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A NEW CLASS OF INTEGRALS INVOLVING HYPERGEOMETRIC FUNCTION

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Abstract. The aim of this research is to provide twenty five integrals involving hypergeometric function in the form of a single integral. Fifty two interesting integrals follow as special cases of our main findings. These results are obtained with the help of generalized Watson's theorem on the sum of a ${}_3F_2$ recently obtained by Lavoie, Grondin and Rathie. The integrals given in this paper are simple, interesting and easily established, and they may be useful.

1. Introduction

We start with an interesting integral due to MacRobert [2, p. 450]

$$\begin{aligned}(1.1)\qquad &\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,[1+cx+d(1-x)]^{-\alpha-\beta}\,dx\\ &\qquad=\frac{\Gamma(\alpha)\Gamma(\beta)}{(1+c)^{\alpha}(1+d)^{\beta}\,\Gamma(\alpha+\beta)}\end{aligned}$$

provided $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$ and the constants c and d are such that none of the expressions $1+c$, $1+d$ and $1+cx+d(1-x)$, where $0\le x\le 1$, are zero.

In 1992, Lavoie, Grondin and Rathie [1] have obtained the following generalization of the well known classical Watson's theorem on the sum of a ${}_3F_2$, *viz.*

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(1.2)

$$
{}_3F_2\left(\begin{matrix} a,\ b,\ c \\ \frac{1}{2}(a+b+i+1),\ 2c+j \end{matrix}\Big|1\right)
= A_{i,j}\frac{2^{a+b+i-2}\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{i}{2}+\frac{1}{2})\Gamma(c+[\frac{j}{2}]+\frac{1}{2})\Gamma(c-\frac{a}{2}-\frac{b}{2}-\frac{|i+j|}{2}+\frac{j}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)}
$$

$$
\times\left\{B_{i,j}\frac{\Gamma(\frac{a}{2}+\frac{(1-(-1)^i)}{4})\Gamma(\frac{b}{2})}{\Gamma(c-\frac{a}{2}+\frac{1}{2}+[\frac{j}{2}]-\frac{(-1)^j(1-(-1)^i)}{4})\Gamma(c-\frac{b}{2}+\frac{1}{2}+[\frac{j}{2}])}
+C_{i,j}\frac{\Gamma(\frac{a}{2}+\frac{(1+(-1)^i)}{4})\Gamma(\frac{b}{2}+\frac{1}{2})}{\Gamma(c-\frac{a}{2}+[\frac{j+1}{2}]+\frac{(-1)^j(1-(-1)^i)}{4})\Gamma(c-\frac{b}{2}+[\frac{j+1}{2}])}\right\}
$$

for $i,\ j = -2,\ -1,\ 0,\ 1,\ 2$. Also $[x]$ is the greatest integer less than or equal to x. The tables of the coefficients $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are given as in [1].

In the same paper [1], the following fifty special cases of (1.2) are given.

$$
{}_3F_2\left(\begin{matrix} -2n,\ a+2n,\ c \\ \frac{1}{2}(a+i+1),\ 2c+j \end{matrix}\Big|1\right)
= D_{i,j}\frac{(\frac{1}{2})_n(\frac{a}{2}-c+\frac{3}{4}-\frac{1}{4}(-1)^i-[\frac{j}{2}+\frac{(1-(-1)^i)}{4}])_n}{(c+\frac{1}{2}+[\frac{j}{2}])_n(\frac{a}{2}+\frac{(1+(-1)^i)}{4})_n} \tag{1.3}
$$

and

$$
{}_3F_2\left(\begin{matrix} -2n-1,\ a+2n+1,\ c \\ \frac{1}{2}(a+i+1),\ 2c+j \end{matrix}\Big|1\right)
= E_{i,j}\frac{(\frac{3}{2})_n(\frac{a}{2}-c+\frac{5}{4}+\frac{(-1)^i}{4}-[\frac{j}{2}+\frac{(1+(-1)^i)}{4}])_n}{(c+\frac{1}{2}+[\frac{j+1}{2}])_n(\frac{a}{2}+\frac{(3-(-1)^i)}{4})_n}, \tag{1.4}
$$

where $i,\ j = -2,\ -1,\ 0,\ 1,\ 2$ and as usual, $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$ $(\lambda \neq 0,\ -1,\ -2,\ \ldots)$, and the tables of the coefficients $D_{i,j}$ and $E_{i,j}$ are given as in [1].

2. Main Results

The integrals to be evaluated are, for $i,\ j = -2,\ -1,\ 0,\ 1,\ 2$,

(2.1)

$$\begin{aligned}
&\int_0^1 x^{e-1}(1-x)^{e+j-1}\left[1+cx+d(1-x)\right]^{-2e-j} \\
&\times {}_2F_1\left(a,\ b;\ \frac{1}{2}(a+b+i+1);\ \frac{(1+c)x}{1+cx+d(1-x)}\right) dx \\
&= A_{i,j}2^{a+b+i-2}\frac{\Gamma(e)\Gamma(e+j)\Gamma\left(e+\left[\frac{j}{2}\right]+\frac{1}{2}\right)}{(1+c)^e\,(1+d)^{e+j}\Gamma(\frac{1}{2})} \\
&\times \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(e-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}|i+j|+\frac{1}{2}j+\frac{1}{2}\right)}{\Gamma(a)\Gamma(b)\Gamma(2e+j)} \\
&\times \left\{B_{i,j}\frac{\Gamma\left(\frac{1}{2}a+\frac{1}{4}(1-(-1)^i)\right)\Gamma(\frac{1}{2}b)}{\Gamma\left(e-\frac{1}{2}a+\frac{1}{2}+\left[\frac{j}{2}\right]-\frac{1}{4}(-1)^j\,(1-(-1)^i)\right)\Gamma\left(e-\frac{1}{2}b+\frac{1}{2}+\left[\frac{j}{2}\right]\right)}\right. \\
&\left. +C_{i,j}\frac{\Gamma\left(\frac{1}{2}a+\frac{1}{4}(1+(-1)^i)\right)\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma\left(e-\frac{1}{2}a+\left[\frac{j+1}{2}\right]+\frac{1}{4}(-1)^j\,(1-(-1)^i)\right)\Gamma\left(e-\frac{1}{2}b+\left[\frac{j+1}{2}\right]\right)}\right\}
\end{aligned}$$

provided $\mathrm{Re}(2e-a-b) > -1-i-2j$ for $i,\ j = -2,\ -1,\ 0,\ 1,\ 2$; $\mathrm{Re}(e) > 0$ for $j = 0,\ 1,\ 2$; and $\mathrm{Re}(e) > -j$ for $j = -2,\ -1$.

The coefficients $A_{i,j}$, and $B_{i,j}$ and $C_{i,j}$ can be obtained from the tables of $A_{i,j}$, and $B_{i,j}$ and $C_{i,j}$ by simply changing c to e. The constants c and d are such that none of the expressions $1+c$, $1+d$ and $1+cx+d(1-x)$, where $0 \leq x \leq 1$, are zero. Also $[x]$ is the greatest integer less than or equal to x.

3. Proof of Integrals

In order to evaluate (2.1), let us consider a more general integral involving hypergeometric function, *viz.*

$$\begin{aligned}(3.1)\qquad &\int_0^1 x^{\mu-1}(1-x)^{\sigma-1}\,[1+cx+d(1-x)]^{-\mu-\sigma} \\ &\quad\times {}_2F_1\left(a,\, b;\, \lambda;\, \frac{(1+c)x}{1+cx+d(1-x)}\right) dx \\ &= \frac{\Gamma(\mu)\Gamma(\sigma)}{(1+c)^{\mu}(1+d)^{\sigma}\,\Gamma(\mu+\sigma)}\, {}_3F_2\,(a,\, b,\, \mu;\, \lambda,\, \mu+\sigma;\, 1)\end{aligned}$$

provided $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\sigma) > 0$, $\mathrm{Re}(\lambda+\sigma-a-b) > 0$ and the constants c and d are such that none of the expressions $1+c$, $1+d$ and $1+cx+d(1-x)$, where $0 \leq x \leq 1$, are zero.

Proof of (3.1). Express the hypergeometric function as a series, change the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series in the interval (0, 1), evaluate the integral with the help of a known result due to MacRobert (1.1) and sum the series. We finally arrive at the right-hand side of (3.1). □

Now we are ready to evaluate (2.1). In (3.1), if we take $\mu = e$, $\sigma = e+j$ and $\lambda = \frac{1}{2}(a+b+i+1)$ $(i, j = -2, -1, 0, 1, 2)$, then we obtain the following form

(3.2)

$$\begin{aligned}&\int_0^1 x^{e-1}(1-x)^{e+j-1}\,[1+cx+d(1-x)]^{-2e-j} \\ &\quad\times {}_2F_1\left(a,\, b;\, \frac{1}{2}(a+b+i+1);\, \frac{(1+c)x}{1+cx+d(1-x)}\right) dx \\ &= \frac{\Gamma(e)\Gamma(e+j)}{(1+c)^{e}(1+d)^{e+j}\,\Gamma(2e+j)}\, {}_3F_2\left(a,\, b,\, e;\, \frac{1}{2}(a+b+i+1),\, 2e+j;\, 1\right)\end{aligned}$$

The series on the right-hand side of (3.2) can be summed with the help of (1.2) and we arrive at (2.1).

4. Special Cases of (2.1)

1. Let $b = -2n$ and replace a by $a+2n$ or let $b = -2n-1$ and replace a by $a+2n+1$, where n is a nonnegative integer. In each case one of the two terms on the right-hand side of (2.1) will vanish and fifty interesting integrals are produced. They can be simply represented as, for i, $j = -2, -1, 0, 1, 2$,

(4.1)

$$\int_0^1 x^{e-1}(1-x)^{e+j-1}\,[1+cx+d(1-x)]^{-2e-j} \times {}_2F_1\left(-2n,\, a+2n\,;\, \frac{1}{2}(a+i+1)\,;\, \frac{(1+c)x}{1+cx+d(1-x)}\right) dx$$
$$= \frac{D_{i,j}\,\Gamma(e)\Gamma(e+j)(\frac{1}{2})_n\left(\frac{1}{2}a-e+\frac{3}{4}-\frac{1}{4}(-1)^i-\left[\frac{j}{2}+\frac{1}{4}\left(1-(-1)^i\right)\right]\right)_n}{(1+c)^e\,(1+d)^{e+j}\,\Gamma(2e+j)\,\left(e+\frac{1}{2}+\left[\frac{j}{2}\right]\right)_n\left(\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)_n}$$

and

(4.2)

$$\int_0^1 x^{e-1}(1-x)^{e+j-1}\,[1+cx+d(1-x)]^{-2e-j} \times {}_2F_1\left(-2n-1,\, a+2n+1\,;\, \frac{1}{2}(a+i+1)\,;\, \frac{(1+c)x}{1+cx+d(1-x)}\right) dx$$
$$= \frac{E_{i,j}\,\Gamma(e)\Gamma(e+j)(\frac{3}{2})_n\left(\frac{1}{2}a-e+\frac{1}{4}(-1)^i+\frac{5}{4}-\left[\frac{j}{2}+\frac{1}{4}\left(1+(-1)^i\right)\right]\right)_n}{(1+c)^e\,(1+d)^{e+j}\,\Gamma(2e+j)\,\left(e+\frac{1}{2}+\left[\frac{j+1}{2}\right]\right)_n\left(\frac{1}{2}a+\frac{1}{4}\left(3-(-1)^i\right)\right)_n}$$

where the coefficients $D_{i,j}$ and $E_{i,j}$ can be obtained from the tables of $D_{i,j}$ and $E_{i,j}$ by simply changing c to e.

2. If the Gegenbauer polynomials [3, p. 279] are considered in the form

$$C_n^a(1-2x) = \frac{(2a)_n}{n!}\,{}_2F_1(-n,\, n+2a;\, \frac{1}{2}+a;\, x) \tag{4.3}$$

then the case $i=0$, $j=-1$ of (4.1) and (4.2) gives the following interesting results.

$$\int_0^1 x^{e-1}(1-x)^{e-2}\,[1+cx+d(1-x)]^{-2e+1} \times C_{2n}^a\left(1-\frac{2(1+c)x}{1+cx+d(1-x)}\right) dx = \frac{\Gamma(e)\Gamma(e-1)(a)_n\left(a-e+\frac{3}{2}\right)_n}{(1+c)^e\,(1+d)^{e-1}\,\Gamma(2e-1)\,\left(e-\frac{1}{2}\right)_n\, n!} \tag{4.4}$$

provided $\mathrm{Re}(e) > 1$, $\mathrm{Re}(2e-2a) > 1$, and

$$\begin{aligned} \int_0^1 & x^{e-1}(1-x)^{e-2}\,[1+cx+d(1-x)]^{-2e+1} \\ & \times C_{2n+1}^{a}\left(1-\frac{2(1+c)x}{1+cx+d(1-x)}\right)dx \\ & = \frac{-a\,\Gamma(\frac{1}{2})\Gamma(e-1)(a+1)_n\left(a-e+\frac{3}{2}\right)_n}{2^{2e-2}\,(1+c)^e\,(1+d)^{e-1}\,\Gamma\left(e+\frac{1}{2}\right)\,\left(e+\frac{1}{2}\right)_n\,n!}. \end{aligned} \tag{4.5}$$

These are two typical illustrations.

3. In (2.1), if we take $i=1$, $j=1$ and $i=-1$, $j=-1$, we get two integrals obtained earlier by Sharma and Rathie [4].

4. In (2.1), if we take $i=0$, $j=1$ and $i=0$, $j=-1$, we get two integrals obtained earlier by Rathie [5].

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References

[1] J. L. Lavoie, F. Grondin and A. K. Rathie, *Generalizations of Watson's theorem on the sum of a* $_3F_2$, Indian J. Math. **34** (1992), no. 1, 23–32.

[2] T. M. MacRobert, *Beta function formulae and integrals involving E-functions*, Math. Ananalean **142** (1961), 450–452.

[3] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.

[4] G. Sharma and A. K. Rathie, *On a new class of integrals involving hypergeometric function*, Ann. Univ. Timisoara Ser. Stiint. Mat., Romania **28** (1990), 83–89.

[5] A. K. Rathie, *A new class of integrals involving hypergeometric function and the H-function*, Soochow J. Math **18** (1992), 47–54.

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