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BLOCK DIAGONAL PRECONDITIONERS FOR THE GALERKIN LEAST SQUARES METHOD IN LINEAR ELASTICITY

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ABSTRACT. In [8], Franca and Stenberg developed several Galerkin least squares methods for the solution of the problem of linear elasticity. That work concerned itself only with the error estimates of the method. It did not address the related problem of finding effective methods for the solution of the associated linear systems. In this work, we propose the block diagonal preconditioners. The preconditioned conjugate residual method is robust in that the convergence is uniform as the parameter, ν, goes to $\frac{1}{2}$. Computational experiments are included.

1. Introduction

Let Ω be a bounded convex polygonal domain in R^2 and $\partial\Omega$ be the boundary of Ω. The pure displacement boundary value problem for planar linear elasticity is given in the form

$$
\begin{aligned}
(1) \qquad & 2\mu\{\underset{\sim}{\nabla}\cdot\underset{\approx}{\varepsilon}(\underset{\sim}{u})+\frac{\nu}{1-2\nu}\underset{\sim}{\nabla}\nabla\cdot\underset{\sim}{u}\}+\underset{\sim}{f}=\underset{\sim}{0} \quad \text{in } \Omega,\\
& \underset{\sim}{u}=\underset{\sim}{0} \quad \text{on } \partial\Omega.
\end{aligned}
$$

Here $\underset{\sim}{u}=(u_1,u_2)$ denotes the displacement, $\underset{\sim}{f}=(f_1,f_2)$ is the body force, ν is Poisson's ratio and μ is the shear modulus given by $\mu=E/\{2(1+\nu)\}$ where E is the Young's modulus. Instead of using Poisson's ratio ν and Young's elasticity modulus E, we can also work with the Lame

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constants λ and μ. These constants are related to each other by the following equations;

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)},$$

$$\mu = \frac{E}{2(1+\nu)}, \quad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}.$$

We restrict Poisson's ratio to $0 \leq \nu < 1/2$ where the upper limit corresponds to an incompressible material.

We use undertildes to denote vector-valued functions, operators and their associated spaces, and double undertildes are used for matrix-valued functions and operators.

We define various standard differential operators as follows (see [6]):

$$\nabla \cdot \underset{\sim}{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y},$$

$$\nabla \cdot \underset{\approx}{\tau} = \begin{pmatrix} \partial\tau_{11}/\partial x & +\partial\tau_{12}/\partial y \\ \partial\tau_{21}/\partial x & +\partial\tau_{22}/\partial y \end{pmatrix}, \quad \underset{\approx}{\nabla}\, \underset{\sim}{v} = \begin{pmatrix} \partial v_1/\partial x & \partial v_1/\partial y \\ \partial v_2/\partial x & \partial v_2/\partial y \end{pmatrix},$$

$$\underset{\approx}{\tau} : \underset{\approx}{\eta} = \sum_{i=1}^{2}\sum_{j=1}^{2} \tau_{ij}\eta_{ij}, \quad \text{and} \quad \underset{\approx}{\varepsilon}(\underset{\sim}{v}) = \frac{1}{2}\left[\underset{\approx}{\nabla}\, \underset{\sim}{v} + (\underset{\approx}{\nabla}\, \underset{\sim}{v})^t\right].$$

Let $H^m(\Omega)$ denote the usual Sobolev space of functions with $L^2(\Omega)$ derivatives up to order m. $H^m(\Omega)$ is equipped with the norm

$$||v||_m := \left(\int_\Omega \sum_{|\alpha|\leq m} |\partial^\alpha v|^2 \, dxdy \right)^{\frac{1}{2}}.$$

We use the following convention for the Sobolev seminorms (see [1]):

$$|v|_m := \left(\int_\Omega \sum_{|\alpha|=m} |\partial^\alpha v|^2 \, dxdy \right)^{\frac{1}{2}}.$$

Let $H_0^m(\Omega) = \{v \in H^m(\Omega) : v|_{\partial\Omega} = 0\}$.

The pure displacement problem (1) can be formulated as a saddle point problem and discretized by the Galerkin least squares methods. This

discretization gives rise to symmetric indefinite linear systems of equations of the form

$$\begin{pmatrix} A & B^t \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \quad \text{or } \mathcal{A}x = b.$$

Here the submatrix A is symmetric and positive definite and C is a symmetric and positive definite matrix. For such indefinite systems, the conjugate gradient method is generally inapplicable, however, the preconditioned conjugate residual method(PCR) can be used (see [3]).

In recent years, several iterative methods have been developed to solve such problems. The oldest algorithm is known as the Uzawa algorithm, see [2]. It is mainly a gradient algorithm applied to the Schur compliment of $C + BA^{-1}B^t$ of the indefinite linear system. When this algorithm is applied, we have to solve a linear system of the form $A\omega = d$. The Uzawa algorithm is quite expensive since A normally is not well conditioned. Therefore, many authors have considered an inner iteration for A^{-1}, see [4] and [7]. Other authors have considered the use of symmetric positive definite block diagonal preconditioners for the indefinite algebraic system to avoid inner and outer iterations, see [11], [12] and [13]. In [13], Sylvester and Wathen developed a simple diagonal preconditioner for a saddle point problem arising from stabilized and unstabilized Stokes flow. They derived estimates of the eigenvalue spectrum of Stokes operator on which the convergence rate of the iteration depends and provides a good criterion for choosing stabilizing parameter α to ensure fast convergence of the iterative method. In [10], Klawonn introduced an optimal preconditioner for a saddle point problem with a penalty term and showed that the condition number of the preconditioned system is bounded independently of the finite element discretization and the penalty parameter. In this paper, we apply the block diagonal preconditioner proposed by Klawonn to a saddle point problem arising from stabilized linear elasticity problem. We can use any combinations of finite element spaces for approximating the displacement and pressure, however, Klawonn's approach is restricted to certain finite element spaces. In section 4, we provide numerical results which show that the convergence rate of the preconditioned conjugate residual methods is independent of the mesh size h and Poisson's ratio ν.

This paper is organized as follows. In section 2, we describe the algebraic form of the problem, the block diagonal preconditioners and the finite element theory thereof. In section 3, we explain the preconditioned

conjugate residual(PCR) method as an example a Krylov space method for indefinite linear systems. In section 4, we give the computational results for our problem.

2. Algebraic Form of the Problem

In this section, we give the algebraic form of the problem and present the block diagonal preconditioners for the problem. We state the theorem of the condition number estimate of the preconditioned system without proof.

For simplicity, we assume that $2\mu = 1$. Let $p = -\frac{1}{\epsilon}\nabla \cdot \underset{\sim}{u}$, where $\epsilon = (1-2\nu)/\nu$. Then (1) is equivalent to

$$
\begin{aligned}
&-\underset{\sim}{\nabla} \cdot \underset{\approx}{\varepsilon}(\underset{\sim}{u}) + \underset{\sim}{\nabla} p = \underset{\sim}{f} \quad \text{in } \Omega, \\
&\epsilon p + \nabla \cdot \underset{\sim}{u} = 0 \quad \text{in } \Omega, \\
&\underset{\sim}{u} = \underset{\sim}{0} \quad \text{on } \partial\Omega.
\end{aligned} \tag{2}
$$

Hence, we have the following weak formulation:

Find $(\underset{\sim}{u}, p) \in \underset{\sim}{H}{}^1_0(\Omega) \times L^2(\Omega)$ such that

$$
\begin{aligned}
&\int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}) : \underset{\approx}{\varepsilon}(\underset{\sim}{v})\, dxdy - \int_\Omega \Big(\nabla \cdot \underset{\sim}{v}\Big) p\, dxdy = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, dxdy, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \underset{\sim}{v} \in \underset{\sim}{H}{}^1_0(\Omega), \\
&\epsilon \int_\Omega pq\, dxdy + \int_\Omega \Big(\nabla \cdot \underset{\sim}{u}\Big) q\, dxdy = 0, \qquad \forall q \in L^2(\Omega).
\end{aligned} \tag{3}
$$

Let $\mathcal{T}^k$ be a family of triangulations of Ω, where $\mathcal{T}^{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}^k$. Let h_T =diam(T) for each $T \in \mathcal{T}^k$ and $h_k = \max_{T \in \mathcal{T}^k} h_T$. Then $h_k = 2h_{k+1}$. Now let's define the conforming finite element spaces.

$$
\begin{aligned}
\underset{\sim}{V}_k &:= \{\underset{\sim}{v} \in \underset{\sim}{C}{}^0(\Omega)\ ;\ \underset{\sim}{v}|_T \text{ is linear for all } T \in \mathcal{T}^k \text{ and } \underset{\sim}{v}|_{\partial\Omega} = \underset{\sim}{0}\} \text{ and} \\
P_k &:= \{q \in C^0(\Omega)\ ;\ q|_T \text{ is linear for all } T\ \in \mathcal{T}^k\}.
\end{aligned}
$$

Then the discretized Galerkin least squares method for (3) is the following:
Find $(\underset{\sim}{u}_k, p_k) \in \underset{\sim}{V}_k \times P_k$ such that

$$(4) \quad \mathcal{B}_k\Big((\underset{\sim}{u}_k, p_k), (\underset{\sim}{v}_k, q_k)\Big) = \mathcal{F}_{\underset{\sim}{f}}(\underset{\sim}{v}_k, q_k) \qquad \forall (\underset{\sim}{v}_k, q_k) \in \underset{\sim}{V}_k \times P_k$$

where

$$\begin{aligned}
&\mathcal{B}_k\Big((\underset{\sim}{u}_k, p_k), (\underset{\sim}{v}_k, q_k)\Big) \\
&= \int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{u}_k) : \underset{\approx}{\varepsilon}(\underset{\sim}{v}_k)\, dxdy - \int_\Omega \Big(\nabla \cdot \underset{\sim}{u}_k\Big) q_k\, dxdy - \int_\Omega \Big(\nabla \cdot \underset{\sim}{v}_k\Big) p_k\, dxdy \\
&\quad -\alpha \sum_{T \in \mathcal{T}^k} h_T^2 \int_T \Big(-\underset{\sim}{\nabla} \cdot \underset{\approx}{\varepsilon}(\underset{\sim}{u}_k) + \underset{\sim}{\nabla} p_k\Big) \cdot \Big(-\underset{\sim}{\nabla} \cdot \underset{\approx}{\varepsilon}(\underset{\sim}{v}_k) + \underset{\sim}{\nabla} q_k\Big)\, dxdy \\
&\quad -\epsilon \int_\Omega p_k q_k\, dxdy
\end{aligned}$$

and

$$\mathcal{F}_{\underset{\sim}{f}}(\underset{\sim}{v}_k, q_k) = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}_k\, dxdy - \alpha \sum_{T \in \mathcal{T}^k} h_T^2 \int_T \underset{\sim}{f} \cdot \Big(-\underset{\sim}{\nabla} \cdot \underset{\approx}{\varepsilon}(\underset{\sim}{v}_k) + \underset{\sim}{\nabla} q_k\Big)\, dxdy.$$

Note that the bilinear form $\mathcal{B}_k$ is symmetric and indefinite.

Introducing the piecewise linear nodal basis functions $\{\underset{\sim}{\varphi}_i\}$ and $\{\psi_j\}$ for $\underset{\sim}{V}_k$ and P_k, respectively, the discrete problem (4) can be written in matrix form as

$$\begin{pmatrix} A & B^t \\ B & -C \end{pmatrix} \begin{pmatrix} \underset{\sim}{u} \\ p \end{pmatrix} = \begin{pmatrix} \underset{\sim}{f} \\ 0 \end{pmatrix} \quad \text{or } \mathcal{A}x = b$$

where $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ and $f = (f_j)$.

Here a_{ij}, b_{ij}, c_{ij} and f_j are given by the following:

$$\begin{aligned}
a_{ij} &= \int_\Omega \underset{\approx}{\varepsilon}(\underset{\sim}{\varphi}_i^k) : \underset{\approx}{\varepsilon}(\underset{\sim}{\varphi}_j^k)\, dxdy, \\
b_{ij} &= -\int_\Omega \Big(\nabla \cdot \underset{\sim}{\varphi}_i^k\Big)\psi_j^k\, dxdy, \\
c_{ij} &= \epsilon \int_\Omega \psi_i^k \psi_j^k\, dxdy + \alpha \sum_{T\in\mathcal{T}^k} h_T^2 \int_T \Big(\underset{\sim}{\nabla}\, \psi_i^k\Big)\cdot\Big(\underset{\sim}{\nabla}\, \psi_j^k\Big)\, dxdy \quad \text{and} \\
f_j &= \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{\varphi}_j^k\, dxdy - \alpha \sum_{T\in\mathcal{T}^k} h_T^2 \int_T \underset{\sim}{f}\cdot\Big(\underset{\sim}{\nabla}\, \psi_j^k\Big)\, dxdy.
\end{aligned}$$

By [8], we guarantee the existence of the solution for the problem (4) by the well-known theory of Babuška (see [5]).

Let us consider the preconditioner of the problem $\mathcal{A}x = b$, where

$$\mathcal{A} = \begin{pmatrix} A & B^t \\ B & -C \end{pmatrix}.$$

Note that if we choose $\alpha = \epsilon$ in C, then $\mathcal{A}$ can be written in the form of

$$\mathcal{A}_* := \begin{pmatrix} A & B^t \\ B & -\epsilon C_* \end{pmatrix}.$$

We choose the positive definite block diagonal preconditioner

$$\tilde{\mathcal{B}}_* := \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{C}_* \end{pmatrix}$$

where $\tilde{A}$ is defined by the multigrid method with a V-cycle including symmetric Gauss-Seidel smoothing and $\tilde{C}_*$ by the diagonal of C.

Since $||p_k||_0^2 \le \sum_{T\in\mathcal{T}^k} h_T^2 ||\nabla p_k||_{0,T}^2 + ||p_k||_0^2$
$\le C||p_k||_0^2 + ||p_k||_0^2$ (by the shape regularity)
$\le (C+1)||p_k||_0^2, \quad \forall p_k \in P_k,$

C_* is spectrally equivalent to the pressure mass matrix M_p. And we know that $\tilde{A}$ and $\tilde{C}_*$ are good preconditioners for A and C_* respectively:

(i) $\exists a_0, a_1 > 0$ such that $a_0^2\, \underset{\sim}{u}_k^t \tilde{A} \underset{\sim}{u}_k \le \underset{\sim}{u}_k^t A \underset{\sim}{u}_k \le a_1^2\, \underset{\sim}{u}_k^t \tilde{A} \underset{\sim}{u}_k, \quad \forall \underset{\sim}{u}_k \in \underset{\sim}{V}_k$

(ii) $\exists c_0, c_1 > 0$ such that $c_0^2\, p_k^t \tilde{C}_* p_k \le p_k^t M_p p_k \le c_1^2\, p_k^t \tilde{C}_* p_k, \quad \forall p_k \in P_k.$

In this case, Klawonn estimates the condition number $\kappa(\tilde{\mathcal{B}}_*^{-1}\mathcal{A}_*)$ (see [10]).

Let

$$\kappa(\tilde{\mathcal{B}}^{-1}\mathcal{A}) := \frac{\lambda_{max}^{abs}}{\lambda_{min}^{abs}} := \frac{\max\{|\lambda| : \lambda \in \sigma(\tilde{\mathcal{B}}^{-1}\mathcal{A})\}}{\min\{|\lambda| : \lambda \in \sigma(\tilde{\mathcal{B}}^{-1}\mathcal{A})\}},$$

where $\sigma(\tilde{\mathcal{B}}^{-1}\mathcal{A})$ denotes the spectrum of $\tilde{\mathcal{B}}^{-1}\mathcal{A}$.

Now we can state the theorem which is in [10].

THEOREM 1.

$$\kappa(\tilde{\mathcal{B}}_*^{-1}\mathcal{A}_*) = \frac{\max\{a_1^2, c_1^2\}}{\min\{a_0^2, c_0^2\}} \frac{1/2 + \sqrt{\beta_1^2 + \frac{1}{4}}}{-1/2 + \sqrt{\beta_0^2 + \frac{1}{4}}}$$

where β_0 is the inf-sup constant of the method and β_1 is the continuity constant of B.

This theorem says that the condition number $\kappa(\tilde{\mathcal{B}}_*^{-1}\mathcal{A}_*)$ is completely determined by the preconditioners $\tilde{A}, \tilde{C}_*$, the condition number of $BA^{-1}B^t$. We also note that the result is independent of the discretization and the constants ϵ, α. Thus we can guarantee that the convergence rate of the Krylov space method will not deteriorate when h, ϵ and α decrease.

3. PCR Method

The PCR(preconditioned conjugate residual) method is an algorithm to solve $\mathcal{A}x = b$ with a symmetric indefinite matrix $\mathcal{A}$ and a positive definite preconditioner $\tilde{\mathcal{B}}$. We will state a stable version that is based on a 3 term recurrence. See [3].

ALGORITHM 1.

Initialization :

$$\begin{aligned} r_0 &:= b - \mathcal{A}x_0 \\ p_{-1} &:= 0 \\ p_0 &:= \tilde{\mathcal{B}}^{-1}r_0 \end{aligned}$$

Iteration :

$$\begin{aligned}
\lambda &:= \frac{r_m^t \tilde{\mathcal{B}}^{-1}\mathcal{A}p_m}{p_m^t \mathcal{A}\tilde{\mathcal{B}}^{-1}\mathcal{A}p_m} \\
x_{m+1} &:= x_m + \lambda p_m \\
r_{m+1} &:= r_m - \lambda \mathcal{A}p_m \\
\alpha_0 &:= \frac{p_m^t \mathcal{A}\tilde{\mathcal{B}}^{-1}\mathcal{A}\tilde{\mathcal{B}}^{-1}\mathcal{A}p_m}{p_m^t \mathcal{A}\tilde{\mathcal{B}}^{-1}\mathcal{A}p_m} \\
\alpha_1 &:= \frac{p_m^t \mathcal{A}\tilde{\mathcal{B}}^{-1}\mathcal{A}\tilde{\mathcal{B}}^{-1}\mathcal{A}p_{m-1}}{p_{m-1}^t \mathcal{A}\tilde{\mathcal{B}}^{-1}\mathcal{A}p_{m-1}} \\
p_{m+1} &:= \tilde{\mathcal{B}}^{-1}\mathcal{A}p_m - \alpha_0 p_m - \alpha_1 p_{m-1}.
\end{aligned}$$

REMARK 1. If we assume that $\tilde{\mathcal{A}}$ is an optimal positive definite preconditioner but choose $\tilde{\mathcal{B}} := c\tilde{\mathcal{A}}, c \in R^+$, then λ will grow in proportional to $c^{-(m+1)}$. This can easily be seen by induction. The easiest way of fixing this phenomenon is to normalize p_{m+1} in every iteration. We have done so in our implementation.

We state the next theorem which can be found in [9].

THEOREM 2. *Let the regular matrix $\mathcal{A}$ be symmetric and $\tilde{\mathcal{B}}$ be positive definite. Then the m-th iterate of* Algorithm 1 *satisfies*

$$||\tilde{\mathcal{B}}^{-1/2}\mathcal{A}(x^m - x^*)||_2 \leq \frac{2c^\mu}{1+c^{2\mu}}||\tilde{\mathcal{B}}^{-1/2}(\mathcal{A}x^0 - b)||_2$$

where $c := \frac{\kappa-1}{\kappa+1}$, $\kappa := \kappa(\tilde{\mathcal{B}}^{-1}\mathcal{A})$ *and* $\frac{m}{2} - 1 < \mu \leq \frac{m}{2}$, $\forall \mu \in Z$.

This theorem says that the convergence rate of the PCR-method is determined by the condition number of the preconditioned system.

4. Experimental Results

We apply our preconditioner to the pure displacement boundary value problem (2). The domain Ω is the unit square, and the body force $f =$

(f_1, f_2) is taken to be as follows:

$$f_1 = \pi^2[2\sin 2\pi y(-1+2\cos 2\pi x) - 0.5\cos\pi(x+y) + \frac{\epsilon}{\epsilon+2}\sin\pi x\sin\pi y]$$
$$f_2 = \pi^2[2\sin 2\pi x(1-2\cos 2\pi y) - 0.5\cos\pi(x+y) + \frac{\epsilon}{\epsilon+2}\sin\pi x\sin\pi y].$$

The initial iterate is $\underset{\sim}{0}$, and the program executes until $||r_k||_2/||r_0||_2$ is less than 10^{-5}, where r_k is the $k-$th residual. The computations were done in double-precision arithmetic. We choose our preconditioners $\tilde{A}$ and $\tilde{C}$ for A and C with a V-cycle multigrid including one symmetric Gauss-Seidel smoothing and a simple diagonal preconditioner, respectively.

First, we show the numerical experiments with the case of $\alpha = \epsilon$ and $\tilde{C}$ =diag (C_*). See Table 1. In this case, Klawonn gives condition number estimates for the preconditioned system and some numerical experiments for the linear system arising from the saddle point problem with the penalty term (see [10]). In [10], Klawonn uses the Taylor-Hood finite element spaces which satisfy the Babuška-Brezzi condition to approximate the velocity and the pressure. In Table 1, we show that the convergence rate of the preconditioned conjugate residual method is independent of the mesh size h for the moderate numbers of Poisson's ratio ν, for example, ν = 0.35, 0.4, 0.45, 0.495, 0.4995. We note that the convergence rate of the preconditioned conjugate residual method is also independent of Poisson's ratio ν. In [10], Klawonn gives the iteration numbers as h decreases only for the case of $\nu = 0.3$.

Next, we give the numerical experiments with the case of $\alpha \neq \epsilon$ and $\tilde{C}$ =diag (C). See Table 2, 3 and 4. In this case, we tested our preconditioned linear system with the various Poisson's ratio ν and the mesh size h. We observe that the convergence rate of the preconditioned conjugate residual method is independent of the mesh size h and Poisson's ratio ν. We show that the preconditioned conjugate residual method is an efficient and robust iterative solver for the pure displacement boundary problem with the least squares term.

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	N=16	N=32	N=64
ν=0.35	30	34	36
ν=0.40	28	31	35
ν=0.45	22	25	28
ν=0.495	23	24	25
ν=0.4995	50	57	56

TABLE 1. Case of $\alpha = \epsilon$

ν	0.3	0.45	0.495	0.4995	0.49995	0.499995	0.4999995
α=1.0	28	39	43	43	43	43	43
α=0.5	23	33	39	39	39	39	39
α=0.1	14	25	30	30	30	30	30
α=0.05	17	22	28	30	30	30	30
α=0.01	21	30	41	42	42	42	42

TABLE 2. Case of $\alpha \neq \epsilon$, $N = 16$, i.e., $h = 1/16$

ν	0.3	0.45	0.495	0.4995	0.49995	0.499995	0.4999995
α=1.0	33	50	54	57	57	57	57
α=0.5	26	40	46	47	47	47	47
α=0.1	15	28	35	36	36	36	36
α=0.05	16	25	32	34	34	34	34
α=0.01	23	34	44	45	45	45	45

TABLE 3. Case of $\alpha \neq \epsilon$, $N = 32$, i.e., $h = 1/32$

ν	0.3	0.45	0.495	0.4995	0.49995	0.499995	0.4999995
α=1.0	34	55	62	65	65	65	65
α=0.5	26	45	51	54	54	54	54
α=0.1	15	30	38	38	38	38	38
α=0.05	18	25	35	36	36	36	36
α=0.01	23	34	45	47	47	47	47

TABLE 4. Case of $\alpha \neq \epsilon$, $N = 64$, i.e., $h = 1/64$

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