Comm. Korean Math. Soc. 15 (2000), No. 1, pp. 173–181

FREE SURFACE WAVES OF A TWO-LAYER FLUID OVER A STEP

JEONGWHAN CHOI AND SUNGIM WHANG

ABSTRACT. The objective of this paper is to study two dimensional steady gravitational waves on the interface between two immiscible, inviscid and incompressible fluids bounded above by a horizontal rigid boundary and below by a rigid step. A KdV equation for the first order perturbation in an asymptotic expansion can appear. However the coefficient of the KdV theory fails in that case. By a unified asymptotic method, we overcome this difficulty and derive a modified KdV equation with forcing. We find homogeneous steady solutions and present numerical solutions.

1. Introduction

Since Peters and Stoker ([1]) studied two-dimensional solitary waves in a two-layer medium of immiscible fluids there have been growing interests in studying interfacial waves in a two layer fluid. Many interesting wave patterns have been found and new mathematical methods have also been developed in various fluid domains. Numerical studies of steady flow of a two-layer incompressible fluid over a semi-circular bump bounded by a free or rigid upper boundary were carried out by Forbes ([2]), and Sha and Vanden-Broeck ([3]), and an asymptotic approach for the case of a rigid upper boundary or a free surface with surface tension were studied by Choi, Sun and Shen ([4],[5]). A linear study for interfacial waves of two-layer fluid over a step was studied by Moni and King ([6]) numerically and nonlinear study based on KdV theory of single layer fluid over a step was studied by Shen ([7]). However, for some special values of the depth

Received March 15, 1999. Revised June 22, 1999.
1991 Mathematics Subject Classification: 76B25.
Key words and phrases: modified KdV, wave, step.
The research reported here was partly supported by Korean Research Foundation made in the program year of 1997 and KOSEF 97-0701-0101-3.

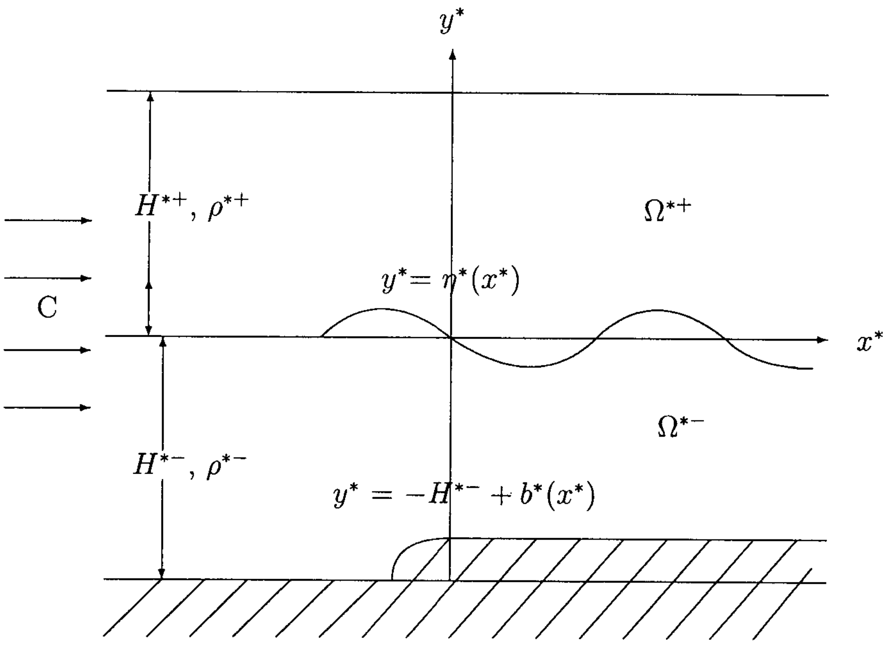

Fig.1. Fluid Domain

and density ratios of the two-layer fluid, the coefficient of the nonlinear term in the KdV equation vanishes and the asymptotic expansion for the KdV theory can fail. In this paper, to overcome this difficulty, we apply a refined asymptotic scheme developed in [5] to derive the so-called modified KdV equations with forcing for the interfacial wave of a two layer fluid over a step under the condition that the nonlinear term in the KdV equation vanishes. The modified KdV equation possesses non-periodic wave solutions, which are obtained as limiting solutions of its periodic solutions. We formulate the problem and derive the modified KdV equation with forcing as a model equation in Sec. 2. In Sec. 3 we present homogeneous solutions and numerical solutions of the model equation.

2. Formulation and Model Equation

We consider a two-layer immiscible ideal fluid flow in a two dimensional channel bounded above by a horizontal rigid boundary and below by a rigid step (Fig. 1). The domains of the upper fluid with a constant density ρ^{*+} and the lower fluid with a constant density ρ^{*-} are denoted by Ω^{*+} and Ω^{*-} respectively. Since a two-dimensional object at the lower boundary is moving with a constant speed C, we choose a coordinate system moving with the object so that, in reference to the coordinate system, the object is stationary and the fluid flow becomes steady. Then the governing equations and boundary conditions are as follows:

In $\Omega^{*\pm}$,

$$
\begin{aligned}
&u^{*\pm}_{x^*} + v^{*\pm}_{y^*} = 0,\\
&u^{*\pm}u^{*\pm}_{x^*} + v^{*\pm}u^{*\pm}_{y^*} = -p^{*\pm}_{x^*}/\rho^{*\pm},\\
&u^{*\pm}v^{*\pm}_{x^*} + v^{*\pm}v^{*\pm}_{y^*} = -p^{*\pm}_{y^*}/\rho^{*\pm} - g;
\end{aligned}
$$

at the interface, $y^* = \eta^*$,

$$
\begin{aligned}
&u^{*\pm}\eta^*_{x^*} - v^{*\pm} = 0,\\
&p^{*+} - p^{*-} = 0;
\end{aligned}
$$

at the rigid boundaries, $y^* = H^{*\pm}(x^*)$,

$$
v^{*-} - u^{*\pm}H^{*\pm}_{x^*} = 0;
$$

where $(u^{*\pm}, v^{*\pm})$ are velocities, $p^{*\pm}$ are pressures, g is the gravitational acceleration constant, $H^{*+}(x^*) = H^{*+}$, $H^{*-}(x^*) = -H^{*-} + b^*(x^*)$, and

$b^*(x^*)$ are the equation of the bottom. We define the following nondimensional variables:

$$\begin{aligned}
&\epsilon = H/L << 1,\ \eta = \epsilon^{-1}\eta^*/H^{*-},\ p^{\pm} = p^{*\pm}/gH^{*-}\rho^{*-},\\
&(x,y) = (\epsilon x^*, y^*)/H^{*-},\ (u^{\pm}, v^{\pm}) = (gH^{*-})^{-1/2}(u^{*\pm},\ \epsilon^{-1}v^{*\pm}),\\
&\rho^+ = \rho^{*+}/\rho^{*-} < 1,\ \rho^- = \rho^{*-}/\rho^{*-} = 1,\ U = C/(gH^{*-})^{1/2},\\
&h = H^{*+}/H^{*-},\ b(x) = b^*(x)(H^{*-}\epsilon^3)^{-1},
\end{aligned}$$

where L is the horizontal scale, H is the vertical scale, H^{*+} and H^{*-} are the equilibrium depths of the upper and lower fluids at $x^* = -\infty$ respectively. We assume that $U = u_0 + \lambda\epsilon^2$, where u_0 and λ are constants, $u^{\pm}$, $v^{\pm}$, and $p^{\pm}$ are functions of x and y near the equilibrium state $u^{\pm} = u_0$, $v^{\pm} = 0$, $p^+ = -\rho^+ y + \rho^+ h$ and $p^- = -\rho^- y + \rho^+ h$, and possess asymptotic expansions:

$$\begin{aligned}
(u^{\pm}, v^{\pm}, p^{\pm}) &= (u_0, 0, -\rho^{\pm}y + \rho^+ h) + \epsilon(u_1^{\pm}, v_1^{\pm}, p_1^{\pm})\\
&\quad + \epsilon^2(u_2^{\pm}, v_2^{\pm}, p_2^{\pm}) + \epsilon^3(u_3^{\pm}, v_3^{\pm}, p_3^{\pm}) + O(\epsilon^4), \qquad (1)
\end{aligned}$$

where $v_0^{\pm} = 0$, and $p_0^{\pm} = -\rho^{\pm}y + \rho^+ h$. As was in [4], we substitute the asymptotic expansions of u, v, p into the nondimensionalized Euler equations and boundary conditions for the successive approximation of the nondimensionalized Euler equations. Then, by solving this sequence of equations, one can easily find $p_1^{\pm}, v_1^{\pm}, u_1^{\pm}, p_2^{\pm}, v_2^{\pm}, u_2^{\pm}, p_3^{\pm}, v_3^{\pm}, u_3^{\pm}$ in terms of η with the assumption $\eta(-\infty) = 0$ and can also derive the following equation of η,

$$\begin{aligned}
&u_0\eta_x - v_1^- + \epsilon(u_1^-\eta_x - \eta v_{1y}^- - v_2^-)\\
&+ \epsilon^2(u_2^-\eta_x + \eta\eta_x u_{1yy}^-\eta^2 + \eta v_{2y}^- - v_3^-) = O(\epsilon^3). \qquad (2)
\end{aligned}$$

The critical speed u_0 is obtained if zeroth order term of (2) vanishes. Hence,

$$u_0^2 = h(1-\rho)/(\rho + h). \qquad (3)$$

Then we put $v_1^{\pm}$, $u_1^{\pm}$, $v_2^{\pm}$, $u_2^{\pm}$, and $v_3^{\pm}$ into (2) to obtain the following equation for η,

$$\begin{aligned}
&3(1-\rho)(\rho - h^2)/(u_0(\rho + h)^2)\eta\eta_x\\
&+ \epsilon(A_1\eta_x + A_2\eta^2\eta_x + A_3\eta_{xxx} + A_4 b_x) = O(\epsilon^2), \qquad (4)
\end{aligned}$$

where

$$
\begin{aligned}
A_1 &= 2\lambda, \\
A_2 &= 6\rho(1-\rho)(1+h)^2/(u_0(\rho+h)^3) > 0, \\
A_3 &= -hu_0(1+\rho h)/3(\rho+h) < 0, \\
A_4 &= -(u_0+1).
\end{aligned}
$$

The nonlinear term $\eta\eta_x$ disappears if ρ equals the square of h and we obtain the following forced modified KdV equation when $\rho = h^2$ by dropping the higher order term of ϵ,

$$\eta_{xxx} = a_1\eta^2\eta_x + a_2\eta_x + a_3 b_x, \tag{5}$$

$$
\begin{aligned}
a_1 &= 6\rho(1-\rho)(1+h)^2/(hu_0^2(\rho+h)^2(1+\rho h)/3) > 0, \\
a_2 &= 6\lambda(\rho+h)/(hu_0(1+\rho h)), \\
a_3 &= -3(u_0+1)(\rho+h)/(hu_0(1+\rho+h)) < 0.
\end{aligned}
$$

We note that if $\rho \neq h^2$ in (4), a KdV equation, which has been studied in [7], can be derived.
Integrating (5) from $-\infty$ to x yields

$$\eta_{xx} = a_1\eta^3/3 + a_2\eta + a_3 b(x), \tag{6}$$

where

$$
b(x) = \begin{cases} 0 & if\ x \leq -R \\ \sqrt{(1-x^2/R^2)} & if\ -R \leq x \leq 0 \\ 1 & if\ 0 \leq x. \end{cases}
$$

3. Homogeneous Solutions and Numerical Solutions

Since $b(x)$ stands for step-shaped obstruction, we first look for the solution of (6) when $b(x)$ is a constant.
By multiplying η_x to both sides of (6) and integrating the resulting equation, we obtain

$$\eta_x^2 = a_1\eta^4/6 + a_2\eta^2 + 2a_3\eta + e \stackrel{\text{def}}{=} f(\eta). \tag{7}$$

Let $c_1 < c_2 < c_3 < c_4$ be four real zeros of $f(\eta) = 0$. Then the solutions of (7) in this case should be larger than or equal to c_2 and less than or equal to c_3 since, otherwise, the right side of (7) becomes negative, and no real

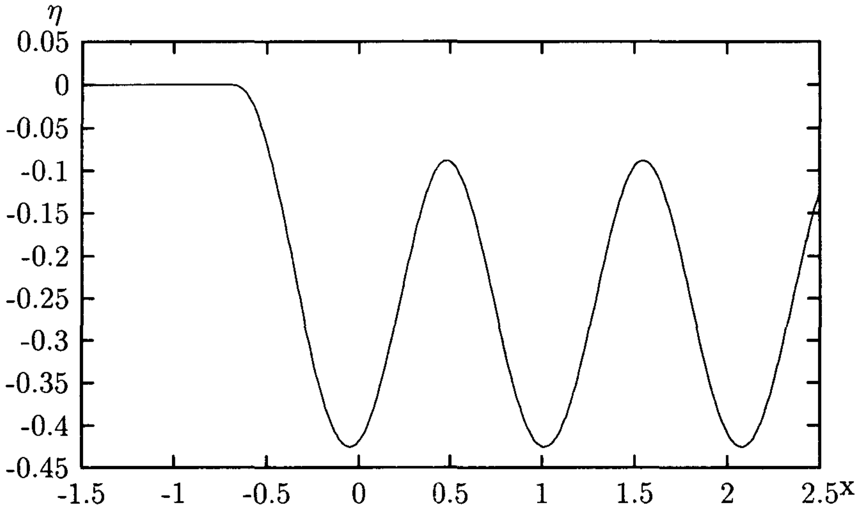

Fig.2. Typical Periodic Wave

ρ = 0.5, λ = -3.5, R = -0.7,

$$b(x) = \begin{cases} 0 & x \leq -R \\ \sqrt{1 - x^2/0.49} & -R \leq x \leq 0 \\ 1 & x \geq 0 \end{cases}$$

solution appears. We first consider the periodic solution of (7), which can be expressed as

$$(8) \qquad \eta = (\alpha_1 \mathrm{sn}^2(\alpha_2(x - x_0), k) - \alpha_3)/(\alpha_4 \mathrm{sn}^2(\alpha_2(x - x_0), k) - \alpha_5),$$

for $c_2 < \eta < c_3$, where

$$k^2 = (c_4 - c_1)(c_3 - c_2)/(c_4 - c_2)(c_3 - c_1),$$
$$\alpha_1 = c_1(c_3 - c_2), \ \alpha_2 = c_1(c_3 - c_2)/(6(c_3 - c_1))^{1/2},$$
$$\alpha_3 = c_2(c_3 - c_1), \ \alpha_4 = c_3 - c_2, \ \alpha_5 = c_3 - c_1,$$

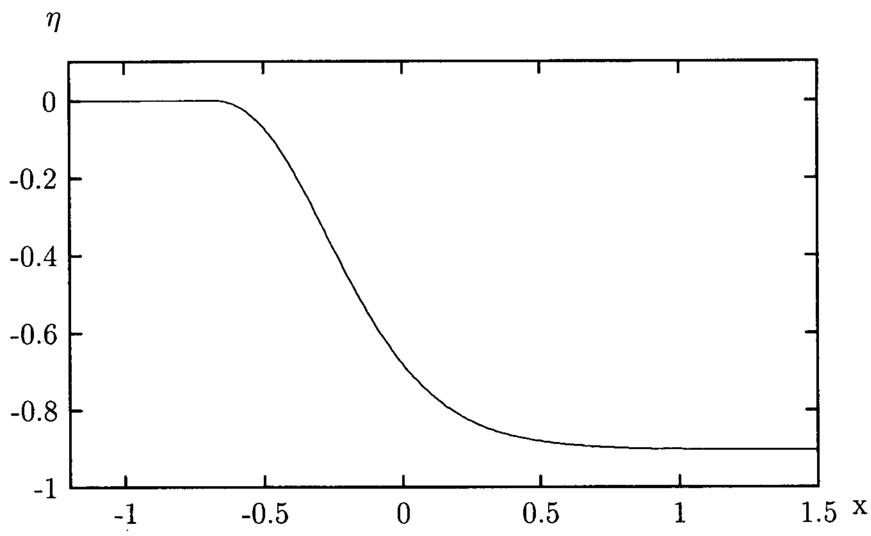

Fig.3. Hydraulic Fall

ρ = 0.5, λ = -2.48417, R = -0.7,

$$b(x) = \begin{cases} 0 & x \leq -R \\ \sqrt{1 - x^2/0.49} & -R \leq x \leq 0 \\ 1 & x \geq 0 \end{cases}$$

and x_0 is a phase shift and $\mathrm{sn}(u, k)$ is a Jacobian Elliptic function. Since $\mathrm{sn}(x, 1) = \tanh x$, (8) has the following limiting solution as $c_4 \to c_3$;

(9) $$\eta = c_1 - E(c_2 - c_1)/(\tanh^2(hM(x - x_0)) - E),$$

where

$$E = (c_3 - c_1)/(c_3 - c_2) > 1, \ h = (c_1/6)^{1/2},$$
$$M^2 = (c_3 - c_1)(c_1 - c_2)/4.$$

If $c_3 = 0$, η in (8) becomes, as $c_4 \to 0$,

(10) $$\eta = -c_1 \mathrm{sech}^2(\alpha_2(x - x_0))/(\tanh^2(\alpha_2(x - x_0)) - c_1/c_2),$$

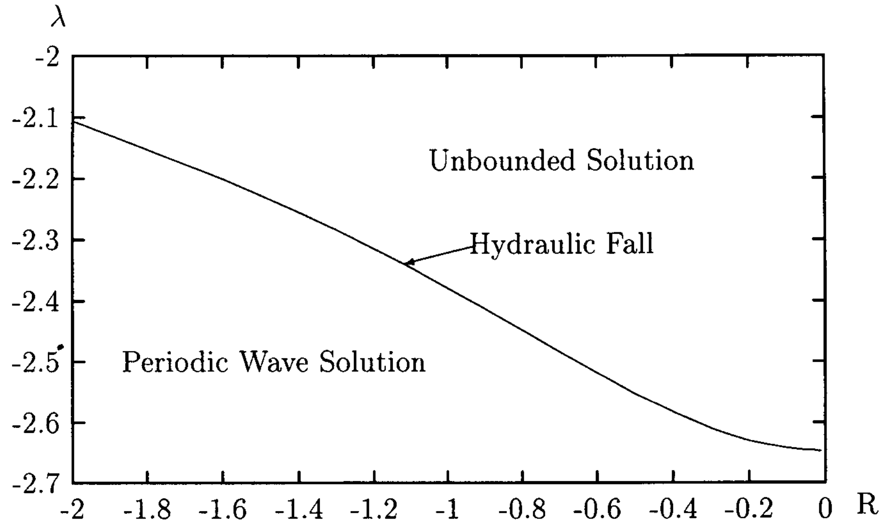

Fig. 4: Branch Curve

$$\rho = 0.5,\ \mathrm{b(x)} = \begin{cases} 0 & \mathrm{x} \le \mathrm{-R} \\ \sqrt{1 - x^2/R^2} & \mathrm{-R} \le \mathrm{x} \le 0 \\ 1 & \mathrm{x} \ge 0 \end{cases}$$

where again $\mathrm{sn}(x,1) = \tanh x$ has been used.
Next we consider the existence of the solution of (6) on $[x^-, x^+]$ with $\eta(x^-) = \eta_x(x^-) = 0$. Define a complete metric space $B = \{f | f \in C[x^-, x^+]\}$, $||f|| = \max_{x^- \le x \le x^+} |f(x)| \le M$ for some given positive constant M. Then (6) can be converted to an integral equation and by using contraction mapping theorem, one can easily show that the solution of (6) exists for $-\lambda$ sufficiently large (Ref. [4]).
Since we have shown the existence of solutions of (6) for $(-\infty, x^-)$, $[x^-, x^+]$ and (x^+, ∞) separately, we find the global solution of (6) numerically by using matching process. Typical periodic cnoidal wave solution and hydraulic fall solution are given in Fig. 2 and 3 for $\rho = 1/2$ and branch curve for hydraulic fall is given in Fig. 4.

References

[1] A. S. Peters and J. J. Stoker, *Solitary waves in liquids haveing nonconstant density,* Comm. Pure and Appl. Math. **13** (1960), 115.

[2] L. b. Forbes, *Two layer critical flow over a semi-circular obstruction,* J. Eng. Math. **23** (1989), 325.

[3] H. Y. Sha and J. M. Vanden-Broeck, *Two layer flows past a semi-circular obstruction,* Phys. Fluids A **5** (1993), 2661.

[4] J. W. Choi, S. M. Sun, and M. C. Shen, *Steady capillary-gravity waves on the interface of a two layer fluid over an obstruction-Forced modified KdV equation,* J. Eng. Math. **28** (1994), 193.

[5] J. W. Choi, S. M. Sun, and M. C. Shen, *Internal capillary-gravity waves of a two-layer fluid with free surface over an obstruction-Forced extended KdV equation,* Phys. Fluid A **8** (1996), 397.

[6] J. N. Moni and A. C. King, *Interfacial flow over a step,* Phys. Fluids A **6** (1994), 2986.

[7] S. S. Shen and L. Gong, *Solitary waves on a shelf,* Phys. Fluids A **5** (1993), 1071.

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