Comm. Korean Math. Soc. 15 (2000), No. 1, pp. 103–109

ON A GENERALIZATION OF FENCHEL'S THEOREM

Y. D. CHAI AND MOONJEONG KIM

ABSTRACT. In this paper, we present the proof of generalized Fenchel's theorem by estimating the Gauss-Kronecker curvature of the tube of a nondegenerate closed curve in R^n.

1. Introduction

Fenchel's well-known theorem on the total curvature $\int \kappa\, ds$ of a simple closed curve α in R^3 consists of the inequality $\int \kappa\, ds \geq 2\pi$, together with the statement that the equality holds if and only if the curve is a plane convex curve. After Fenchel's original work [2], there has been a variety of settings ([3], [4], [5], [6], [7]).

In this paper, we take natural coordinates of the tube of a given nondegenerate curve in Euclidean n-space to estimate total curvature of the tube. This estimate provides a proof of generalized Fenchel's theorem:

Let α be a closed curve in R^n. Then $\int \kappa ds \geq 2\pi$, with the equality if and only if the curve is a convex plane curve.

2. Preliminaries

Let α be a curve in R^{n+1}, which, for convenience, is supposed to be parametrized by arc length, so that $\|\alpha(s)\| = 1$. Suppose that the curve

---

Received February 25, 1999. Revised May 26, 1999.

1991 Mathematics Subject Classification: 53C20, 52A10.

Key words and phrases: tube, nondegenerate curve, total curvature, Gauss-Kronecker curvature.

This research is supported by the Korea Research Foundation made in the program of 1998, Project No. 1998-015-D00034.

is non-degenerate, or equivalently

$$\alpha(s) \wedge \alpha'(s) \wedge \cdots \wedge \alpha^{(n)}(s)$$

is not identically zero.
For a Frenet frame

$$(\alpha(s), e_1(s), \cdots, e_n(s))$$

of the curve α, the Frenet equations are

$$\begin{aligned} \frac{d\alpha}{ds} &= e_1 \\ \frac{de_i}{ds} &= -\kappa_{i-1}(s)e_{i-1} + \kappa_i(s)e_{i+1}. \end{aligned}$$

The κ_i $(i = 1, \cdots, n)$ uniquely determine α up to rigid motion. Furthermore, $\{e_2, e_3, \cdots, e_n\}$ span the normal space to the curve.

Let S be an oriented hypersurface in R^{n+1} and let $p \in S$. Let Z be any non-zero normal vector field on S such that $N = Z/\|Z\|$ and let $\{\nu_1, \cdots \nu_n\}$ be any basis for S_p and K be the Gauss - Kronecker curvature. Using the fact that $dN(\nu)$ and $\nabla_\nu N$ have the same vector part for all $\nu \in S_p$, $p \in S$, we find that

$$\begin{aligned} K(p) &= (-1)^n \det \begin{pmatrix} \nabla_{\nu_1} N \\ \vdots \\ \nabla_{\nu_n} N \\ N(p) \end{pmatrix} \Bigg/ \det \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_n \\ N(p) \end{pmatrix} \\ &= \det \begin{pmatrix} dN(\nu_1) \\ \vdots \\ dN(\nu_n) \\ N^{S^n}(N(p)) \end{pmatrix} \Bigg/ \det \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_n \\ N(p) \end{pmatrix}, \end{aligned}$$

where N^{S^n} is the standard orientation on S^n.

LEMMA 1. *Let T be a torus in R^{n+1}. The total curvature of T satisfies the inequality;*

$$\int_T |K| d\sigma \geq 2Vol(S^n),$$

where $d\sigma$ is the volume form on T.

PROOF. Let T^+ be the region of T where K is nonnegative and let T^- be the region of T where K is nonpositive. Then

$$\int_T |K|d\sigma = \int_{T^+} Kd\sigma + \int_{T^-} |K|d\sigma.$$

Note that $\int_{T^-} Kd\sigma$ represents the area of the image under the Gauss map (counting multiplicities) of the part of T where $K \geq 0$.
But each half-line through the origin in R^{n+1} appears at least once as a normal direction of T^+. So the Gauss map $G : T^+ \longrightarrow S^n$ covers the entire unit sphere S^n; hence $\int_{T^+} Kd\sigma \geq Vol(S^n)$.
Since $\int_T Kd\sigma = \int_{T^+} Kd\sigma + \int_{T^-} Kd\sigma$ and $\int_T Kd\sigma = 0$,

$$\int_{T^-} Kd\sigma = -\int_{T^+} Kd\sigma \leq -Vol(S^n).$$

But $\int_{T^-} |K|d\sigma = -\int_{T^-} Kd\sigma$. Thus $\int_{T^-} |K|d\sigma \geq Vol(S^n)$. Therefore $\int_T |K|d\sigma = \int_{T^+} Kd\sigma + \int_{T^-} |K|d\sigma \geq 2Vol(S^n)$. □

LEMMA 2. *Let T be a torus in R^{n+1}. Then $\int_T |K|d\sigma = 2Vol(S^n)$ if and only if T is a regular torus.*

PROOF. ("Only if "part)
Note that

$$2Vol(S^n) = \int_T |K|d\sigma = \int_{T^+} Kd\sigma + \int_{T^-} |K|d\sigma.$$

From the fact that $\int_T Kd\sigma = \int_{T^+} Kd\sigma + \int_{T^-} Kd\sigma$ and $\int_T Kd\sigma = 0$, we have

$$\int_{T^-} Kd\sigma = -\int_{T^+} Kd\sigma$$

and

$$\int_{T^-} Kd\sigma = -\int_{T^-} Kd\sigma = \int_{T^-} -Kd\sigma = \int_{T^-} |K|d\sigma.$$

Thus $\int_{T^-} Kd\sigma = -Vol(S^n)$ and $\int_{T^+} Kd\sigma = Vol(S^n)$.
Hence, the spherical image of Gauss map on T^- covers S^n without overlapping and the spherical image of Gauss map on T^+ also covers S^n without overlapping. This is possible only when T is a regular torus. "If" part is obvious. □

3. Result

Now we will derive the lower bound of the total curvature of α in R^{n+1}.

THEOREM 1. *For a simple closed curve α in R^{n+1},*

$$\int_\alpha k\, ds \geq 2\pi,$$

with the equality if and only if α is a convex plane curve.

PROOF. Let T be the tube of radius t around $\alpha(s)$. For a sufficiently small t, T is a smooth hypersurface parametrized by

$$X(s,\theta_1,\cdots,\theta_{n-1}) = \alpha(s) + tN(s,\theta_1,\cdots,\theta_{n-1}),$$
$$0 \leq \theta_1 \leq 2\pi,\ 0 \leq \theta_2,\cdots,\theta_{n-1} \leq \pi,$$

where

$$\begin{aligned} N(s,\theta_1,\cdots,\theta_{n-1}) \;=\; & (\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1})e_2(s) \\ & +(\cos\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1})e_3(s) \\ & +\cdots+(\cos\theta_{i-2}\sin\theta_{i-1}\cdots\sin\theta_{n-1})e_i(s) \\ & +\cdots+\cos\theta_{n-1}e_{n+1}(s). \end{aligned}$$

Then

$$d\sigma = \det\begin{pmatrix} X_s \\ X_{\theta_1} \\ \vdots \\ X_{\theta_{n-1}} \\ N \end{pmatrix} ds d\theta_1\cdots d\theta_{n-1} \quad \text{and}$$

$$K = \det\begin{pmatrix} N_s \\ N_{\theta_1} \\ \vdots \\ N_{\theta_{n-1}} \\ N \end{pmatrix} \Bigg/ \det\begin{pmatrix} X_s \\ X_{\theta_1} \\ \vdots \\ X_{\theta_{n-1}} \\ N \end{pmatrix}$$

Hence

$$Kd\sigma = \begin{pmatrix} N_s \\ N_{\theta_1} \\ \vdots \\ N_{\theta_{n-1}} \\ N \end{pmatrix} ds d\theta_1 \cdots d\theta_{n-1}.$$

On the other hand,

$$\begin{aligned} N_s &= (-\kappa_1 \sin\theta_1 \cdots \sin\theta_{n-1})e_1 + \cdots \\ &\quad + (\kappa_{i-1}\cos\theta_{i-3}\sin\theta_{i-2}\cdots\sin\theta_{n-1} - \kappa_i \cos\theta_{i-1}\sin\theta_i \cdots \sin\theta_{n-1})e_i \\ &\quad + \cdots + (\kappa_n \cos\theta_{n-2}\sin\theta_{n-1})e_{n+1}. \end{aligned}$$

$$\begin{aligned} N_{\theta_i} &= (\sin\theta_1\sin\theta_2\cdots\cos\theta_i\sin\theta_{i+1}\cdots\sin\theta_{n-1})e_2 + \cdots \\ &\quad + (-\sin\theta_i \sin\theta_{i+1}\cdots\sin\theta_{n-1})e_{i+2}, \end{aligned}$$

where $1 \le i \le n-1$.

$$\det \begin{pmatrix} N_s \\ N_{\theta_1} \\ \vdots \\ N_{\theta_{n-1}} \\ N \end{pmatrix} = -\kappa_1 \sin\theta_1 \sin^2\theta_2 \cdots \sin^{n-1}\theta_{n-1}.$$

So for the region T^+ where the Gauss-Kronecker curvature of T is non-negative, we have

$$\begin{aligned} &\int_{T^+} Kd\sigma \\ &= \int_0^\ell \kappa_1 ds \int_0^\pi \int_0^\pi \cdots \int_0^\pi \int_0^\pi \sin\theta_1 \sin^2\theta_2 \cdots \sin^{n-1}\theta_{n-1} d\theta_1 \cdots d\theta_{n-1}. \end{aligned}$$

Hence,

$$(1) \quad \int_{T^+} K\, d\sigma = \begin{cases} \dfrac{2^k \cdot \pi^{k-1}}{1\cdot 3\cdot 5\cdots(2k-1)} \displaystyle\int_0^\ell \kappa_1 ds, & n = 2k \\ \dfrac{\pi^k}{1\cdot 2\cdot 3\cdots k} \displaystyle\int_0^\ell \kappa_1 ds, & n = 2k+1 \end{cases}$$

where $k = 1, 2, \cdots$. On the other hand, we have the following inequality from lemma 1

$$\int_{T^+} K \, d\sigma \geq Vol(S^n). \tag{2}$$

From (1) and (2), in R^{n+1},

$$\int_0^\ell \kappa_1 \, ds \geq \begin{cases} \dfrac{1 \cdot 3 \cdot 5 \cdots (2k-1)\pi^{\frac{3}{2}}}{2^{k-1}(k-\frac{1}{2})!}, & n = 2k \\ \dfrac{2 \cdot 1 \cdot 2 \cdot 3 \cdots k\pi}{k!}, & n = 2k+1 \end{cases} \tag{3}$$

where $k = 1, 2, \cdots$.

Either case of the right hand side of (3) gives 2π. Hence $\int_0^l \kappa_1 ds \geq 2\pi$. Observe that if α is not a plane convex curve, T cannot be a regular torus. So from lemma 1 and lemma 2, $\int_{T^+} kd\sigma > Vol(S^n)$.

Now it follows $\int_0^l k_1 d\sigma > 2\pi$ from (1) and (2). This completes the proof of our theorem. □

THEOREM 2. *If $\kappa(s) \leq 1/R$ for a closed curve α in R^n, R being a constant, then α has a length $L \geq 2\pi R$. The shortest closed curve with curvature $\kappa(s) \leq 1/R$ is a circle of radius R.*

PROOF. Note that

$$L = \int_0^L ds \geq \int_0^L R\kappa ds = R \int_0^L \kappa ds \geq 2\pi R.$$

So any such a curve has length at least $2\pi R$. The length of the shortest closed curve with curvature $k(s) \leq \frac{1}{R}$ must be $2\pi R$ or $\int_0^L kds = 2\pi$, since a circle of radius R satisfies $L = 2\pi R$. From our theorem 1, the shortest curve must be a plane convex curve. Now the Schur's theorem [1] says that the equality of the distance between their end points holds if and only if two arcs are congruent. Since the circle of radius R is the case, we conclude that it must itself be a circle because both of the distances between their end points are 0. □

References

[1] S. S. Chern, *Studies in global geometry and analysis*, M.A.A. **4** (1967), 16-56.

[2] W. Fenchel, *Uber Krummung und Windung geschlossener Raumkurven*, Math. Ann. **101** (1929), 238 - 252.

[3] ———, *The differential geometry of closed space curves,* Bull. Amer. Math. Soc. **57** (1951), 44-54.
[4] H. W. Guggenheimer, *Differential Geometry,* McGraw-Hill, New York, 1963, pp. 251-252.
[5] D. Laugwitz, *Differential and Riemannian Geometry,* Academic Press, New York, 1965, pp. 203-204.
[6] R. Osserman, *Curvature in the Eighties,* Amer. Monthly (1990), 731-756.
[7] K. Voss, *Eine Bemerkung über die Totalkrümmung geschlossener Raumkurven,* Arch. Math. **6** (1955), 259-263.

Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea
E-mail: ydchai@yurim.skku.ac.kr
kmj@math.skku.ac.kr