to improve the accuracy of the approximation by constructing new basis functions while keeping the wavelet coefficients. For

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any given scaling function ϕ and accuracy n, we construct basis function β with compact support so that for smooth function f,

$$||f(x) - \sum_{j=-\infty}^{\infty} f_j^k \beta_j^k(x)|| = \mathcal{O}(h^n), \tag{3}$$

where β_j^k is defined as in (2) and

$$f_j^k := \langle f, \phi_j^k \rangle.$$

The basis function β recovers point values of f for comparable accuracy. In this paper, we derive a condition for smoothness of the basis function β.

The *continuous moments* of ϕ is defined by

$$\mathcal{M}_i = \int_{-\infty}^{\infty} x^i \phi(x)\, dx.$$

We define the *shifted continuous moments* of ϕ by

$$\mathcal{M}_{i,j} = \int_{-\infty}^{\infty} x^i \phi(x-j)\, dx = \int_{-\infty}^{\infty} (x+j)^i \phi(x)\, dx. \tag{4}$$

Let $C^p(\mathbf{R})$ be the p times continuously differentiable functions on $\mathbf{R}$.

2. Construction of High Accuracy Wavelet Basis

In this section we review construction of high accuracy wavelet basis ([2]). Assume that for some function f and some orthogonal wavelet basis we know the projection $\mathcal{P}_k f$ onto V_k

$$\mathcal{P}_k f(x) = \sum_{j=-\infty}^{\infty} f_j^k \phi_j^k(x). \tag{5}$$

How well can we recover the point values of f from this?

Assume that wavelet ψ has M vanishing moments. If $f \in C^M$, then for any point x

$$||f(x) - \mathcal{P}_k f(x)|| = \mathcal{O}(h^M),$$

where $h = 2^{-k}$.

There are more detailed results available about the convergence rate of $\mathcal{P}_k f$ to f under various conditions of f and ϕ (see for example [3, 5]), but they are all based on the original wavelet series (5).

We propose instead to use a different series

$$B_k f(x) = \sum_{j=-\infty}^{\infty} f_j^k \beta_j^k(x), \tag{6}$$

where

$$\beta_j^k(x) = 2^{k/2}\beta(2^k x - j) \tag{7}$$

in analogy to (2). For any choice of ϕ and for any $n \in \mathbf{N}$, we will construct a basis function $\beta(x)$ with compact support so that for $f \in C^n$,

$$||f(x) - B_k f(x)|| = \mathcal{O}(h^n). \tag{8}$$

For simplicity, the dependence of β on n and ϕ is not usually expressed in the notation. If necessary, we will denote the basis function for a particular n as $\beta(x; n)$, and similarly for the dependence on other parameters.

To construct β, fix a scaling function ϕ, a level number $k \in \mathbf{Z}$, and a positive integer n. Assume that we are given the wavelet coefficients f_j^k, $j = 0, 1, \ldots, n-1$.

Following a standard approach in numerical analysis, we first attempt to find $c_j^k(x)$, $j = 0, 1, \ldots, n-1$, so that

$$x^p = \sum_{j=0}^{n-1} \langle x^p, \phi_j^k(x)\rangle c_j^k(x) \qquad \text{for } p = 0, \ldots, n-1. \tag{9}$$

For $f(x) = x^p$,

$$f_j^k = \langle f, \phi_j^k \rangle = \int_{-\infty}^{\infty} x^p 2^{k/2} \phi(2^k x - j)\, dx = h^{p+(1/2)} \mathcal{M}_{p,j}.$$

Let us define two $n \times n$ matrices $\mathbf{H}$ and $\mathbf{M}$ by

$$\mathbf{H} = h^{1/2} \begin{pmatrix} h^0 & 0 & \cdots & 0 \\ 0 & h^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h^{n-1} \end{pmatrix},$$

and

$$\mathbf{M} = \begin{pmatrix} \mathcal{M}_{0,0} & \mathcal{M}_{0,1} & \cdots & \mathcal{M}_{0,n-1} \\ \mathcal{M}_{1,0} & \mathcal{M}_{1,1} & \cdots & \mathcal{M}_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}_{n-1,0} & \mathcal{M}_{n-1,1} & \cdots & \mathcal{M}_{n-1,n-1} \end{pmatrix}.$$

Equation (9) leads to a system of linear equations for $c_j^k(x)$, $j = 0, \dots, n-1$:

$$\mathcal{A}\vec{c} = \vec{d}, \tag{10}$$

where

$$\begin{aligned} \mathcal{A} &= \mathbf{HM}, \\ \vec{c}(x) &= (c_0^k(x), c_1^k(x), \dots, c_{n-1}^k(x))^T, \\ \vec{d}(x) &= (1, x, x^2, \dots, x^{n-1})^T. \end{aligned} \tag{11}$$

It is easy to show that the matrix $\mathbf{M}$ is nonsingular for all choices of ϕ, n. From (4) , it follows immediately that $\mathbf{M} = \mathbf{B} \cdot \mathbf{V}$, where $\mathbf{B}$ is the lower triangular matrix with entries

$$b_{is} = \binom{i}{s} \mathcal{M}_{i-s}, \qquad 0 \le s \le i,$$

and $\mathbf{V}$ is the Vandermonde matrix with entries $v_{sj} = j^s$. Hence,

$$\det(\mathbf{M}) = \det(\mathbf{B}) \cdot \det(\mathbf{V}) = \mathcal{M}_0^{\mathbf{p}} \left(\prod_{\mathbf{k}=1}^{\mathbf{n}-1} \mathbf{k}! \right) \neq \mathbf{0}.$$

From

$$\vec{c}(x) = \mathbf{M}^{-1}\mathbf{H}^{-1}\vec{d}(x) = h^{-1/2}\mathbf{M}^{-1}\vec{d}(x/h),$$

we observe that each $c_j^k(x)$ is a polynomial of degree $n-1$ in (x/h), and that

$$c_j^k(x) = h^{-1/2} c_j^0(x/h) = 2^{k/2} c_j^0(2^k x). \tag{12}$$

To put this approach into a wavelet-like setting, we select a unit interval $I = [x_0, x_0 + 1)$ for some (as yet undetermined) point x_0. Scaled and translated versions of I are denoted by $I_{k,l} = [(x_0 + l)h, (x_0 + l + 1)h)$.

We restrict the use of formula (9) to the interval $I_{k,0}$. Values of f on a translated interval $I_{k,l}$ are recovered by using the same coefficients c_j^k on a translated set of scaling functions

$$f(x) \approx \sum_{j=0}^{n-1} f_{j+l}^k c_j^k(x - l) = \sum_{j=l}^{n-1+l} f_j^k c_{j-l}^k(x - l) \qquad \text{for } x \in I_{0,l}. \tag{13}$$

We can write (13) in the desired form (6), (7) by defining

$$\beta(x) = \begin{cases} c^0_{n-1}(x+n-1) & \text{if } x \in [x_0-n+1, x_0-n+2), \\ \vdots & \\ c^0_1(x+1) & \text{if } x \in [x_0-1, x_0), \\ c^0_0(x) & \text{if } x \in [x_0, x_0+1), \\ 0 & \text{the others.} \end{cases} \tag{14}$$

Note that $\beta(x)$ is a piecewise polynomial of degree $n-1$.

3. Main Results

In this section we derive a condition for smoothness of basis function $\beta(x)$ for level $k=0$. For notational simplicity we use c_j instead of c^0_j in this section. The result is stated in the Theorem 3.6.

The following lemma provides a preliminary result which is used in the proof of Lemma 3.3.

LEMMA 3.1. *If $i+k = j+l$ for nonnegative integers i, j, k, l with $k \le l, j \le i$, then*

$$\binom{i}{j}\binom{j}{k} = \binom{i}{l}\binom{l}{k}.$$

PROOF.

$$\begin{aligned} \binom{i}{j}\binom{j}{k} &= \frac{i!}{j!(i-j)!}\,\frac{j!}{k!(j-k)!} = \frac{i!}{(i-j)!k!(j-k)!} \\ &= \frac{i!}{(l-k)!k!(i-l)!} = \frac{i!}{l!(i-l)!}\,\frac{l!}{k!(l-k)!} = \binom{i}{l}\binom{l}{k}. \end{aligned}$$

□

The following two lemmas provide preliminary results which are used in the proof of Theorem 3.4.

LEMMA 3.2. *For $r \le i$,*

$$\sum_{k=0}^{r}\binom{i}{k}\left[\sum_{l=0}^{k}\binom{k}{l} j^l \mathcal{M}_{i-k}\right] = \sum_{l=0}^{r}\left[\sum_{k=l}^{r}\binom{i}{k}\binom{k}{l}\mathcal{M}_{i-k}\right] j^l \tag{15}$$

PROOF. Just expand LHS for $k = 0, 1, \ldots, r$ and collect j^l terms with the same index l first. That is, interchange the order of the double summations. □

LEMMA 3.3.

$$\sum_{k=0}^{i}\binom{i}{k}\left[\sum_{l=0}^{k}\binom{k}{l}j^{l}\mathcal{M}_{k-l}\right]=\sum_{s=0}^{i}\left[\sum_{r=0}^{s}\binom{i}{s}\binom{s}{r}j^{r}\right]\mathcal{M}_{i-s} \tag{16}$$

PROOF. Let $t=i-k$ and $r=s-t$. Expand LHS for $k=i,i-1,\dots,0$ and collect $\mathcal{M}_m$ terms with the same index m first. We have

$$\begin{aligned}
\sum_{k=0}^{i}\binom{i}{k}\left[\sum_{l=0}^{k}\binom{k}{l}j^{l}\mathcal{M}_{k-l}\right] &= \sum_{s=0}^{i}\left[\sum_{t=0}^{s}\binom{i}{i-t}\binom{i-t}{s-t}j^{s-t}\right]\mathcal{M}_{i-s}\\
&= \sum_{s=0}^{i}\left[\sum_{r=0}^{s}\binom{i}{i+r-s}\binom{i+r-s}{r}j^{r}\right]\mathcal{M}_{i-s}\\
&= \sum_{s=0}^{i}\left[\sum_{r=0}^{s}\binom{i}{s}\binom{s}{r}j^{r}\right]\mathcal{M}_{i-s}.
\end{aligned}$$

Hence the proof is completed. □

Let $b^{(p)}$ be the pth derivative of each component of a vector b.

THEOREM 3.4. *Let*

$$\mathcal{A}\begin{pmatrix}\gamma_0\\ \gamma_1\\ \vdots\\ \gamma_{p-1}\\ \gamma_p\\ \gamma_{p+1}\\ \gamma_{p+2}\\ \vdots\\ \gamma_{n-1}\end{pmatrix}=\begin{pmatrix}1\\ x\\ \vdots\\ x^{p-1}\\ x^{p}\\ x^{p+1}\\ x^{p+2}\\ \vdots\\ x^{n-1}\end{pmatrix}^{(p)}=\begin{pmatrix}0\\ 0\\ \vdots\\ 0\\ p!\\ (p+1)!x\\ \frac{(p+2)!}{2!}x^{2}\\ \vdots\\ \frac{(n-1)!}{(n-1-p)!}x^{n-1-p}\end{pmatrix}, \tag{17}$$

be given. Then

$$(18)\qquad \mathcal{A}\begin{pmatrix} \gamma_{n-1} \\ \gamma_0 \\ \vdots \\ \gamma_{p-2} \\ \gamma_{p-1} \\ \gamma_p \\ \gamma_{p+1} \\ \vdots \\ \gamma_{n-2} \end{pmatrix} = \begin{pmatrix} 1 \\ (x+1) \\ \vdots \\ (x+1)^{p-1} \\ (x+1)^p \\ (x+1)^{p+1} \\ (x+1)^{p+2} \\ \vdots \\ (x+1)^{n-1} \end{pmatrix}^{(p)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ p! \\ (p+1)!(x+1) \\ \frac{(p+2)!}{2!}(x+1) \\ \vdots \\ \frac{(n-1)!}{(n-1-p)!}(x+1)^{n-1-p} \end{pmatrix}$$

if and only if

$$\gamma_{n-1} = 0.$$

PROOF. Consider the $(i+1)$st component of each side of (17) for $i = 0, 1, \dots, p-1$. We obtain

$$(19)\qquad \sum_{j=0}^{n-1} a_{ij}\gamma_j = \sum_{j=0}^{n-1}\sum_{l=0}^{i} \binom{i}{l} j^l \mathcal{M}_{i-l}\gamma_j = 0 \qquad \text{for} \qquad i = 0, 1, \dots, p-1.$$

Note that (19) implies

$$(20)\qquad \sum_{j=0}^{n-1} j^i \gamma_j = 0 \qquad \text{for} \qquad i = 0, 1, \dots, p-1.$$

Consider the $(i+1)$st component of each side of (17) for $i = p, p+1, \dots, n-1$. We obtain

$$\frac{i!}{(i-p)!}x^{i-p} = \sum_{j=0}^{n-1} a_{ij}\gamma_j \qquad \text{for} \qquad i = p, p+1, \dots, n-1.$$

By letting $k = i - p$, we obtain

$$(21)\qquad x^k = \frac{k!}{(k+p)!}\sum_{j=0}^{n-1} a_{k+p,j}\gamma_j \qquad \text{for} \qquad k = 0, 1, \dots, n-1-p.$$

Consider the $(i+1)$st component of each side of (18) for $i = 0, 1, \dots, p-1$. We obtain

$$(RHS)_{i+1} = 0,$$

and

$$\begin{aligned}
(LHS)_{i+1} &= a_{i,0}\gamma_{n-1} + \sum_{j=0}^{n-2} a_{i,j+1}\gamma_j \\
&= \mathcal{M}_i\gamma_{n-1} + \sum_{j=0}^{n-2}\left[\sum_{k=0}^{i}\binom{i}{k}(j+1)^k\mathcal{M}_{i-k}\right]\gamma_j \\
&= \mathcal{M}_i\gamma_{n-1} + \sum_{j=0}^{n-2}\left[\sum_{k=0}^{i}\binom{i}{k}\sum_{l=0}^{k}\binom{k}{l}j^l\mathcal{M}_{i-k}\right]\gamma_j \\
&= \mathcal{M}_i\gamma_{n-1} + \sum_{j=0}^{n-2}\left[\sum_{l=0}^{i}\left[\sum_{k=l}^{i}\binom{i}{k}\binom{k}{l}\mathcal{M}_{i-k}\right]j^l\right]\gamma_j, \qquad \text{by (15)} \\
&= \mathcal{M}_i\gamma_{n-1} + \sum_{l=0}^{i}\sum_{k=l}^{i}\binom{i}{k}\binom{k}{l}\mathcal{M}_{i-k}\left[\sum_{j=0}^{n-2}j^l\gamma_j\right] \\
&= \mathcal{M}_i\gamma_{n-1} + \sum_{l=0}^{i}\sum_{k=l}^{i}\binom{i}{k}\binom{k}{l}\mathcal{M}_{i-k}\left[-(n-1)^l\gamma_{n-1}\right], \qquad \text{by (20)} \\
&= \left[\mathcal{M}_i - \sum_{l=0}^{i}\sum_{k=l}^{i}\binom{i}{k}\binom{k}{l}\mathcal{M}_{i-k}(n-1)^l\right]\gamma_{n-1}.
\end{aligned}$$

We obtain, for $i = 0, 1, \dots, p-1$,

(22)

$$(LHS)_{i+1} - (RHS)_{i+1} = \left[\mathcal{M}_i - \sum_{l=0}^{i}\sum_{k=l}^{i}\binom{i}{k}\binom{k}{l}\mathcal{M}_{i-k}(n-1)^l\right]\gamma_{n-1}.$$

Consider the $(i+1)$st component of each side of (18) for $i = p, p+1, \ldots, n-1$:

$$
\begin{aligned}
(RHS)_{i+1} &= \frac{i!}{(i-p)!}(x+1)^{i-p} \\
&= \frac{i!}{(i-p)!}\sum_{k=0}^{i-p}\binom{i-p}{k}x^k \\
&= \sum_{k=0}^{i-p}\frac{i!}{(i-p)!}\binom{i-p}{k}\frac{k!}{(k+p)!}\sum_{j=0}^{n-1}a_{k+p,j}\,\gamma_j, \qquad \text{by (21)} \\
&= \sum_{j=0}^{n-1}\left[\sum_{k=0}^{i-p}\binom{i}{k+p}\sum_{l=0}^{k+p}\binom{k+p}{l}j^l\mathcal{M}_{k+p-l}\right]\gamma_j \\
&= \sum_{j=0}^{n-1}\left[\sum_{m=p}^{i}\binom{i}{m}\sum_{l=0}^{m}\binom{m}{l}j^l\mathcal{M}_{m-l}\right]\gamma_j, \text{ by letting } m = k+p \\
&= \sum_{j=0}^{n-1}\left[\sum_{m=0}^{i}\binom{i}{m}\sum_{l=0}^{m}\binom{m}{l}j^l\mathcal{M}_{m-l}\right]\gamma_j \\
&\quad -\sum_{j=0}^{n-1}\left[\sum_{m=0}^{p-1}\binom{i}{m}\sum_{l=0}^{m}\binom{m}{l}j^l\mathcal{M}_{m-l}\right]\gamma_j \\
&= \sum_{j=0}^{n-1}\left[\sum_{s=0}^{i}\binom{i}{s}\sum_{r=0}^{s}\binom{s}{r}j^r\mathcal{M}_{i-s}\right]\gamma_j \\
&\quad -\sum_{j=0}^{n-1}\left[\sum_{m=0}^{p-1}\binom{i}{m}\sum_{l=0}^{m}\binom{m}{l}j^l\mathcal{M}_{m-l}\right]\gamma_j, \qquad \text{by (16)} \\
&= \sum_{j=0}^{n-1}\left[\sum_{s=0}^{i}\binom{i}{s}(j+1)^s\mathcal{M}_{i-s}\right]\gamma_j \\
&\quad -\sum_{j=0}^{n-1}\left[\sum_{l=0}^{p-1}\left[\sum_{m=l}^{p-1}\binom{i}{m}\binom{m}{l}\mathcal{M}_{m-l}\right]j^l\right]\gamma_j, \qquad \text{by (15)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{n-1}\left[\sum_{s=0}^{i}\binom{i}{s}(j+1)^s\mathcal{M}_{i-s}\right]\gamma_j \\
&\quad -\sum_{l=0}^{p-1}\sum_{m=l}^{p-1}\binom{i}{m}\binom{m}{l}\mathcal{M}_{m-l}\left[\sum_{j=0}^{n-1} j^l\gamma_j\right] \\
&= \sum_{j=0}^{n-1}\left[\sum_{s=0}^{i}\binom{i}{s}(j+1)^s\mathcal{M}_{i-s}\right]\gamma_j, \qquad \text{by (20)} \\
&= \sum_{j=0}^{n-2}\left[\sum_{s=0}^{i}\binom{i}{s}(j+1)^s\mathcal{M}_{i-s}\right]\gamma_j + \left[\sum_{s=0}^{i}\binom{i}{s}n^s\mathcal{M}_{i-s}\right]\gamma_{n-1} \\
&= \sum_{j=0}^{n-2} a_{i,j+1}\gamma_j + \left[\sum_{s=0}^{i}\binom{i}{s}n^s\mathcal{M}_{i-s}\right]\gamma_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
(LHS)_{i+1} &= a_{i,0}\gamma_{n-1} + \sum_{j=0}^{n-2} a_{i,j+1}\gamma_j \\
&= \mathcal{M}_i\gamma_{n-1} + \sum_{j=0}^{n-2} a_{i,j+1}\gamma_j.
\end{aligned}
$$

We obtain, for $i = p, p+1, \ldots, n-1$,

$$
(RHS)_{i+1} - (LHS)_{i+1} = \left[\sum_{s=0}^{i}\binom{i}{s}n^s\mathcal{M}_{i-s} - \mathcal{M}_i\right]\gamma_{n-1}. \tag{23}
$$

By (22) and (23), we obtain that (18) if and only if $\gamma_{n-1} = 0$, or

$$
\mathcal{M}_i = \begin{cases} \displaystyle\sum_{l=0}^{i}\sum_{k=l}^{i}\binom{i}{k}\binom{k}{l}\mathcal{M}_{i-k}(n-1)^l & \text{for } i = 0, 1, \ldots, p-1, \\ \displaystyle\sum_{s=0}^{i}\binom{i}{s}n^s\mathcal{M}_{i-s} & \text{for } i = p, p+1, \ldots, n-1 \end{cases}
$$

In fact, for $i = 1$, if $p \geq 2$,

$$\sum_{l=0}^{1}\sum_{k=l}^{1}\binom{1}{k}\binom{k}{l}\mathcal{M}_{1-k}(n-1)^{l} = \mathcal{M}_1 + 1 + (n-1) \neq \mathcal{M}_1,$$

and if $p \leq 1$,

$$\sum_{s=0}^{1}\binom{i}{s}n^{s}\mathcal{M}_{i-s} = \mathcal{M}_1 + n \neq \mathcal{M}_1.$$

Therefore, (18) holds if and only if $\gamma_{n-1} = 0$. □

THEOREM 3.5. *In the equation* $\mathcal{A}(\vec{c})^{(p)} = (\vec{d})^{(p)}$, *there exists an* $x_0 \in \mathbf{R}$ *such that* $(c_{n-1})^{(p)}(x_0) = 0$ *if and only if*

$$(c_i)^{(p)}(x_0+1) = \begin{cases} (c_{n-1})^{(p)}(x_0) & \text{for } i = 0, \\ (c_{i-1})^{(p)}(x_0) & \text{for } i = 1, \dots, n-1 \end{cases} \tag{24}$$

PROOF. In the equation $\mathcal{A}\vec{c}^{(p)} = \vec{d}^{(p)}$, we take $x = x_0$ and $x = x_0 + 1$. Then we have

$$\mathcal{A}\begin{pmatrix} c_0(x_0) \\ c_1(x_0) \\ c_2(x_0) \\ \vdots \\ c_{n-1}(x_0) \end{pmatrix}^{(p)} = \begin{pmatrix} 1 \\ x_0 \\ {x_0}^2 \\ \vdots \\ {x_0}^{n-1} \end{pmatrix}^{(p)}, \tag{25}$$

and

$$\mathcal{A}\begin{pmatrix} c_0(x_0+1) \\ c_1(x_0+1) \\ c_2(x_0+1) \\ \vdots \\ c_{n-1}(x_0+1) \end{pmatrix}^{(p)} = \begin{pmatrix} 1 \\ x_0+1 \\ (x_0+1)^2 \\ \vdots \\ (x_0+1)^{n-1} \end{pmatrix}^{(p)}. \tag{26}$$

By Theorem 3.4, there exists $x_0 \in \mathbf{R}$ such that $(c_{n-1})^{(p)}(x_0) = 0$ if and only if

$$\mathcal{A}\begin{pmatrix} c_{n-1}(x_0) \\ c_0(x_0) \\ c_1(x_0) \\ \vdots \\ c_{n-2}(x_0) \end{pmatrix}^{(p)} = \begin{pmatrix} 1 \\ x_0+1 \\ (x_0+1)^2 \\ \vdots \\ (x_0+1)^{n-1} \end{pmatrix}^{(p)}. \tag{27}$$

Subtract the equation (27) from the equation (26):

$$\mathcal{A}\begin{pmatrix} c_0(x_0+1) - c_{n-1}(x_0) \\ c_1(x_0+1) - c_0(x_0) \\ c_2(x_0+1) - c_1(x_0) \\ \vdots \\ c_{n-1}(x_0+1) - c_{n-2}(x_0) \end{pmatrix}^{(p)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{28}$$

Since $\mathcal{A}$ is nonsingular, (28) holds if and only if (24) holds. Hence the proof is completed. □

THEOREM 3.6. *$\beta(x) \in C^p(\mathbf{R})$ if and only if there exists $x_0 \in \mathbf{R}$ such that $(c_{n-1})^{(i)}(x_0) = 0$ for $i = 0, 1, \ldots, p$.*

PROOF. From the definition for $\beta(x)$, $\beta(x) \in C^p(\mathbf{R})$ if and only if it is C^p function at all interior nodes. It comes directly from the Theorem 3.4. □

REMARKS. 1. One of the advantages of Theorem 3.4 is that, even if we define $\beta(x)$ on $[x_0-n+1, x_0+1]$, $\beta(x)$ is a C^p function on $[x_0-n+1, x_0+1]$ if and only if there exists $x_0 \in \mathbf{R}$ such that $(c_{n-1})^{(p)}(x_0) = 0$ for $i = 0, 1, \ldots, p$.

2. Note that $\beta(x)$ is a piecewise polynomial of degree $\leq n-1$ and the coefficients of the degree $n-1$ for $c_i(x)$ for $i = 0, 1, \ldots, n-1$ are not all equal in general. Hence the highest regularity we can achieve for $\beta(x)$ is C^{n-2}.

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Department of Mathematics Education
Sunchon National University
Sunchon 540-742, Korea
E-mail: sgkwon@sunchon.ac.kr