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NOTE ON THREE OF RAMANUJAN'S THEOREMS

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ABSTRACT. The object of this note is to introduce three Ramanujan's formulae of similar nature among his many curious ones and to prove them by making use of the theory of generalized hypergeometric series.

1. Introduction and Preliminaries

The intention of this note is to prove the following curious formulae of Ramanujan which were recorded in [3, p. 495]:

$$1+3\left(\frac{x-1}{x+1}\right)^2+5\left\{\frac{(x-1)(x-2)}{(x+1)(x+2)}\right\}^2+\cdots=\frac{x^2}{2x-1}, \tag{1.1}$$

$$1-3\left(\frac{x-1}{x+1}\right)^2+5\left\{\frac{(x-1)(x-2)}{(x+1)(x+2)}\right\}^2-\cdots=\frac{\{\Gamma(x+1)\}^2}{\Gamma(2x)}, \tag{1.2}$$

$$1-3\left(\frac{x-1}{x+1}\right)^3+5\left\{\frac{(x-1)(x-2)}{(x+1)(x+2)}\right\}^3-\cdots=\frac{\{\Gamma(x+1)\}^3\Gamma(3x-1)}{\{\Gamma(2x)\}^3}, \tag{1.3}$$

by considering special cases of known transformation formulas involving generalized hypergeometric series.

The generalized hypergeometric series with p numerator and q denominator parameters is defined by

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{matrix}\middle| z\right] &= {}_pF_q\left[\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z\right] \\ &:= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}, \end{aligned} \tag{1.4}$$

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where $(\lambda)_n$ denotes the Pochhammer symbol (or *the shifted factorial*, since $(1)_n = n!$) defined by, for any complex number λ,

$$(1.5)\quad (\lambda)_n = \begin{cases} \lambda(\lambda+1)\dots(\lambda+n-1), & \text{if } n \in \mathbf{N} := \{1, 2, 3, \dots\}, \\ 1 & \text{if } n = 0, \end{cases}$$

which, in virtue of the fundamental relation for the Gamma function Γ :

$$(1.6)\qquad \Gamma(\lambda+1) = \lambda\Gamma(\lambda) \quad \text{and} \quad \Gamma(1) = 1,$$

is written in the following equivalent form:

$$(1.7)\qquad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \neq 0, -1, -2, \dots).$$

A useful well-known asymptotic formula for Gamma function is also provided:

$$(1.8)\qquad \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1 + \mathrm{O}\left(\frac{1}{z}\right)\right] \quad (z \to \infty;\ |\arg z| < \pi).$$

By making use of the Pochhammer symbol and the following elementary identities

$$(x-1)(x-2)\cdots(x-n) = (-1)^n(-x+1)_n \quad \text{and} \quad 2n+1 = \frac{(3/2)_n}{(1/2)_n},$$

the three formulae concerned are expressed in terms of generalized hypergeometric series as follows:

$$(1.1')\qquad {}_4F_3\left[\begin{matrix} 1, & \frac{3}{2}, & -x+1, & -x+1 \\ & \frac{1}{2}, & x+1, & x+1 \end{matrix} \;\middle|\; 1\right] = \frac{x^2}{2x-1},$$

$$(1.2')\qquad {}_4F_3\left[\begin{matrix} 1, & \frac{3}{2}, & -x+1, & -x+1 \\ & \frac{1}{2}, & x+1, & x+1 \end{matrix} \;\middle|\; -1\right] = \frac{\{\Gamma(x+1)\}^2}{\Gamma(2x)},$$

$$(1.3')\qquad {}_5F_4\left[\begin{matrix} 1, & \frac{3}{2}, & -x+1, & -x+1, & -x+1 \\ & \frac{1}{2}, & x+1, & x+1, & x+1 \end{matrix} \;\middle|\; 1\right] = \frac{\{\Gamma(x+1)\}^3\Gamma(3x-1)}{\{\Gamma(2x)\}^3}.$$

2. Proof of (1.1′), (1.2′), and (1.3′)

We start with a transformation formula due to Whipple [4, p. 253]

(2.1)

$$
{}_7F_6\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ d,\ e,\ -m \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a+m \end{matrix}\ \middle|\ 1\right]
= \frac{(1+a)_m(1+a-d-e)_m}{(1+a-d)_m(1+a-e)_m}
{}_4F_3\left[\begin{matrix} 1+a-b-c,\ d,\ e,\ -m \\ 1+a-b,\ 1+a-c,\ d+e-a-m \end{matrix}\ \middle|\ 1\right],
$$

which transforms a terminating well-poised ${}_7F_6$ into a Saalschützian ${}_4F_3$, and conversely transforms any terminating Saalschützian ${}_4F_3$ into a well-poised ${}_7F_6$.

Taking the limit as $m \to \infty$ in (2.1) with the aid of (1.8), we obtain the formula

(2.2)

$$
{}_6F_5\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c\ d,\ e \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e \end{matrix}\ \middle|\ -1\right]
= \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}{}_3F_2\left[\begin{matrix} 1+a-b-c,\ d,\ e \\ 1+a-b,\ 1+a-c \end{matrix}\ \middle|\ 1\right],
$$

which expresses a well-poised ${}_6F_5$ with argument -1 in terms of ${}_3F_2$ with argument 1 and vice versa.

If we make $c \to \infty$ in (2.2) with the aid of (1.8) and use Gauss's summation theorem [2]:

$$
{}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix}\ \middle|\ 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{2.3}
$$

and, in the resulting equation, replace d and e by c and d respectively, we derive the formula

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ d \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d \end{matrix}\ \middle|\ 1\right] \\
&= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)},
\end{aligned} \tag{2.4}
$$

which, for $d = \frac{1}{2}a$, yields Dixon's theorem [1]:

$$(2.5)\quad {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+a-b,\ 1+a-c \end{matrix}\ \middle|\ 1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)}.$$

Then making $d \to \infty$ in (2.4) with the help of (1.8), we obtain the formula

$$(2.6)\quad {}_4F_3\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ c \\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c \end{matrix}\ \middle|\ -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}.$$

Now setting $a = 1$, $b = c = d = -x+1$ in (2.6) and (2.4) yields the desired formulas (1.2′) and (1.3′).

To prove (1.1′), recall the formulas [5, pp. 261-262]

$$(2.7)\quad \begin{aligned} &{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d \\ 1+a-b,\ 1+a-c,\ 1+a-d \end{matrix}\ \middle|\ 1\right] \\ &= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-c-d)\Gamma(1+a-b-d)} \\ &\times {}_4F_3\left[\begin{matrix} \frac{1}{2},\ b,\ c,\ d \\ 1+\frac{1}{2}a,\ \frac{1}{2}+\frac{1}{2}a,\ b+c+d-a \end{matrix}\ \middle|\ 1\right], \end{aligned}$$

$$(2.8)\quad {}_3F_2\left[\begin{matrix} b,\ c,\ d \\ k,\ b+c+d-k+1 \end{matrix}\ \middle|\ 1\right] = \frac{\Gamma(k)\Gamma(k-b-c)\Gamma(k-b-d)\Gamma(k-c-d)}{\Gamma(k-b)\Gamma(k-c)\Gamma(k-d)\Gamma(k-b-c-d)}.$$

If we replace a, b, c, and d by 1, $\frac{3}{2}$, $-x+1$ and $-x+1$ in (2.7) respectively, we obtain

(2.9)

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} 1,\ \frac{3}{2},\ -x+1,\ -x+1 \\ \frac{1}{2},\ x+1,\ x+1 \end{matrix}\ \middle|\ 1\right] \\ &= \frac{\Gamma(1/2)\left\{\Gamma(x+1)\right\}^2\Gamma\left(2x-\frac{3}{2}\right)}{\Gamma(2x)\left\{\Gamma\left(x-\frac{1}{2}\right)\right\}^2}\,{}_3F_2\left[\begin{matrix} \frac{1}{2},\ -x+1,\ -x+1 \\ 1,\ \frac{5}{2}-2x \end{matrix}\ \middle|\ 1\right], \end{aligned}$$

which, for $k = 1$, $b = 1/2$, and $c = -x + 1 = d$ in (2.8), and using (1.6), leads immediately to (1.1′).

Finally, setting $a = 1$ and $b = c = d = e = -x + 1$ in (2.2), we obtain a formula of similar nature as those considered here

$$\begin{aligned} &1 - 3\left(\frac{x-1}{x+1}\right)^4 + 5\left\{\frac{(x-1)(x-2)}{(x+1)(x+2)}\right\}^4 - \cdots \\ &= {}_6F_5\left[\begin{matrix}1, & \frac{3}{2}, & -x+1, & -x+1 & -x+1 & -x+1 \\ & \frac{1}{2}, & x+1, & x+1, & x+1, & x+1\end{matrix}\,\middle|\, -1\right] \\ &= \frac{\{\Gamma(x+1)\}^2}{\Gamma(2x)}\,{}_3F_2\left[\begin{matrix}2x, & -x+1, & -x+1 \\ & x+1, & x+1\end{matrix}\,\middle|\, 1\right], \end{aligned} \tag{2.10}$$

the ${}_3F_2$ of which does not seem easy to be expressed as a closed form.

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