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HÖLDER CONTINUITY OF H-SSSI SαS PROCESSES

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ABSTRACT. Let $\{X(t) : t \geq 0\}$ be a symmetric α stable and H-self-similar process with stationary increments. We examine a.s. Hölder unboundedness of $S\alpha S$ H-sssi Chentsov processes and H-sssis Chentsov fields for order $\gamma > H$. Finally, we prove a.s. Hölder continuity of $S\alpha S$ H-sssi processes with ergodic scaling transformations for the case of $H > 1/\alpha$.

1. Introduction

We are interested in $S\alpha S$ H-sssi processes which are symmetric α stable (SαS) and H-self-similar (ss) processes with index H and have stationary increments (si). We know that the existence of moments limits the possible values of H and, consequently, if an H-sssi process is $S\alpha S$ process, $0 < \alpha \leq 2$, then its self-similarity index H is restricted to the interval $(0, 1/\alpha)$ if $\alpha < 1$ and to the interval $(0, 1]$ if $\alpha \geq 1$ ([2], [7]).

Nolan ([6]) gave a necessary and sufficient condition for the Hölder continuity of sample paths of $S\alpha S$ processes when $0 < \alpha < 1$. Takashima ([8]) studied the Hölder continuity of the linear fractional stable processes of the continuous sample paths and Kono and Maejima ([5]) studied the Hölder continuity of the sample paths of the hamonizable fractional stable processes as an application of the Lepage representation.

Chapter 2 is to review some definitions and properties of $S\alpha S$ H-sssis Chentsov random fields. It was introduced by Paul Lévy in 1948 and given a geometric construction by Chentsov in 1957 ([1]). Chentsov's construction allows the field to be defined as $M(V_t)$, $t \geq 0$, where M is a Gaussian random measure and V_t is the set of all hyperplanes separating the origin zero from the point t.

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Shigeo Takenaka ([9]) gives a geometric construction for the Lévy fractional Brownian field with $0 < H \leq 1/2$. In Takenaka ([10]), he defines the (α,H)-Takenaka fields. We define the Chentsov fields by generalizing Chentsov's construction. We let the measure M be $S\alpha S$, $0 < \alpha \leq 2$, and consider measurable set V_t.

In chapter 3, we prove a.s. Hölder unboundedness of $S\alpha S$ H-sssis Chentsov fields for all order $\gamma > H$ and a.s. Hölder continuity of $S\alpha S$ H-sssi processes with ergodic scaling transformations for the case of $H > 1/\alpha$.

Self-similar processes are also related to many problems in time series analysis, i.e., modeling for network traffic and estimating for the intensity of long range dependence ([3], [4], [12]). Readers who are interested in self-similar traffic modeling and analysis are referred to bibliographical guide by Taqqu ([11]) and Willinger, Taqqu and Erramilli ([13]).

2. Preliminaries

A stochastic process $X = \{X(t) : t \geq 0\}$ and for $H > 0, a > 0$, a scaling transformation $S_{H,a}$ of X is defined by

$$(S_{H,a}X)(t) = a^{-H}X(at), \quad t \geq 0.$$

DEFINITION 2.1. A stochastic process $\{X(t) : t \geq 0\}$ is called a self-similar with index $H > 0$ (H-ss) if for any $a > 0$,

$$(S_{H,a}X)(t) \stackrel{d}{=} X(t),$$

where, $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions, we write simply S_H for $S_{H,a}$.

DEFINITION 2.2. The stochastic process $\{X(t) : t \geq 0\}$ has stationary increments (si) if

$$\{X(t+h) - X(h) : t \geq 0\} \stackrel{d}{=} \{X(t) - X(0) : t \geq 0\}, \quad \text{for all } h \geq 0.$$

A positive self-similarity H and stationary increments imply that $X(0) = 0$ a.s. and X is continuous in probability:

$$X(t+h) - X(t) \stackrel{d}{=} X(h) \stackrel{d}{=} h^H X(sgn\ h) \stackrel{d}{\to} 0 \quad \text{as } h \to 0.$$

Thus, we can take a separable version of X to obtain criteria for sample boundedness and sample continuity in terms of finite-dimensional distributions.

DEFINITION 2.3. A stochastic process $\{X(t) : t \geq 0\}$ is called a self-similar with ergodic scaling transformation if S_H is ergodic.

We consider $S\alpha S$ random fields of the form

$$X(t) = \int_E 1_{V_t}(x) M(dx), \quad t \geq 0,$$

where M is a $S\alpha S$ random measure with control measure m and the $V_t^{'}$s are sets parametrized by t.

DEFINITION 2.4. Let $0 < \alpha \leq 2$, $(E, \mathcal{E}, m)$ be a measure space, $\mathcal{E}$ be non-trivial σ-field on E, M be a $S\alpha S$ random measure with control measure m and $\{V_t : t \geq 0\}$ be a family of measurable subsets satisfying

$$m(V_t) < \infty \qquad \text{for all } t \geq 0.$$

The process

$$X(t) = M(V_t), \quad t \geq 0$$

is called a $S\alpha S$ Chentsov process.

LEMMA 2.1. *Let $\{X(t) : t \geq 0\}$ be a $S\alpha S$ H-sssi Chentsov process with control measure m. Then*
(i) $m(V_{at}) = a^{\alpha H} m(V_t), \quad t \geq 0.$
(ii) $m(V_{t+h} \Delta V_h) = m(V_t \Delta V_0), \quad t \geq 0,$
where Δ denotes the symmetric difference.
(iii) $m(V_t \Delta V_s) = c|t-s|^{\alpha H}$, *where* $c = m(V_1 \Delta V_0)$.
(iv) $H \leq \frac{1}{\alpha}$.

PROOF. [7, Proposition 8.2.3, 8.2.4, Corollary 8.2.5] □

Lemma 2.1 (iv) implies that the Lévy fractional Brownian motion can not be represented as a Chentsov process when $1/2 < H < 1$. The Lévy-Chentsov and (α,H)-Takenaka processes provided examples of $S\alpha S$ and H-sssis Chentsov processes with $H = 1/\alpha$ and $0 < H < 1/\alpha$, respectively.

LEMMA 2.2. *Let $\{X(t) : t \geq 0\}$ be a $S\alpha S$ Chentsov process with control measure m. Then*

$$-log E \exp\{i(X(t) - X(s))\} = m(V_t \Delta V_s).$$

PROOF. We know that

$$\begin{aligned} M(V_t) - M(V_s) &= M(V_t \cap V_s) + M(V_t \cap V_s^c) - M(V_s \cap V_t) - M(V_s \cap V_t^c) \\ &= M(V_t \cap V_s^c) - M(V_s \cap V_t^c) \end{aligned}$$

Since the last two terms are independent, we obtain

$$\begin{aligned} -\log E\exp\{i(M(V_t)-M(V_s))\} &= m(V_t\cap V_s^c)+m(V_s\cap V_t^c) \\ &= m((V_t\cap V_s^c)\cup(V_s\cap V_t^c)) \\ &= m(V_t\Delta V_s). \end{aligned}$$

□

The extension of the notion of stationary increments to $\mathcal{R}^n, n>1$, is more delicate. Stationary increments in $\mathcal{R}^1$ means the finite-dimensional distributions of the increments are invariant under translation. Translations are the only Euclidean rigid body motions in $\mathcal{R}^1$, but in $\mathcal{R}^n$, the Euclidean rigid body motions include all rotations and translations.

Let $\mathcal{G}(\mathcal{R}^n)$ denote the group of Euclidean rigid body motions in $\mathcal{R}^n$.

DEFINITION 2.5. The random field $\{X(t):t\in\mathcal{R}^n\}$ has stationary increments in the strong sense (sis) if

$$\{X(g(t))-X(g(0)):t\in\mathcal{R}^n\}\overset{d}{=}\{X(t)-X(0):t\in\mathcal{R}^n\},$$

for all Euclidean rigid body motions $g\in\mathcal{G}(\mathcal{R}^n)$.

Let $0<\alpha\le 2$, $(E, \mathcal{E}, m)$ be a measure space, $\mathcal{E}$ be non-trivial σ-field on E, M be a $S\alpha S$ random measure with control measure m and $\{V_t, t\in\mathcal{R}^n\}$ be a family of measurable subsets satisfying

$$m(V_t)<\infty \qquad \text{for all } t\in\mathcal{R}^n.$$

The random field

$$X(t)=M(V_t),\ \ t\in\mathcal{R}^n$$

is called a $S\alpha S$ Chentsov field.

3. Hölder continuity of $S\alpha S$ H-sssi processes

3.1. Unboundedness of the Chentsov field

We define

$$\| X \|_\alpha=[-\log E\exp iX]^{1/\alpha},$$

for $S\alpha S$ process X. We will consider sample path property of $S\alpha S$ H-sssi Chentsov process $X(t)$. We examine whether a stochastic Hölder condition of order γ, i.e., there is an a.s. finite, positive random variable $C(\omega)$ such that whenever h is small and $t\ge 0$,

$$|X(t+h)-X(t)|\le C(\omega)\cdot h^\gamma$$

holds or not.

THEOREM 3.1. (i) *Let $\{X(t) : t \geq 0\}$ be $S\alpha S$ H-sssi Chentsov process. Then*

$$\frac{|X(t+h) - X(t)|}{h^{\gamma}}$$

is a.s. unbounded as $h \to 0$ for all $\gamma > H$.
(ii) *Let $\{X(t) : t \in \mathcal{R}^n\}$ be $S\alpha S$ H-sssis Chentsov field. Then*

$$\frac{|X(t+h) - X(t)|}{\| h \|^{\gamma}}$$

is a.s. unbounded as $\| h \| \to 0$ for all $\gamma > H$.

PROOF. (i) Since $X(t)$ is $S\alpha S$ Chentsov process,

$$\begin{aligned} \| X(t) \|_{\alpha} &= [-\log E \exp iX(t)]^{1/\alpha} \\ &= [-\log E \exp iM(V_t)]^{1/\alpha} \\ &= (m(V_t))^{1/\alpha}. \end{aligned}$$

By Lemma 2.1 (iii) and Lemma 2.2,

$$\begin{aligned} \| X(t+h) - X(t) \|_{\alpha} &= [-\log E \exp i\{X(t+h) - X(t)\}]^{1/\alpha} \\ &= (m(V_{t+h} \Delta V_t))^{1/\alpha} \\ &= (ch^{\alpha H})^{1/\alpha} = c^{1/\alpha} h^H. \end{aligned}$$

Therefore,

$$h^{\gamma} = o(\| X(t+h) - X(t) \|_{\alpha})$$

for all $\gamma > H$.
By [6, Theorem 3.1], a uniform stochastic Hölder condition of order γ fails. Hence, $\dfrac{|X(t+h) - X(t)|}{h^{\gamma}}$ is a.s. unbounded as $h \to 0$ for all $\gamma > H$.

(ii) We get $m(V_{g(t)} \Delta V_{g(0)}) = m(V_t \Delta V_0)$, for all $g \in \mathcal{G}(\mathcal{R}^n)$, $t \in \mathcal{R}^n$ and $m(V_t \Delta V_s) = c \| t - s \|^{\alpha H}$, where $c = m(V_{e_0} \Delta V_0)$ and $e_0 = (1, 0, \cdots, 0)$. Therefore, by the same argument as (i), we can prove the assertion. □

3.2. Hölder continuity of $S\alpha S$ H-sssi processes

Chentsov fields lives in $H \leq 1/\alpha$. Now, we consider Hölder continuity of $S\alpha S$ H-sssi processes for the case of $H > 1/\alpha$.

For any $\gamma \geq 0$ and $c \geq 0$, define

$$E_{\gamma,c} = \{\text{there is } \delta > 0 \text{ such that } |X(t)| \leq ct^{\gamma} \text{ for all } 0 < t < \delta\}.$$

LEMMA 3.1. *Let $\{X(t) : t \geq 0\}$ be an H-ss with ergodic scaling transformation S_H. Then*
(i) $P(E_{\gamma,c}) = 0$ or 1,
(ii) $\limsup_{t\to 0} \dfrac{|X(t)|}{t^\gamma} = 0$ *a.s. or* ∞ *a.s.*

PROOF. (i) Suppose that for some $\delta > 0$, $|X(t)| \leq ct^\gamma$ for all $0 < t < \delta$. Then

$$|X(t)| = |X(at/a)| = a^H|X(t/a)| \leq ct^\gamma a^{H-\gamma} \quad \text{for any } a > 0.$$

Put $u = \dfrac{t}{a}$. Then

$$|(S_H X)(u)| = a^{-H}|X(au)| \leq a^{-H}c(au)^\gamma a^{H-\gamma} = cu^\gamma$$

for $0 < u < \dfrac{\delta}{a}$. Thus, we know that

$$E_{\gamma,c} \subset S_H^{-1}E_{\gamma,c} \text{ and } P(E_{\gamma,c}\Delta S_H^{-1}E_{\gamma,c}) = 0.$$

Since S_H is ergodic, we get $P(E_{\gamma,c}) = 0$ or 1.

(ii) Let $c_\gamma = \sup\{c \geq 0 : P(E_{\gamma,c}) = 0\}$. Then $\limsup_{t\to 0} \dfrac{|X(t)|}{t^\gamma} = c_\gamma$ a.s. By (i),

$$\limsup_{t\to 0} \frac{|X(t)|}{t^\gamma} = 0 \text{ a.s. or } \infty \text{ a.s.}$$

□

THEOREM 3.2. *Let $\{X(t) : t \geq 0\}$ be $S\alpha S$ H-sssi process and $H > 1/\alpha$. Suppose that S_H is ergodic. Then*

$$P\left\{\lim_{\delta\to 0}\sup_{|h|<\delta} \frac{|X(t+h) - X(t)|}{h^\gamma} = 0\right\} = 1$$

for any $\gamma < H$ and $t \geq 0$; so the sample paths of X are γ-Hölder continuous.

PROOF. Let

$$Y(t) = \limsup_{h\to 0} \frac{|X(t+h) - X(t)|}{h^\gamma}.$$

Then

$$
\begin{aligned}
Y(at) &\stackrel{d}{=} \limsup_{h\to 0} \frac{|X(at+h)-X(at)|}{h^{\gamma}} \\
&\stackrel{d}{=} \limsup_{h\to 0} \frac{a^{H}\{|X(t+h/a)-X(t)|\}}{a^{\gamma}(h/a)^{\gamma}} \\
&\stackrel{d}{=} \limsup_{u\to 0} a^{H-\gamma}\frac{|X(t+u)-X(t)|}{u^{\gamma}} \\
&\stackrel{d}{=} a^{H-\gamma}Y(t).
\end{aligned}
$$

By stationary increment property of X, $\{Y(t) : t \geq 0\}$ is $(H-\gamma)$-sssi, i.e.,

$$Y(at+b) \stackrel{d}{=} a^{H-\gamma}Y(t).$$

With $a = 1$ at 0, we know that $Y(t) \stackrel{d}{=} Y(0)$. By Lemma 3.1, it is enough to prove that $Y(0) < \infty$ a.s.

$$
\begin{aligned}
&\sum_{n=1}^{\infty} P(\max_{2^{-n-1}\leq t\leq 2^{-n}} |X(t)| \geq 2^{-n\gamma}) \\
&\leq \sum_{n=1}^{\infty} P(\max_{2^{-n-1}\leq t\leq 2^{-n}} |X(t)-X(2^{-n-1})| \geq 2^{-n\gamma-1}) \\
&\quad +P(|X(2^{-n-1})| \geq 2^{-n\gamma-1}).
\end{aligned}
$$

Choose $\beta \in (1/H, \alpha)$. Then the first probability of last term is

$$
\begin{aligned}
P &\ (\max_{2^{-n-1}\leq t\leq 2^{-n}} |X(t)-X(2^{-n-1})| \geq 2^{-n\gamma-1}) \\
&= P(\max_{0\leq t\leq 2^{-n-1}} |X(t)| \geq 2^{-n\gamma-1}) \\
&= P(\max_{0\leq t\leq 1} |X(2^{-n-1}t)| \geq 2^{-n\gamma-1}) \\
&= P(\max_{0\leq t\leq 1} |X(t)| \geq 2^{-n\gamma-1}2^{(n+1)H}) \\
&\leq \frac{C'_{\beta H}E[\max_{0\leq t\leq 1}|X(t)|]^{\beta}}{2^{n(H-\gamma)\beta}} \\
&\leq \frac{C''_{\beta H}E[|X(1)|^{\beta}]}{2^{n(H-\gamma)\beta}}.
\end{aligned}
$$

The second probability is

$$\begin{aligned} P(|X(1/2^{-n-1})| \geq 2^{-n\gamma-1}) &= P((1/2^{-n-1})^H |X(1)| \geq 2^{-n\gamma-1}) \\ &= P(|X(1)| \geq 2^{-n\gamma-1} \cdot 2^{nH+H}) \\ &\leq C'''_{\beta H} 2^{-n(H-\gamma)\beta}. \end{aligned}$$

That is, for any $\gamma < H$,

$$\sum_{n=1}^{\infty} P(\max_{2^{-n-1} \leq t \leq 2^{-n}} |X(t)| \geq 2^{-n\gamma}) < C_{\beta H} \sum_{n=1}^{\infty} 2^{-n(H-\gamma)\beta} < \infty.$$

Applying Borel-Cantelli's lemma, with probability one, there is a number N such that

$$\max_{2^{-n-1} \leq t \leq 2^{-n}} |X(t)| \leq 2^{-n\gamma} \quad \text{for } n \geq N.$$

That is,

$$\frac{|X(t)|}{t^\gamma} \leq 2^\gamma \quad \text{for } 0 < t \leq 2^{-N}.$$

Thus, we get

$$Y(0) = \limsup_{t \to 0} \frac{|X(t)|}{t^\gamma} < \infty \text{ a.s.}$$

□

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