Comm. Korean Math. Soc. **15** (2000), No. 1, pp. 77–90

# BANACH ALGEBRA OF FUNCTIONALS OVER PATHS IN ABSTRACT WIENER SPACE

YEON-HEE PARK

ABSTRACT. In this paper, we will establish the existence theorem of the operator valued function space integral over paths in abstract Wiener space under the general conditions rather than the known conditions.

## 1. Introduction

In their paper ([3]), Johnson and Lapidus introduced a family of $\{\mathcal{A}_t, t > 0\}$ of commutative Banach algebras of functionals on Wiener space and showed that for every $F \in \mathcal{A}_t$, the functional integral $K_\lambda^t(F)$ exists and is given by a time ordered perturbation expansion which serves to disentangle, in the sense of Feynman's operational calculus. In [4], Kuelbs and LaPage suggested the existence of mean zero, stationary increment, Gaussian measure over paths in abstract Wiener space $C_0(B)$. In [5], Ryu found the formula similar to Wiener integration formula and established the existence theorem of the operator-valued function space integral of functionals on $C_0(B)$. In [7], Yoo and Ryu introduced the (s-w)-integral and proved the existence of the operator-valued function space integral represented by (s-w)-integral.

In this paper, we show that for certain functions $F$ given by (1) in section 3, the operator valued function space integral $K_\lambda^t(F)$ exists and can be disentangled by a time ordered perturbation expansion or generalized Dyson series (see Theorem 16 below). Also we introduce a family $\{\mathcal{B}_t, t > 0\}$ of Banach algebras of paths in abstract Wiener space and show that for every $F \in \mathcal{B}_t$, the functional integral $K_\lambda^t(F)$ exists and $\|K_\lambda^t(F)\| \le \|F\|_t$.

Received August 9, 1999. Revised October 18, 1999.
1991 Mathematics Subject Classification: 28B05, 46G12.
Key words and phrases: abstract Wiener space, scale-invariant measurable subset, (s-w)-integral.

We restrict attention to real $\lambda > 0$ whereas that restriction is not made in [3].

## 2. Definitions and Preliminaries

Let $(B, \beta(B), m)$ be an abstract wiener space and let $C_0(B)$ denote the set of all $B$-valued continuous functions on [0,t] into $B$ which vanish at origin. From [4] it follows that $C_0(B)$ is a real separable Banach space with the norm

$$\|x\|_{C_0(B)} = \sup_{s\in[0,t]} \|x(s)\|_B$$

and the minimal $\sigma$- algebra making the mapping $x \longrightarrow x(s)$ measurable consists of the Borel subsets of $C_0(B)$. Moreover the Brownian motion in $B$ induces a probability measure $m_B$ on $(C_0(B), \beta(C_0(B))$ which is mean zero Gaussian.

We begin with introducing a concrete form of $m_B$ in [5]. Let $\vec{s} = (s_1, \cdots, s_n)$ be given with $0 = s_0 < s_1 < \cdots < s_n < t$ and let $T_{\vec{s}} : B^n \longrightarrow B^n$ be defined by

$$\begin{aligned} &T_{\vec{s}}(y_1, \cdots, y_n) \\ &= \left(\sqrt{s_1 - s_0}y_1, \sqrt{s_1 - s_0}y_1 + \sqrt{s_2 - s_1}y_2, \cdots, \sum_{k=1}^{n} \sqrt{s_k - s_{k-1}}y_k\right). \end{aligned}$$

Then we define a Borel measure $\nu_{\vec{s}}$ on $\beta(B^n)$ by $\nu_{\vec{s}}(E) = (\times_1^n m)(T_{\vec{s}}^{-1}(E))$ for every $E \in \beta(B^n)$. Let $f_{\vec{s}} : C_0(B) \longrightarrow B^n$ be the function with

$$f_{\vec{s}}(y) = (y(s_1), \cdots, y(s_n)).$$

For Borel subsets $E_1, \cdots, E_n$ of $B$, $f_{\vec{s}}^{-1}(\times_1^n E_i)$ is called the $I$-set and then the collection $\mathcal{I}$ of all $I$-sets is an algebra. We define a set function $m_B$ on $\mathcal{I}$ by $m_B(f_{\vec{s}}^{-1}(\times_1^n E_i)) = \nu_{\vec{s}}(\times_1^n E_i)$. Then $m_B$ is well defined and countably additive on $\mathcal{I}$. Using the Caratheodory process, we have a Borel measure $m_B$ on $\beta(C_0(B))$.

Now we introduce integration formula which plays a key role in this paper. We easily obtain by the change of variable theorem.

LEMMA 1. *Let $\vec{s} = (s_1, \cdots, s_n)$ be given with $0 = s_0 < s_1 < \cdots < s_n \leq t$ and let $f : B^n \longrightarrow C$ be a Borel measurable function. Then*

$$\int_{C_0(B)} f(x(s_1), \cdots, x(s_n))dm_B(x) \stackrel{*}{=} \int_{B^n} f \circ T_{\vec{s}}(y_1, \cdots, y_n)d(\times_1^n m)(y_1, \cdots, y_n)$$

*where by $\stackrel{*}{=}$ we mean that if either side exists then both sides exist and they are equal.*

Next we give some definitions and facts from [7].

DEFINITION 2. Let $L_{p,\infty}(B)(1 \leq p < \infty)$ be the class of all $C$-valued Borel measurable function $\psi$ on $B$ such that for each $\lambda > 0$, $\psi(\lambda(\cdot))$ is $m$-integrable and

$$\|\psi\|_{p,\infty} := \sup_{\lambda>0} \|\psi(\lambda(\cdot))\|_p := \sup \left[ \int_B |\psi(\lambda x)|^p dm(x) \right]^{\frac{1}{p}} < \infty.$$

For $f$ and $g$ in $L_{p,\infty}(B)$, we say that $f$ is equivalent to $g$, denoted by $f \sim g$, if $\{\lambda x \in B | f(x) \neq g(x)\}$ is an $m$-null set for all $\lambda > 0$. Clearly $\sim$ is an equivalent relation on $L_{p,\infty}(B)$. Hence we obtain a quotient space $L_{p,\infty}(B)/\sim$ which we denote by $L_{p,\infty}(B)$.

THEOREM 3. *$L_{p,\infty}(B)$ is a Banach space with norm $\|\cdot\|_{p,\infty}$, but it is not separable.*

DEFINITION 4. For $\lambda > 0$, we define an operator $C_\lambda$ on $L_{p,\infty}(B)$ given by

$$(C_\lambda \psi)(x) = \int_B \psi(\lambda^{-1/2} x_1 + x)dm(x_1)$$

for $\psi \in L_{p,\infty}(B)$.

LEMMA 5. *For $\lambda > 0$, $C_\lambda$ is a bounded linear operator from $L_{p,\infty}(B)$ into itself. Moreover $\|C_\lambda\| \leq 1$.*

REMARK 6. 1. $C_\lambda$ is not strongly continuous.
2. Let $\lambda, \mu > 0$, let $\psi \in L_{p,\infty}(B)$ and $x \in B$. Then $[(C_\lambda \circ C_\mu)\psi](x) = C_{\frac{\lambda\mu}{\lambda+\mu}}(\psi)(x)$.
3. Let $C_\lambda = C^*_{1/\lambda}$ for $\lambda > 0$. Then $C^*_\lambda$ has the semigroup property with respect to $\lambda$.

DEFINITION 7. For a bounded Borel measurable functional $\theta_1$ on $B$, we define the multiplication operator $M_{\theta_1}$ by $(M_{\theta_1}\psi)(x) = \theta_1(x)\psi(x)$ for $\psi \in L_{p,\infty}(B)$. Let $\theta : [0,t] \times B \longrightarrow C$ be a bounded Borel measurable function. Let $\theta(s)$ denote the operator $M_{\theta(s,\cdot)}$ of multiplication by $\theta(s,\cdot)$, acting in $L_{p,\infty}(B)$.

REMARK 8. Let $C(B)$ be the space of all $B$-valued continuous functions on [0,t]. $z$ in $C(B)$ has a unique decomposition $z = x + y$ where $x \in C_0(B)$ and $y \in B$.

DEFINITION 9. Fix $t > 0$. Let $F : C(B) \longrightarrow C$ be a function, and $\lambda > 0$, $\psi \in L_{p,\infty}(B)$ and $y \in B$. We consider the expression

$$(K_\lambda^t(F)\psi)(y) = \int_{C_0(B)} F(\lambda^{-1/2}x + y)\psi(\lambda^{-1/2}x(t) + y)dm_B(x).$$

If $K_\lambda^t(F)$ exists and $K_\lambda^t(F)$ is a bounded linear operator from $L_{p,\infty}(B)$ into itself for all $\lambda > 0$, then we say that the operator valued function space integral $K_\lambda^t(F)$ exists for all $\lambda > 0$.

DEFINITION 10. Let $(\Omega, \mu)$ be a measure space and let $f : \Omega \longrightarrow \mathcal{L}(L_{p,\infty}(B), L_{p,\infty}(B))$, the space of all bound linear operator from $L_{p,\infty}(B)$ $(1 \le p < \infty)$ to itself, be a function. We say that $f$ is (s-w)-integrable if there exists $U \in \mathcal{L}(L_{p,\infty}(B), L_{p,\infty}(B))$ such that for $\psi \in L_{p,\infty}(B)$, $\varphi \in L_{q,\infty}(B)$ with $\frac{1}{p} + \frac{1}{q} = 1$, $\nu$ a Borel measure on $(0, +\infty), \lambda > 0$,

$$\begin{aligned}\int_{(0,+\infty)} \int_{\Omega_\lambda} &[U\psi](x)\varphi(x)dm_\lambda(x)d\nu(\lambda) \\ &= \int_\Omega \int_{(0,+\infty)} \int_{\Omega_\lambda} [f(\omega)\psi](x)\varphi(x)dm_\lambda(x)d\nu(\lambda)d\mu(\omega).\end{aligned}$$

In this case, we write $U = (s-w) - \int_\Omega f(\omega)d\mu(\omega)$.

THEOREM 11. *The (s-w)-integral is well defined.*

THEOREM 12. *If $f$ is (s-w)-integrable on $\Omega$, $f$ is bounded and $A$ is a measurable subset of $\Omega$, then $f$ is (s-w)-integrable on $A$. Moreover for $\psi$ in $L_{p,\infty}(B)$,*

$$\left[(s-w) - \int_A f(\omega)d\mu(\omega)\right]\psi(x) = \int_A [f(\omega)\psi](x)d\mu(\omega) \quad s - a.e.x.$$

THEOREM 13. *Let $f$ be (s-w)-integrable on $\Omega$ such that $\|f\|$ is bounded. Then*

$$\left\|(s-w)-\int_{\Omega} f(\omega)d\mu(\omega)\right\| \le \|f\|_{\infty}|\mu|(\Omega).$$

THEOREM 14. *If $f$ is (s-w)-integrable, $f$ is bounded and $(A_i)$ is a sequence of pairwise disjoint Borel measurable subsets of $\Omega$, then*

$$(s-w)-\int_{\cup_{i=1}^{\infty}A_i} f(\omega)(x)d\mu(\omega) = \sum_{i=1}^{\infty}(s-w)-\int_{A_i} f(\omega)(x)d\mu(\omega)$$

*in the uniform topology.*

THEOREM 15. *Let $(f_n)$ be a sequence of (s-w)-integrable functions on $\Omega$ with such that $\|f_n\|$ is bounded and $(f_n)$ converges to $f$ uniformly in the uniform operator topology. Then $f$ is (s-w)-integrable on $\Omega$. Moreover,*

$$\lim_{n\to\infty}(s-w)-\int_{\Omega} f_n(\omega)d\mu(\omega) = (s-w)-\int_{\Omega} f(\omega)d\mu(\omega).$$

## 3. Operator valued function space integral and Banach algebra

In this section we adopt the following notations and assumptions. Let $\theta_u : [0,t]\times B \longrightarrow C$ be a bounded Borel measurable function for $u = 1,\cdots,m$ and let $M(0,t)$ be the space of complex Borel measures $\eta$ on the open interval $(0,t)$. Then every measure $\eta \in M(0,t)$ has a unique decomposition, $\eta = \mu + \nu$, into a continuous part $\mu$ and a discrete part $\nu$ [3].

By contrast to [5], many measures and potentials may be involved; moreover, the discrete part of each measure is unrestricted. First we consider the functionals of type

$$F(x) := \prod_{u=1}^{m}\int_{(0,t)} \theta_u(s,x(s))d\eta_u(s), \tag{1}$$

where $\eta_u \in M(0,t)$, $\theta_u$ are given as above for $u = 1,\cdots,m$ and $x \in C(B)$. Let $\eta_u = \mu_u + \nu_u$. For each $u = 1,\cdots,m$, we write

$$\nu_u = \sum_{p=1}^{\infty} \omega_{p:u}\delta_{\tau_{p:u}} \tag{2}$$

where $\{\tau_{p;u}\}_{p=1}^{\infty}$ is a sequence from $(0,t)$ and $\{\omega_{p:u}\}_{p=1}^{\infty}$ is a sequence from C such that

$$\|\nu_u\| = \sum_{p=1}^{\infty} |\omega_{p:u}| < \infty.$$

Given $k$ between 0 and m, $[k:m]$ will denote the collection of all subsets of size $k$ of the set of integer $\{1,\cdots,m\}$. If $\{\alpha_1,\cdots,\alpha_k\} \in [k:m]$, we shall always write $\{\alpha_{k+1},\cdots,\alpha_m\} = \{1,\cdots,m\}/\{\alpha_1,\cdots,\alpha_k\}$.

Now we state and prove one theorem of our main results.

THEOREM 16. *Let $F$ be defined by* (1). *Then $K_\lambda^t(F)$ exists for all $\lambda > 0$ such that*

$$\begin{aligned}(K_\lambda^t(F))\\ =\sum_{k=0}^{m} \sum_{\{\alpha_1,\cdots,\alpha_k\}\in[k:m]} \sum_{p_{k+1}=1} \cdots \sum_{p_m=1} \sum_{\rho\in S_k} \sum_{j_1+\cdots+j_{m-k+1}=k} \Big(\prod_{u=k+1}^{m} \omega_{p_{\sigma(u)}:\alpha_{\sigma(u)}}\Big)\\ (s-w) - \int_{\Delta_{k;j_1,\cdots,j_{m-k+1}}(\rho)} [L_k\cdots L_m(s_1,\cdots,s_k)]d(\times_{u=1}^{k}\mu_{\alpha_{\rho(u)}})(s_{\rho(u)})\end{aligned} \tag{3}$$

*where for each fixed $k \in \{0,\cdots,m\}$, $\rho$ ranges through the group $S_k$ of permutations of $\{1,\cdots,m\}$ and*

$$\begin{aligned}\Delta_{k;j_1,\cdots,j_{m-k+1}}(\rho) &= \{(s_1,\cdots,s_k)\in(0,t)^k | 0 < s_{\rho(1)} < \cdots < s_{\rho(j_1)}\\ &< \tau_{p_{\sigma(k+1)};\alpha_{\sigma(k+1)}} < s_{\rho(j_1+1)} < \cdots < s_{\rho(j_1+j_2)} < \tau_{p_{\sigma(k+2)};\alpha_{\sigma(k+2)}}\\ &< s_{\rho(j_1+j_2+1)} < \cdots < s_{\rho(j_1+j_2+\cdots+j_{m-k})} < \tau_{p_{\sigma(m)};\alpha_{\sigma(m)}}\\ &< s_{\rho(j_1+\cdots+j_{m-k}+1)} < \cdots < s_{\rho(k)} < t\}.\end{aligned} \tag{4}$$

*In addition, for $(s_1,\cdots,s_k) \in \Delta_{k;j_1,\cdots,j_{m-k+1}}(\rho)$ and $r = k,\cdots,m$,*

$$\begin{aligned}L_r &= \theta_{\alpha_{\sigma(r)}}(\tau_{p_{\sigma(r)};\alpha_{\sigma(r)}}) \circ C_{d_{r,1}} \circ \theta_{\alpha_{\rho(j_1+\cdots+j_{r-k}+1)}}(s_{\rho(j_1+\cdots+j_{r-k}+1)}) \circ C_{d_{r,2}}\\ &\circ\theta_{\alpha_{\rho(j_1+\cdots+j_{r-k}+2)}}(s_{\rho(j_1+\cdots+j_{r-k}+2)}) \circ \cdots \circ C_{d_{r,j_{r-k+1}}}\\ &\circ\theta_{\alpha_{\rho(j_1+\cdots+j_{r-k+1})}}(s_{\rho(j_1+\cdots+j_{r-k+1})}) \circ C_{d_{r+1,0}}\end{aligned} \tag{5}$$

*where*

$$
\begin{aligned}
&d_{r,1} = \lambda[s_{\rho(j_1+\cdots+j_{r-k}+1)} - \tau_{p_{\sigma(r)}}]^{-1},\\
(6)\quad &d_{r+1,0} = \lambda[\tau_{p_{\sigma(r+1)}} - (s_{\rho(j_1+\cdots+j_{r-k+1})}]^{-1},\\
&d_{r,i} = \lambda[s_{\rho(j_1+\cdots+j_{r-k}+i)} - s_{\rho(j_1+\cdots+j_{r-k}+i-1)}]^{-1} \quad (i = 2,\cdots,j_{r-k+1}).
\end{aligned}
$$

*Here, $\sigma$ is a permutation of $\{k+1,\cdots,m\}$ such that*

$$\tau_{p_{\sigma(k+1)};\alpha_{\sigma(k+1)}} \le \cdots \le \tau_{p_{\sigma(m)};\alpha_{\sigma(m)}}.$$

*Conventions: $\tau_{p_{\sigma(k)};\alpha_{\sigma(k)}} = 0$; $\tau_{p_{\sigma(m+1)};\alpha_{\sigma(m+1)}} = t$ and $\theta_{\alpha_{\sigma(k)}}(\tau_{p_{\sigma(k)};\alpha_{\sigma(k)}}) = 1$, the identity operator. We take $j_0 = 0$; then when $r = k$, it is reasonable to interpret $j_1 + \cdots + j_{r-k} + 1$ as 1 and $j_{r-k} = 0$. Further, for all $\lambda > 0$*

$$\|K_\lambda^t(F)\| \le \prod_{u=1}^{m}(\sup|\theta_u|\|\eta_u\|). \tag{7}$$

PROOF. Let $\lambda > 0$ be given and $\psi$ be in $L_{p\infty}(B)$. Then for $y$ in $B$ except for some Borel scale invariant null subset,

$$|F(\lambda^{-1/2}x + y)\psi(\lambda^{-1/2}x(t) + y)| \le \prod_{u=1}^{m}(\sup|\theta_u|\|\eta_u\|)|\psi(\lambda^{-1/2}x(t) + y)|$$

for $x \in C_0(B)$. Using by Lemma 1 and the generalized Minkowski's inequality and the Jensen's inequality, we have

$$
\begin{aligned}
&\|K_\lambda^t(F)\psi\|_{p\infty}\\
&= \sup_{\mu>0}\Big[\int_B |(K_\lambda^t(F)\psi)(\mu y)|^p dm(y)\Big]^{\frac{1}{p}}\\
&\le \prod_{u=1}^{m}(\sup|\theta_u|\|\eta_u\|)\sup_{\mu>0}\Big[\int_B\Big(\int_{C_0(B)}|\psi(\lambda^{-1/2}x(t) + \mu y)|dm_B(x)\Big)^p dm(y)\Big]^{\frac{1}{p}}\\
&= \prod_{u=1}^{m}(\sup|\theta_u|\|\eta_u\|)\sup_{\mu>0}\Big[\int_B\Big(\int_B|\psi(\lambda^{-1/2}\sqrt{t}x_1 + \mu y)|dm(x_1)\Big)^p dm(y)\Big]^{\frac{1}{p}}\\
&\le \prod_{u=1}^{m}(\sup|\theta_u|\|\eta_u\|)\sup_{\mu>0}\Big[\int_B\int_B|\psi(\lambda^{-1/2}\sqrt{t}x_1 + \mu y)|^p dm(x_1)dm(y)\Big]^{\frac{1}{p}}\\
&= \prod_{u=1}^{m}(\sup|\theta_u|\|\eta_u\|)\sup_{\mu>0}\Big[\int_B|\psi(\sqrt{t/\lambda + \mu^2}z)|^p dm(z)\Big]^{\frac{1}{p}}\\
&= \prod_{u=1}^{m}(\sup|\theta_u|\|\eta_u\|)\|\psi\|_{p\infty}.
\end{aligned}
$$

Therefore we obtain (7).

Also for $y$ in $B$ except for some Borel scale invariant null subset, for $\varphi$ in $L_{q\infty}(B)$ and for any Borel measure $\nu$ on $(0,\infty)$

$$\int_0^\infty \int_{\Omega_\lambda} (K_\lambda^t(F)\psi)(y)\varphi(y)dm_\lambda(y)d\nu(\lambda)$$

$$\overset{(1)}{=} \int_0^\infty \int_{\Omega_\lambda} \int_{C_0(B)} \prod_{u=1}^m \Big[\int_{(0,t)} \theta_u(s, \lambda^{-1/2}x(s)+y)d\eta_u(s)\Big] \cdot \psi(\lambda^{-1/2}x(t)+y)\varphi(y)dm_B(x)dm_\lambda(y)d\nu(\lambda)$$

$$\overset{(2)}{=} \int_0^\infty \int_{\Omega_\lambda} \int_{C_0(B)} \prod_{u=1}^m \Big[\int_{(0,t)} \theta_u(s, \lambda^{-1/2}x(s)+y)d\mu_u(s) + \sum_{p=1}^\infty \omega_{p;u}\theta_u(\tau_{p;u}, \lambda^{-1/2}x(\tau_{p;u})+y)\Big] \cdot \psi(\lambda^{-1/2}x(t)+y)\varphi(y)dm_B(x)dm_\lambda(y)d\nu(\lambda)$$

$$\overset{(3)}{=} \int_0^\infty \int_{\Omega_\lambda} \int_{C_0(B)} \Big\{\sum_{k=0}^m \sum_{\{\alpha_1,\cdots,\alpha_k\}\in[k:m]} \sum_{p_{k+1}=1} \cdots \sum_{p_m=1} \Big[\prod_{u=1}^k \int_{(0,t)} \theta_{\alpha_u}(s_u, \lambda^{-1/2}x(s_u)+y)d\mu_{\alpha_u}(s_u)\Big] \cdot \Big[\prod_{u=k+1}^m \omega_{p_{\sigma(u)};\alpha_{\sigma(u)}}\theta_{\alpha_{\sigma(u)}}(\tau_{p_{\sigma(u)};\alpha_{\sigma(u)}}, \lambda^{-1/2}x(\tau_{p_{\sigma(u)};\alpha_{\sigma(u)}})+y)\Big] \cdot \psi(\lambda^{-1/2}x(t)+y)\Big\}\varphi(y)dm_B(x)dm_\lambda(y)d\nu(\lambda)$$

$$\overset{(4)}{=} \int_0^\infty \int_{\Omega_\lambda} \sum_{k=0}^m \sum_{\{\alpha_1,\cdots,\alpha_k\}\in[k:m]} \sum_{p_{k+1}=1} \cdots \sum_{p_m=1} \int_{C_0(B)} \int_{(0,t)^k} \prod_{u=1}^k \theta_{\alpha_u}(s_u, \lambda^{-1/2}x(s_u)+y) \times_{u=1}^k d\mu_{\alpha_u}(s_u) \cdot \prod_{u=k+1}^m \omega_{p_{\sigma(u)};\alpha_{\sigma(u)}}\theta_{\alpha_{\sigma(u)}}(\tau_{p_{\sigma(u)};\alpha_{\sigma(u)}}, \lambda^{-1/2}x(\tau_{p_{\sigma(u)};\alpha_{\sigma(u)}})+y) \cdot \psi(\lambda^{-1/2}x(t)+y)\varphi(y)dm_B(x)dm_\lambda(y)d\nu(\lambda)$$

$$\overset{(5)}{=} \sum_{k=0}^m \sum_{\{\alpha_1,\cdots,\alpha_k\}\in[k:m]} \sum_{p_{k+1}=1} \cdots \sum_{p_m=1} \sum_{\rho\in S_k} \sum_{j_1+\cdots+j_{m-k+1}=k} \Big(\prod_{u=k+1}^m \omega_{p_{\sigma(u)};\alpha_{\sigma(u)}}\Big) \int_0^\infty \int_{\Omega_\lambda} \int_{\Delta_{k;j_1,\cdots,j_{m-k+1}}(\rho)} \int_{C_0(B)} \prod_{u=1}^k \theta_{\alpha_{\rho(u)}}(s_{\rho(u)}, \lambda^{-1/2}x(s_{\rho(u)})+y) \cdot \prod_{u=k+1}^m \theta_{\alpha_{\sigma(u)}}(\tau_{p_{\sigma(u)};\alpha_{\sigma(u)}}, \lambda^{-1/2}x(\tau_{p_{\sigma(u)};\alpha_{\sigma(u)}})+y) \cdot \psi(\lambda^{-1/2}x(t)+y)\varphi(y)dm_B(x) \times_{u=1}^k d\mu_{\alpha_{\rho(u)}}(s_{\rho(u)})dm_\lambda(y)d\nu(\lambda).$$

Step(2) results from (2) and carrying out the integral with respect to $\sum_{p=1}^\infty \omega_{p:u}\delta_{\tau_{p;u}}$. Step(3) follows from multinomial theorem. For $\rho \in S_k$, define the simplex

$$\Delta_k(\rho) = \{(s_1,\cdots,s_k) \in (0,t)^k | 0 < s_{\rho(1)} < \cdots < s_{\rho(k)} < t\}.$$

Since $\mu$ is continuous measure, it can be seen by a sectioning argument that

$$(0,t)^k = \bigcup_{\rho \in S_k} \Delta_k(\rho) = \bigcup_{\rho \in S_k} \bigcup_{j_1+\cdots+j_{m-k-1}=k} \Delta_{k;j_1,\cdots,j_{m-k-1}}(\rho)$$

except for a set of $\mu \times \cdots \times \mu$ measure zero. And by the Fubini theorem, we have step(4) and step(5). Let

$$\begin{aligned} A &= \int_{C_0(B)} \prod_{u=1}^{k} \theta_{\alpha_{\rho(u)}}(s_{\rho(u)}, \lambda^{-1/2}x(s_{\rho(u)}) + y) \\ &\quad \cdot \prod_{u=k+1}^{m} \theta_{\alpha_{\sigma(u)}}(\tau_{p_{\sigma(u)};\alpha_{\sigma(u)}}, \lambda^{-1/2}x(\tau_{p_{\sigma(u)};\alpha_{\sigma(u)}}) + y)\psi(\lambda^{-1/2}x(t) + y)dm_B(x). \end{aligned}$$

Let

$$a_{q,0} = \lambda^{-1/2}\sqrt{\tau_{p_{\sigma(k-q)}} - s_{\rho(j_1+\cdots+j_q)}}x_{q,0},$$

$$a_{q-1,1} = \lambda^{-1/2}\sqrt{s_{\rho(j_1+\cdots+j_{q-1}+1)} - \tau_{p_{\sigma(k-q-1)}}}x_{q-1,1}$$

where $q = 1, \cdots, m-k+1$. By Lemma 1, (5),(6), and Definition 4, we have

$$\begin{aligned} A &= \int_{B^{m-1}} \theta_{\alpha_{\rho(1)}}(s_{\rho(1)}, a_{0,1} + y)\theta_{\alpha_{\rho(2)}}\big(s_{\rho(2)}, a_{0,1} + \lambda^{-1/2}\sqrt{s_{\rho(2)} - s_{\rho(1)}}x_{0,2} + y\big) \\ &\quad \cdots \theta_{\alpha_{\rho(j_1)}}\left(s_{\rho(j_1)}, a_{0,1} + \lambda^{-1/2}\sum_{i=2}^{j_1} \sqrt{s_{\rho(i)} - s_{\rho(i-1)}}x_{0,i} + y\right) \\ &\quad \cdot \theta_{\alpha_{\rho(j_1-1)}}\left(s_{\rho(j_1+1)}, a_{0,1} + a_{1,0} + a_{1,1} + \lambda^{-1/2}\sum_{i=2}^{j_1} \sqrt{s_{\rho(i)} - s_{\rho(i-1)}}x_{0,i} + y\right) \end{aligned}$$

$$
\begin{aligned}
&\cdots\theta_{\alpha_{\rho(j_1+j_2)}}\Bigg(s_{\rho(j_1+j_2)}, \lambda^{-1/2}\sum_{q=1}^{2}\sum_{i=2}^{j_q}\sqrt{s_{\rho(j_1+\cdots+j_{q-1}+i)} - s_{\rho(j_1+\cdots+j_{q-1}+i-1)}}x_{q-1,i} \\
&+\sum_{q=1}^{2} a_{q-1,1} + a_{1,0} + y\Bigg)\cdots\theta_{\alpha_{\rho(j_1+\cdots+j_{m-k}+1)}}\Bigg(s_{\rho(j_1+\cdots+j_{m-k}+1)}, \\
&\lambda^{-1/2}\sum_{q=1}^{m-k}\sum_{i=2}^{j_q}\sqrt{s_{\rho(j_1+\cdots+j_{q-1}+i)} - s_{\rho(j_1+\cdots+j_{q-1}+i-1)}}x_{q-1,i} \\
&+\sum_{q=1}^{m-k}(a_{q-1,1}+a_{q,0}) + a_{m-k,1} + y\Bigg)\cdots\theta_{\alpha_{\rho(j_1+\cdots+j_{m-k+1})}}\Bigg(s_{\rho(j_1+\cdots+j_{m-k+1})}, \\
&\lambda^{-1/2}\sum_{q=1}^{m-k+1}\sum_{i=2}^{j_q}\sqrt{s_{\rho(j_1+\cdots+j_{q-1}+i)} - s_{\rho(j_1+\cdots+j_{q-1}+i-1)}}x_{q-1,i} \\
&+\sum_{q=1}^{m-k}(a_{q-1,1}+a_{q,0}) + a_{m-k,1} + y\Bigg)\cdot\prod_{u=k+1}^{m}\theta_{\alpha_{\sigma(u)}}\Bigg(\tau_{p_{\sigma(u)};\alpha_{\sigma(u)}}, \\
&\lambda^{-1/2}\sum_{q=1}^{u-k}\sum_{i=2}^{j_q}\sqrt{s_{\rho(j_1+\cdots+j_{q-1}+i)} - s_{\rho(j_1+\cdots+j_{q-1}+i-1)}}x_{q-1,i} \\
&+\sum_{q=1}^{u-k}(a_{q-1,1}+a_{q,0}) + y\Bigg) \\
&\cdot\psi\Bigg(\lambda^{-1/2}\sum_{q=1}^{m-k+1}\sum_{i=2}^{j_q}\sqrt{s_{\rho(j_1+\cdots+j_{q-1}+i)} - s_{\rho(j_1+\cdots+j_{q-1}+i-1)}}x_{q-1,i} \\
&+\sum_{q=1}^{m-k+1}(a_{q-1,1}+a_{q,0}) + y\Bigg)d\big(\times_{p=1}^{m-k+1}\times_{i=1}^{j_p+1} m\big)(x_{p-1,i}) \\
= {} &\Bigg(\Big[\theta_{\alpha_{\sigma(k)}}(\tau_{p_{\sigma(k)};\alpha_{\sigma(k)}})\circ C_{d_{k,1}}\circ\theta_{\alpha_{\rho(1)}}(s_{\rho(1)})\circ C_{d_{k,2}}\circ\cdots\circ\theta_{\alpha_{\rho(j_1)}}(s_{\rho(j_1)})\circ C_{d_{k-1,0}}\Big] \\
&\circ\quad\Big[\theta_{\alpha_{\sigma(k-1)}}(\tau_{p_{\sigma(k-1)};\alpha_{\sigma(k-1)}})\circ C_{d_{k-1,1}}\circ\theta_{\alpha_{\rho(j_1+1)}}(s_{\rho(j_1+1)})\circ C_{d_{k-1,2}}\circ\cdots \\
&\circ\theta_{\alpha_{\rho(j_1+j_2)}}(s_{\rho(j_1+j_2)})\circ C_{d_{k-2,0}}\Big]\circ\cdots\circ\Big[\theta_{\alpha_{\sigma(m)}}(\tau_{p_{\sigma(m)};\alpha_{\sigma(m)}})\circ C_{d_{m,1}} \\
&\circ\theta_{\alpha_{\rho(j_1+\cdots+j_{m-k}+1)}}(s_{\rho(j_1+\cdots+j_{m-k}+1)})\circ C_{d_{m,2}}\circ\cdots\circ\theta_{\alpha_{\rho(k)}}(s_{\rho(k)})\circ C_{d_{m-1,0}}\Big]\psi\Bigg)(y) \\
= {} &(L_kL_{k+1}\cdots L_m\psi)(y)
\end{aligned}
$$

where $x_{p,0} := x_{p-1,j_p+1}, p = 1, \cdots, m-k+1$. Hence from the similar method as in proof of Theorem 2.7 in [7], we conclude that (3) holds for all $\lambda > 0$. □

Next we prove the existence of the operator valued function space integral $K_\lambda^t(F)$ for the series of functional of the form (1) and write each of the factors in (1) as an absolutely convergent infinite series.

COROLLARY 17. *Let $(F_n)$ be a sequence of functionals, each given by*

$$F_n(x) := \prod_{u=1}^{m_n} \int_{(0,t)} \theta_{n,u}(s, x(s)) d\eta_{n,u}(s) \tag{8}$$

*where $\eta_{n,u} \in M(0,t)$ and $\theta_{n,u}$ is as above. Assume that $\sum_{n=0}^{\infty} \prod_{u=1}^{m_n} \sup(|\theta_{n,u}| \|\eta_{n,u}\|) < \infty$ and $F(x) := \sum_{n=0}^{\infty} F_n(x)$. Then for all $\lambda > 0$, $K_\lambda^t(F)$ exists and is given by $K_\lambda^t(F) = \sum_{n=0}^{\infty} K_\lambda^t(F_n)$ where $K_\lambda^t(F_n)$ is defined by (3), with the functional $F$ from Theorem 16 replaced by $F_n$. The series converges in operator norm. Furthermore, for all $\lambda > 0$ we have the estimate*

$$\|K_\lambda(F)\| \le \sum_{n=0}^{\infty} \prod_{u=1}^{m_n} (\sup |\theta_{n,u}| \|\eta_{n,u}\|).$$

PROOF. Since

$$\sum_{n=0}^{\infty} |F_n(\lambda^{-1/2}x + y)| \le \sum_{n=0}^{\infty} \prod_{u=1}^{m_n} (\sup |\theta_{n,u}| \|\eta_{n,u}\|) < \infty,$$

the individual terms are defined and the series absolutely converges for $m_B \times m$ a.e. $(x, y) \in C_0(B) \times B$. Also for $y \in B$ except for some Borel scale invariant null subsets,

$$\sum_{n=0}^{\infty} |F_n(\lambda^{-1/2}x + y)\psi(\lambda^{-1/2}x(t) + y)|$$
$$\le \sum_{n=0}^{\infty} \prod_{u=1}^{m} (\sup |\theta_{n,u}| \|\eta_{n,u}\|) |\psi(\lambda^{-1/2}x(t) + y)|$$

and $|\psi(\lambda^{-1/2}x(t)+y)|$ is integrable in $x$. Hence by dominated convergence theorem, for $\varphi \in L_{q,\infty}(B)$ and for any Borel measure $\nu$ on $(0,+\infty)$,

$$\begin{aligned}
&\int_{(0,\infty)}\int_{\Omega_\lambda}(K_\lambda^t(F)\psi)(y)\varphi(y)dm_\lambda(y)d\nu(\lambda)\\
&= \int_{(0,\infty)}\int_{\Omega_\lambda}\int_{C_0(B)}\Big(\sum_{n=0}^{\infty}F_n(\lambda^{-1/2}x+y)\psi(\lambda^{-1/2}x(t)+y)\Big)\varphi(y)dm_B(x)dm_\lambda(y)d\nu(\lambda)\\
&= \sum_{n=0}^{\infty}\int_{(0,\infty)}\int_{\Omega_\lambda}\int_{C_0(B)}(F_n(\lambda^{-1/2}x+y)\psi(\lambda^{-1/2}x(t)+y))\varphi(y)dm_B(x)dm_\lambda(y)d\nu(\lambda)\\
&= \int_{(0,\infty)}\int_{\Omega_\lambda}\sum_{n=0}^{\infty}(K_\lambda^t(F_n)\psi)(y)\varphi(y)dm_\lambda(y)d\nu(\lambda).
\end{aligned}$$

Now the inequality $\|K_\lambda^t(F_n)\| \le \prod_{u=1}^{m}(\sup|\theta_{n,u}|\|\eta_{n,u}\|)$ from (7) in Theorem 16 assures that the series $\sum_{n=0}^{\infty}(K_\lambda^t(F_n))$ converges in operator norm, uniformly for $\lambda > 0$. Thus by Theorem 15 $K_\lambda^t(F)$ exists and $K_\lambda^t(F) = \sum_{n=0}^{\infty}(K_\lambda^t(F_n))$. Further

$$\|K_\lambda^t(F)\| \le \sum_{n=0}^{\infty}\|K_\lambda^t(F_n)\| \le \sum_{n=0}^{\infty}\prod_{u=1}^{m_n}(\sup|\theta_{n,u}|\|\eta_{n,u}\|).$$

□

We show that the general class of functionals treated in Corollary 17 forms a commutative Banach algebra. The proof is not much different from [3] except for scaling property.

DEFINITION 18. Let $F$ and $G$ be $C$-valued Borel measurable functions on $C_0(B)$. We say that $F$ is equivalent to $G$, write, $F \sim G$ if for all $\lambda > 0$ $F(\lambda^{-1/2}x+y) = G(\lambda^{-1/2}x+y)$ for $m_B \times m$-a.e. $(x,y) \in C_0(B) \times B$.

DEFINITION 19. Let $(F_n)$ be a sequence of functionals each of which is given by an expression of the following form

$$F_n(x) = \prod_{u=1}^{m_n}\int_{(0,t)}\theta_{n,u}(s,x(s))d\eta_{n,u}(s) \tag{9}$$

satisfying

$$\sum_{n=0}^{\infty}\prod_{u=1}^{m}(\sup|\theta_{n,u}|\|\eta_{n,u}\|) < \infty. \tag{10}$$

Let $F$ be defined by

$$F(\lambda^{-1/2}x+y)=\sum_{n=0}^{\infty}F_n(\lambda^{-1/2}x+y). \tag{11}$$

It is shown in Corollary 17 that for every $\lambda>0$, the series in (11) converges absolutely for $m_B\times m$-a.e.$(x,y)\in C_0(B)\times B$. We define $\mathcal{B}_t$ to be the collection of equivalence classes of functional $F$ obtained in this manner. For $F$ in $\mathcal{B}_t$, we define $\|F\|_t$ be the infimum of the left hand side of (10) over all representations of $F$ of the form (11).

THEOREM 20. *For each $t>0$ the space $(\mathcal{B}_t,\|\cdot\|_t)$ is a commutative Banach algebra with identity. Moreover given $F\in\mathcal{B}_t$, $K_\lambda^t(F)$ exists for all $\lambda>0$ and satisfies the norm estimate $\|K_\lambda^t(F)\|\le\|F\|_t$.*

PROOF. Clearly, $0\le\|F\|_t<\infty$ for each $F\in\mathcal{B}_t$. Let $F$ and $G$ in $\mathcal{B}_t$. Then given $\varepsilon>0$, take a representation for $F$ defined by (9) and (11) such that the left hand side of (10) is less than $\|F\|_t+\varepsilon/2$. Choose a similar representation for $G$. Since the series are absolutely convergent, $(F+G)(\lambda^{-1/2}x+y)=\sum_{n=0}^{\infty}(F_n+G_n)(\lambda^{-1/2}x+y)$ and $\|F+G\|_t\le\|F\|_t+\|G\|_t$. Also $\|\alpha F\|_t=|\alpha|\|F\|_t$ for $\alpha\in R$. Let $\|F\|_t=0$ and $F\in\mathcal{B}_t$. Then for any positive integer p, there exists a representation for $F$ given by (9) and (11) and $\sum_{n=0}^{\infty}\prod_{u=1}^{m_n}(\sup|\theta_{n,u}|\|\eta_{n,u}\|)<1/p$. Hence for any $\lambda>0$, $|F_n(\lambda^{-1/2}x+y)|<1/p$ for all $n$. Thus for $\lambda>0$, $F(\lambda^{-1/2}x+y)=0$ for a.e. $(x,y)\in C_0(B)\times B$. Hence $F$ is equivalent to 0. Therefore $(\mathcal{B}_t,\|\cdot\|_t)$ is a normed space. Also for every $\lambda>0$ $\sum_{i+j=k}F_i(\lambda^{-1/2}x+y)G_j(\lambda^{-1/2}x+y)$ converges absolutely a.e.$(x,y)$ and has the sum $F(\lambda^{-1/2}x+y)G(\lambda^{-1/2}x+y)$. Further each term is of the type (9). Consequently $FG\in\mathcal{B}_t$ and $\|FG\|_t\le(\|F\|_t+\varepsilon/2)(\|G\|_t+\varepsilon/2)$. Since $\varepsilon$ was arbitray, $\|FG\|_t\le\|F\|_t\|G\|_t$. Therefore $\mathcal{B}_t$ is an algebra. To show completeness, it suffices to show that every absolute summable series is summerable in the norm of the space. Given a sequence $(F_n)$ in $\mathcal{B}_t$ with $\sum_{n=1}^{\infty}\|F_n\|_t<\infty$, we can choose representation of the form (9) and (11), as follows

$$F_1=F_{11}+F_{12}+\cdots,F_2=F_{21}+F_{22}+\cdots,\cdots,F_n=F_{n1}+F_{n2}+\cdots$$

such that for each $n$, the left hand side of (10) is less than $\|F_n\|_t+1/2^n$. The terms of $\sum F_{ij}$ are of the form (9) and corresponding series of the form (10) converges to a number less than $\sum_{n=1}^{\infty} \|F_n\|_t + 1$. Let

$$F(\lambda^{-1/2}x + y) := \sum_{i,j=1}^{\infty} F_{ij}(\lambda^{-1/2}x + y).$$

Then $F \in \mathcal{B}_t$ and we can choose $N'$ large so that $\sum_{n=N'+1}^{\infty} \|F_n\|_t + 1/2^n < \varepsilon$. So

$$\|F - \sum_{n=1}^{N'} F_n\|_t = \| \sum_{n=N'+1}^{\infty} F_n\|_t \leq \sum_{n=N'+1}^{\infty} \|F_n\|_t + 1/2^n < \varepsilon.$$

Thus $(\mathcal{B}_t, \|\cdot\|_t)$ is a Banach algebra. By Corollary 17, $K_\lambda^t(F)$ exists for all $\lambda > 0$ and $\|K_\lambda^t(F)\| \leq \|F\|_t$. □

COROLLARY 21. *For each $\lambda > 0$, the mapping $K_\lambda^t : \mathcal{B}_t \longrightarrow \mathcal{L}(L_{p\infty}(B), L_{p\infty}(B))$ which associates $K_\lambda^t(F)$ to $F \in \mathcal{B}_t$, is a bounded linear operator of norm at most* 1.

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Department of Mathematics Education
Chonbuk National University
Chonju 561-756, Korea
*E-mail*: yhpark@moak.chonbuk.ac.kr