Comm. Korean Math. Soc. 15 (2000), No. 1, pp. 91–102

REMARKS ON DENJOY-DUNFORD AND DENJOY-PETTIS INTEGRALS

Chun-Kee Park

Abstract. In this paper we generalize some results of R. A. Gordon ([4]) and J. L. Gamez and J. Mendoza ([3]) and prove some convergence theorems for Denjoy-Dunford and Denjoy-Pettis integrable functions.

1. Introduction

In 1989 Gordon ([4]) introduced the concepts of Denjoy-Dunford and Denjoy-Pettis integrals for Banach-valued functions and proved some properties of those integrals. Gamez and Mendoza improved some results of Gordon. Gordon ([5]) also obtained some convergence theorems for Denjoy integrable real-valued functions. In this paper we generalize some results of Gordon ([4]) and Gamez and Mendoza ([3]) and obtain some convergence theorems for Denjoy-Dunford and Denjoy-Pettis integrable functions.

2. Preliminaries

Throughout this paper X will denote a real Banach space and X^* its dual.

Definition 2.1 ([4]). Let $F : [a,b] \to X$ and let E be a subset of $[a,b]$.

(a) The function F is BV on E if $\sup \left\{ \sum_i \|F(d_i) - F(c_i)\| \right\}$ is finite where the supremum is taken over all finite collections $\{[c_i, d_i]\}$ of

Received August 18, 1999. Revised December 22, 1999.
1991 Mathematics Subject Classification: 28B05, 46G10, 26A39.
Key words and phrases: Denjoy-Dunford integrability, Denjoy-Pettis integrability, Denjoy integrability.

nonoverlapping intervals that have endpoints in E.

(b) The function F is AC on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_i \|F(d_i) - F(c_i)\| < \epsilon$ whenever $\{[c_i, d_i]\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_i (d_i - c_i) < \delta$.

(c) The function F is BVG on E if E can be expressed as a countable union of sets on each of which F is BV.

(d) The function F is ACG on E if F is continuous on E and if E can be expressed as a countable union of sets on each of which F is AC.

DEFINITION 2.2 ([4]). Let $\{F_\alpha\}$ be a family of functions from $[a, b]$ to X and let E be a subset of $[a, b]$. The family $\{F_\alpha\}$ is uniformly BVG (ACG) on E if each F_α is BVG (ACG) on E and if each perfect set in E contains a portion on which every F_α is BV (AC).

DEFINITION 2.3 ([4]). Let $F : [a, b] \to X$ and let $t \in (a, b)$. A vector z in X is the approximate derivative of F at t if there exists a measurable set $E \subset [a, b]$ that has t as a point of density such that $\lim\limits_{\substack{s \to t \\ s \in E}} \frac{F(s) - F(t)}{s - t} = z$. We will write $F'_{ap}(t) = z$.

A function $f : [a, b] \to \mathbb{R}$ is Denjoy integrable on $[a, b]$ if there exists an ACG function $F : [a, b] \to \mathbb{R}$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$. The function f is Denjoy integrable on the set $E \subset [a, b]$ if $f\chi_E$ is Denjoy integrable on $[a, b]$.

DEFINITION 2.4 ([4]). (a) A function $f : [a, b] \to X$ is Denjoy-Dunford integrable on $[a, b]$ if for each x^* in X^* the function x^*f is Denjoy integrable on $[a, b]$ and if for every interval I in $[a, b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for all x^* in X^*.

(b) A function $f : [a, b] \to X$ is Denjoy-Pettis integrable on $[a, b]$ if f is Denjoy-Dunford integrable on $[a, b]$ and if $x_I^{**} \in X$ for every interval I in $[a, b]$.

Throughout this paper $(DD) \int_a^b f$ and $(DP) \int_a^b f$ will denote the Denjoy-Dunford integral and the Denjoy-Pettis integral of f on $[a, b]$, respectively.

3. Denjoy-Dunford and Denjoy-Pettis Integrability

In this section we obtain some properties of Denjoy-Dunford and Denjoy-Pettis integrable functions.

THEOREM 3.1. (a) *If* $f : [a,b] \to X$ *is Denjoy-Dunford integrable on* $[a,b]$*, then* f *is weakly measurable.*

(b) *If* $f : [a,b] \to X$ *is bounded and Denjoy-Dunford integrable on* $[a,b]$*, then* f *is Dunford integrable on* $[a,b]$*.*

PROOF. (a) If $f : [a,b] \to X$ is Denjoy-Dunford integrable on $[a,b]$, then $x^*f : [a,b] \to \mathbb{R}$ is Denjoy integrable on $[a,b]$ for all $x^* \in X^*$. Hence x^*f is measurable for all $x^* \in X^*$ ([4, Theorem 12 (a)]). Therefore f is weakly measurable.

(b) If $f : [a,b] \to X$ is bounded and Denjoy-Dunford integrable on $[a,b]$, then $x^*f : [a,b] \to \mathbb{R}$ is bounded and Denjoy integrable on $[a,b]$ for all $x^* \in X^*$. Hence x^*f is Lebesgue integrable on $[a,b]$ for all $x^* \in X^*$([5, Theorem 15.9]). Therefore f is Dunford integrable on $[a,b]$. □

It follows immediately from Pettis Measurability Theorem and Theorem 3.1 that if X is a separable Banach space and $f : [a,b] \to X$ is Denjoy-Dunford integrable on $[a,b]$ then f is measurable.

THEOREM 3.2 ([3]). *A function* $f : [a,b] \to X$ *is Denjoy-Dunford integrable on* $[a,b]$ *if and only if* x^*f *is Denjoy integrable on* $[a,b]$ *for all* $x^* \in X^*$.

THEOREM 3.3. *Suppose that* $f : [a,b] \to X$ *is Denjoy-Dunford integrable on each interval* $[c,d] \subset (a,b)$*. If* $\lim_{\substack{c\to a^+\\ d\to b^-}} (DD)\int_c^d f$ *exists in norm in* X^{**}*, then* f *is Denjoy-Dunford integrable on* $[a,b]$ *and* $(DD)\int_a^b f = \lim_{\substack{c\to a^+\\ d\to b^-}} (DD)\int_c^d f$*.*

PROOF. Let $\lim_{\substack{c\to a^+\\ d\to b^-}} (DD)\int_c^d f = x_0^{**}$, where $x_0^{**} \in X^{**}$. By hypothesis, for each $x^* \in X^*$, $x^*f : [a,b] \to \mathbb{R}$ is Denjoy integrable on each interval $[c,d] \subset (a,b)$ and

$$\langle x^*, x_0^{**} \rangle = \lim_{\substack{c \to a^+ \\ d \to b^-}} \left\langle x^*, (DD) \int_c^d f \right\rangle = \lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d x^* f.$$

Hence for each $x^* \in X^*$, $x^* f$ is Denjoy integrable on $[a,b]$ and $\int_a^b x^* f = \lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d x^* f$ ([5, Theorem 15.12]). Thus f is Denjoy-Dunford integrable on $[a,b]$ by Theorem 3.2 and

$$\langle x^*, x_0^{**} \rangle = \lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d x^* f = \int_a^b x^* f = \left\langle x^*, (DD) \int_a^b f \right\rangle$$

for all $x^* \in X^*$. Hence $(DD) \int_a^b f = x_0^{**} = \lim_{\substack{c \to a^+ \\ d \to b^-}} (DD) \int_c^d f$. □

DEFINITION 3.4. Let $\{f_\alpha\}$ be a family of Denjoy-Dunford integrable functions from $[a,b]$ to X. The family $\{f_\alpha\}$ is uniformly Denjoy-Dunford integrable on $[a,b]$ if for each perfect set $E \subset [a,b]$ there exists an interval $[c,d] \subset [a,b]$ with $c,d \in E$ and $E \cap (c,d) \neq \phi$ such that every f_α is Dunford integrable on $E \cap [c,d]$ and for every α the series $\sum_n \left\| (DD) \int_{c_n}^{d_n} f_\alpha \right\|$ converges where $[c,d] - E = \cup_n (c_n, d_n)$.

THEOREM 3.5. *Let $\{f_\alpha\}$ be a family of Denjoy-Dunford integrable functions from $[a,b]$ to X and let $F_\alpha(t) = (DD) \int_a^t f_\alpha$ for each α. If the family $\{F_\alpha\}$ is uniformly ACG on $[a,b]$, then the family $\{f_\alpha\}$ is uniformly Denjoy-Dunford integrable on $[a,b]$.*

PROOF. Suppose that the family $\{F_\alpha\}$ is uniformly ACG on $[a,b]$ and let E be a perfect set in $[a,b]$. Then there exists an interval $[c,d] \subset [a,b]$ with $c,d \in E$ and $E \cap (c,d) \neq \phi$ such that every F_α is AC on

$E \cap [c,d]$. Fix α. For each $x^* \in X^*$ the function $F_\alpha x^*$ is also AC on $E \cap [c,d]$ and $\langle x^*, F_\alpha(t)\rangle = \int_a^t x^* f_\alpha, t \in [a,b]$. For each $x^* \in X^*$ let $G_{\alpha,x^*} : [c,d] \to \mathbb{R}$ be the function that equals $F_\alpha x^*$ on E and is linear on the intervals contiguous to E. Then the function G_{α,x^*} is AC on $[c,d]$ for each $x^* \in X^*$ ([4, Theorem 3]). Hence G'_{α,x^*} exists almost everywhere on $[c,d]$ and is Lebesgue integrable on $[c,d]$ for each $x^* \in X^*$. Since $G'_{\alpha,x^*} = (F_\alpha x^*)' = x^* f_\alpha$ almost everywhere on $E \cap [c,d]$ for each $x^* \in X^*$, $x^* f_\alpha$ is Lebesgue integrable on $E \cap [c,d]$ for each $x^* \in X^*$. Thus f_α is Dunford integrable on $E \cap [c,d]$. Since F_α is BV on $E \cap [c,d]$, the series $\sum_n \|F_\alpha(d_n) - F_\alpha(c_n)\| = \sum_n \left\|(DD)\int_{c_n}^{d_n} f_\alpha\right\|$ converges where $[c,d] - E = \cup_n (c_n, d_n)$. Since this is valid for each α, the family $\{f_\alpha\}$ is uniformly Denjoy-Dunford integrable on $[a,b]$. □

THEOREM 3.6 ([5]). *Let E be a bounded, closed subset of $\mathbb{R}$ with bounds a and b and let $((a_k, b_k))$ be the sequence of intervals contiguous to E in $[a,b]$. Suppose that $f : [a,b] \to \mathbb{R}$ is Denjoy integrable on E and on each interval $[a_k, b_k]$. If $\lim_{k\to\infty} \omega\left(\int_{a_k}^t f, [a_k, b_k]\right) = 0$ and the series $\sum_{k=1}^{\infty} \left|\int_{a_k}^{b_k} f\right|$ converges, then f is Denjoy integrable on $[a,b]$ and*

$$\int_a^b f = \int_a^b f\chi_E + \sum_{k=1}^{\infty} \int_{a_k}^{b_k} f.$$

THEOREM 3.7. *Let E be a bounded, closed subset of $\mathbb{R}$ with bounds a and b and let $((a_k, b_k))$ be the sequence of intervals contiguous to E in $[a,b]$. Suppose that $f : [a,b] \to X$ is Denjoy-Dunford integrable on E and on each interval $[a_k, b_k]$. If $\lim_{k\to\infty} \omega\left((DD)\int_{a_k}^t f, [a_k, b_k]\right) = 0$ and the series $\sum_{k=1}^{\infty} \left\|(DD)\int_{a_k}^{b_k} f\right\|$ converges, then f is Denjoy-Dunford integrable on $[a,b]$ and*

$$(DD)\int_a^b f = (DD)\int_a^b f\chi_E + \sum_{k=1}^{\infty}(DD)\int_{a_k}^{b_k} f.$$

PROOF. For each $x^* \in X^*$, x^*f satisfies the hypothesis of Theorem 3.6. Hence by Theorem 3.6, for each $x^* \in X^*$, x^*f is Denjoy integrable on $[a,b]$ and

$$\int_a^b x^*f = \int_a^b x^*f\chi_E + \sum_{k=1}^{\infty}\int_{a_k}^{b_k} x^*f.$$

By Theorem 3.2, f is Denjoy-Dunford integrable on $[a,b]$ and

$$\left\langle x^*, (DD)\int_a^b f\right\rangle = \left\langle x^*, (DD)\int_a^b f\chi_E\right\rangle + \sum_{k=1}^{\infty}\left\langle x^*, (DD)\int_{a_k}^{b_k} f\right\rangle$$

for each $x^* \in X^*$. Since $\sum_{k=1}^{\infty}\left\|(DD)\int_{a_k}^{b_k} f\right\|$ converges, we have

$$\sum_{k=1}^{\infty}\left\langle x^*, (DD)\int_{a_k}^{b_k} f\right\rangle = \left\langle x^*, \sum_{k=1}^{\infty}(DD)\int_{a_k}^{b_k} f\right\rangle$$

for each $x^* \in X^*$. Hence we have $(DD)\int_a^b f = (DD)\int_a^b f\chi_E + \sum_{k=1}^{\infty}(DD)\int_{a_k}^{b_k} f$. □

DEFINITION 3.8. Let $\{F_\alpha\}$ be a family of functions from $[a,b]$ to X and let E be a subset of $[a,b]$. The family $\{F_\alpha\}$ is equi AC on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_i \|F_\alpha(d_i) - F_\alpha(c_i)\| < \epsilon$ for all α whenever $\{[c_i, d_i]\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_i (d_i - c_i) < \delta$.

DEFINITION 3.9. Let $\{F_\alpha\}$ be a family of functions from $[a,b]$ to X and let E be a closed subset of $[a,b]$ with its bounds c and d. The family $\{F_\alpha\}$ is equi BV on E if each F_α is BV on E and for each $\epsilon > 0$ there exists a positive interger N such that $\sum_{n=N}^{\infty} \|F_\alpha(d_n) - F_\alpha(c_n)\| < \epsilon$ for all α where $[c,d] - E = \cup_{n=1}^{\infty}(c_n, d_n)$.

LEMMA 3.10. *Let $\{F_\alpha\}$ be a family of functions from $[a,b]$ to X and let E be a closed subset of $[a,b]$ with its bounds c and d. If $\{F_\alpha\}$ is equi AC on E, then $\{F_\alpha\}$ is equi BV on E.*

PROOF. Suppose that $\{F_\alpha\}$ is equi AC on E. Then each F_α is BV on E. Let $\epsilon > 0$ be given and let $[c,d] - E = \cup_{i=1}^{\infty}(c_i, d_i)$. Since $\{F_\alpha\}$ is equi AC on E, there exists $\delta > 0$ such that $\sum_i \|F_\alpha(d_i') - F_\alpha(c_i')\| < \epsilon/2$ for all α whenever $\{[c_i', d_i']\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_i (d_i' - c_i') < \delta$. Since $\sum_{i=1}^{\infty}(d_i - c_i) < \infty$, there exists a positive integer N such that $\sum_{i=N}^{\infty}(d_i - c_i) < \delta$. Hence we have

$$n \geq N \Rightarrow \sum_{i=N}^{n} \|F_\alpha(d_i) - F_\alpha(c_i)\| < \frac{\epsilon}{2}$$

for all α. Letting $n \to \infty$, we have

$$\sum_{i=N}^{\infty} \|F_\alpha(d_i) - F_\alpha(c_i)\| \leq \frac{\epsilon}{2} < \epsilon$$

for all α. Therefore $\{F_\alpha\}$ is equi BV on E. □

DEFINITION 3.11. Let $\{F_\alpha\}$ be a family of functions from $[a,b]$ to X. The family $\{F_\alpha\}$ is equi ACG on a subset E of $[a,b]$ if each F_α is ACG on E and if each perfect set in E contains a portion on which the family $\{F_\alpha\}$ is equi AC.

THEOREM 3.12. *Let $\{f_\alpha\}$ be a family of Denjoy-Dunford integrable functions from $[a,b]$ to X and let $F_\alpha(t) = (DD)\int_a^t f_\alpha$ for each α. If the family $\{F_\alpha\}$ is equi ACG on $[a,b]$, then for each perfect set $E \subset [a,b]$ there exists a portion $E \cap (c,d)$ of E such that every f_α is Dunford integrable on $E \cap [c,d]$ and $\sum_n \left\| (DD)\int_{c_n}^{d_n} f_\alpha \right\|$ converges uniformly on α where $[c,d] - E = \cup_n (c_n, d_n)$.*

PROOF. Suppose that $\{F_\alpha\}$ is equi ACG on $[a,b]$ and let $E \subset [a,b]$ be a perfect set. Then $\{F_\alpha\}$ is uniformly ACG on $[a,b]$. By Theorem 3.5, there exists a portion $E \cap (c',d') \neq \phi$ of E with $c', d' \in E$ such that every f_α is Dunford integrable on $E \cap [c',d']$. Since $\{F_\alpha\}$ is equi ACG on $[a,b]$, for the perfect set $E \cap [c',d']$ there exists a portion $E \cap (c,d) \neq \phi$ of $E \cap [c',d']$ with $c, d \in E$ such that $\{F_\alpha\}$ is equi AC on $E \cap [c,d]$. Each f_α is also Dunford integrable on $E \cap [c,d]$. By Lemma 3.10, $\{F_\alpha\}$ is equi BV on $E \cap [c,d]$. Hence for each $\epsilon > 0$ there exists a positive integer N such that $\sum_{n=N}^{\infty} \|F_\alpha(d_n) - F_\alpha(c_n)\| < \epsilon$ for all α where $[c,d] - E = \cup_n (c_n, d_n)$. Therefore $\sum_n \left\| (DD)\int_{c_n}^{d_n} f_\alpha \right\|$ converges uniformly on α where $[c,d] - E = \cup_n (c_n, d_n)$. □

4. Convergence Theorems

In this section we obtain some results of the convergence of Denjoy-Dunford and Denjoy-Pettis integrable functions.

THEOREM 4.1 ([5]). *Let (f_n) be a sequence of Denjoy integrable functions from $[a,b]$ to $\mathbb{R}$, and let $F_n(t) = \int_a^t f_n$ for each n, and suppose that (f_n) converges pointwise to f on $[a,b]$. If (F_n) is equicontinuous and equi ACG on $[a,b]$, then f is Denjoy integrable on $[a,b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.*

THEOREM 4.2. *Let (f_n) be a sequence of Denjoy-Dunford integrable functions from $[a,b]$ to X, and let $F_n(t) = (DD)\int_a^t f_n$ for each n, and suppose that (f_n) converges pointwise to f on $[a,b]$. If (F_n) is equicontinuous and equi ACG on $[a,b]$, then f is Denjoy-Dunford integrable on $[a,b]$ and $(DD)\int_a^b f = \lim_{n\to\infty}(DD)\int_a^b f_n$ in the weak* topology of X^{**}.*

PROOF. We note that (x^*f_n) and (x^*F_n) satisfy the hypothesis of Theorem 4.1 for every $x^* \in X^*$. Hence x^*f is Denjoy integrable on $[a,b]$ and $\int_a^b x^*f = \lim_{n\to\infty}\int_a^b x^*f_n$ for every $x^* \in X^*$. By Theorem 3.2, f is Denjoy-Dunford integrable on $[a,b]$ and $\left\langle x^*, (DD)\int_a^b f\right\rangle = \lim_{n\to\infty}\left\langle x^*, (DD)\int_a^b f_n\right\rangle$ for every $x^* \in X^*$. Hence $(DD)\int_a^b f = \lim_{n\to\infty}(DD)\int_a^b f_n$ in the weak* topology of X^{**}. □

THEOREM 4.3 ([4]). *Let X be weakly sequentially complete and let $f : [a,b] \to X$ be Denjoy-Dunford integrable on $[a,b]$. If f is measurable, then f is Denjoy-Pettis integrable on $[a,b]$.*

THEOREM 4.4. *Let X be weakly sequentially complete, and let (f_n) be a sequence of measurable Denjoy-Dunford integrable functions from $[a,b]$ to X, and let $F_n(t) = (DD)\int_a^t f_n$ for each n, and suppose that (f_n) converges pointwise to f on $[a,b]$. If (F_n) is equicontinuous and equi ACG on $[a,b]$, then f is Denjoy-Pettis integrable on $[a,b]$ and $(DP)\int_a^b f = \lim_{n\to\infty}(DP)\int_a^b f_n$ in the weak topology of X.*

PROOF. By Theorem 4.2, f is Denjoy-Dunford integrable on $[a,b]$ and $\left\langle x^*, (DD)\int_a^b f\right\rangle = \lim_{n\to\infty}\left\langle x^*, (DD)\int_a^b f_n\right\rangle$ for every $x^* \in X^*$. By Theorem 4.3, f_n is Denjoy-Pettis integrable on $[a,b]$ for each n. Since each f_n is measurable and (f_n) converges pointwise to f on $[a,b]$, f is also measurable on $[a,b]$. By Theorem 4.3, f is also Denjoy-Pettis integrable on $[a,b]$ and $(DP)\int_a^b f = \lim_{n\to\infty}(DP)\int_a^b f_n$ in the weak topology of X.□

THEOREM 4.5. *Let (f_n) be a sequence of Denjoy-Dunford integrable functions from $[a,b]$ to a reflexive Banach space X, and let $F_n(t) = (DD)\int_a^t f_n$ for each n, and suppose that (f_n) converges pointwise to f on $[a,b]$. If (F_n) is equicontinuous and equi ACG on $[a,b]$, then f is Denjoy-Dunford integrable on $[a,b]$ and there is a sequence (g_n) with $g_n \in co\{f_n | n = 1,2,3,\dots\}$ such that $(DD)\int_a^b f = \lim_{n\to\infty}(DD)\int_a^b g_n$ in norm.*

PROOF. By Theorem 4.2, f is Denjoy-Dunford integrable on $[a,b]$ and $(DD)\int_a^b f = \lim_{n\to\infty}(DD)\int_a^b f_n$ in the weak* topology of X^{**}. Since X is reflexive, $(DD)\int_a^b f = \lim_{n\to\infty}(DD)\int_a^b f_n$ weakly in X^{**}. Thus $\lim_{n\to\infty}\left((DD)\int_a^b f_n - (DD)\int_a^b f\right) = 0$ weakly in X^{**}. By Corollary 2[1, p11], there is a sequence (x_n^{**}) of convex combinations of the $(DD)\int_a^b f_n - (DD)\int_a^b f$ such that $\lim_{n\to\infty}\|x_n^{**}\| = 0$. For each n, let $x_n^{**} = \sum_{i=1}^{k(n)} \alpha_{n_i}\left((DD)\int_a^b f_{n_i} - (DD)\int_a^b f\right)$, where $\alpha_{n_i} \geq 0$ for each i and $\sum_{i=1}^{k(n)} \alpha_{n_i} = 1$. Then

$$\begin{aligned}
\lim_{n\to\infty}\|x_n^{**}\| &= \lim_{n\to\infty}\left\|\sum_{i=1}^{k(n)} \alpha_{n_i}\left((DD)\int_a^b f_{n_i} - (DD)\int_a^b f\right)\right\| \\
&= \lim_{n\to\infty}\left\|(DD)\int_a^b\left(\sum_{i=1}^{k(n)} \alpha_{n_i} f_{n_i}\right) - (DD)\int_a^b f\right\| \\
&= 0.
\end{aligned}$$

For each n, let $g_n = \sum_{i=1}^{k(n)} \alpha_{n_i} f_{n_i}$. Then for each n, $g_n \in co\{f_n | n = 1,2,3,\dots\}$ and $(DD)\int_a^b f = \lim_{n\to\infty} (DD)\int_a^b g_n$ in norm. $\square$

THEOREM 4.6. *Let X be weakly sequentially complete, and let (f_n) is a sequence of measurable Denjoy-Dunford integrable functions from $[a,b]$ to X, and let $F_n(t) = (DD)\int_a^t f_n$ for each n, and suppose that (f_n) converges pointwise to f on $[a,b]$. If (F_n) is equicontinuous and equi ACG on $[a,b]$, then f is Denjoy-Pettis integrable on $[a,b]$ and there is a sequence (g_n) with $g_n \in co\{f_n | n = 1,2,3,\dots\}$ such that $(DP)\int_a^b f = \lim_{n\to\infty} (DP)\int_a^b g_n$ in norm.*

PROOF. By Theorem 4.4, f is Denjoy-Pettis integrable on $[a,b]$ and $(DP)\int_a^b f = \lim_{n\to\infty} (DP)\int_a^b f_n$ weakly in X. By Corollary 2 ([1, p. 11]), there is a sequence (x_n) of convex combinations of the $(DP)\int_a^b f_n - (DP)\int_a^b f$ such that $\lim_{n\to\infty} \|x_n\| = 0$. Using the same method in the proof of Theorem 4.5, we obtain a sequence (g_n) with $g_n \in co\{f_n | n = 1,2,3,\dots\}$ such that $(DP)\int_a^b f = \lim_{n\to\infty} (DP)\int_a^b g_n$ in norm. $\square$

References

[1] J. Diestel, *Sequences and Series in Banach spaces*, Grad. Texts in Math., vol. 92, Springer, 1984.

[2] J. Diestel and J. J. Uhl Jr., *Vector measures*, Math. Surveys No. 15, Amer. Math. Soc., 1977.

[3] J. L. Gamez and J. Mendoza, *On Denjoy-Dunford and Denjoy-Pettis integrals*, Studia Math. **130** (1998), 115-133.

[4] R. A. Gordon, *The Denjoy extension of the Bochner, Pettis and Dunford integrals*, Studia Math. **92** (1989), 73-91.

[5] ———, *The integrals of Lebesgue, Denjoy, Perron and Henstock*, Grad. Stud. Math., vol. 4, Amer. Math. Soc., 1994.

[6] D. W. Solomon, *Denjoy integration in abstract spaces*, Mem. Amer. Math. Soc vol. 85, Amer. Math. Soc., 1969.

Department of Mathematics
Kangwon National University
Chuncheon 200-701, Korea
E-mail: ckpark@cc.kangwon.ac.kr