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THE STABILITY OF THE GENERALIZED FORM FOR THE GAMMA FUNCTIONAL EQUATION

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ABSTRACT. The modified Hyers-Ulam-Rassias stability of the generalized form $g(x+p) = \varphi(x)g(x)$ for the Gamma functional equation shall be proved. As a consequence we obtain the stability theorems for the gamma functional equation.

1. Introduction

The stability problem of functional equations was originally raised by S. M. Ulam [5] in 1940. He posed the following problem: Under what conditions does there exist an additive mapping near an approximately additive mapping? In 1941, this problem was solved by D. H. Hyers [1] in the case of Banach space as follows:

THEOREM A. *Let $f : E_1 \longrightarrow E_2$ be a mapping between Banach spaces satisfying the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \delta$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then there exists a unique additive mapping $T : E_1 \longrightarrow E_2$ such that

$$\|f(x) - T(x)\| \le \delta$$

holds true for all $x \in E_1$, and if $f(tx)$ is continuous in t for each fixed x, then T is a linear mapping.

Thereafter we call the type of Theorem A the Hyers-Ulam stability. In 1978, Th. M. Rassias [4] extended the result of Hyers (Theorem A) by

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considering the cases where the Cauchy difference $f(x+y)-f(x)-f(y)$ is not bounded:

THEOREM B. *Let E_1, E_2 be Banach spaces, and let $f : E_1 \longrightarrow E_2$ be a mapping. Assume that there exist $\theta \geq 0$ and $0 \leq p < 1$ such that*

$$(1) \quad \|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p), \quad \text{for any } x, y \in E_1.$$

Then there exists a unique additive mapping $T : E_1 \longrightarrow E_2$ for which the inequality

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p,$$

holds true for any $x \in E_1$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the mapping T is linear.

By regarding a large influence of Theorem B on the study of stability problems of several functional equations, the stability phenomenon of such type is called the Hyers-Ulam-Rassias stability. If the inequality (1) whose right-hand side is replaced by some suitable mapping $\varphi(x, y)$ which is stable, then the additive Cauchy equation is said to have the modified Hyers-Ulam-Rassias stability. These terminologies are similarly applied to the cases of other functional equations.

The aim of the present note is to give modified Hyers-Ulam-Rassias stability of the generalized form for the gamma functional equation

$$(2) \qquad g(x+p) = \varphi(x)g(x),$$

where p is a natural number, and also it is same throughout this paper. Note that the gamma function is a solution for special case of (2).

Let mappings φ and $\phi : (0,\infty) \longrightarrow (0,\infty)$ satisfy the inequality

$$(3) \qquad \Phi(x) = \sum_{j=0}^{\infty} \phi(x+jp) \prod_{i=0}^{j} \frac{1}{\varphi(x+ip)} < \infty$$

for all $x \in (0,\infty)$.

Throughout this paper, let $\delta, p > 0$ be fixed. By using an idea in [4] we can prove the following theorem:

THEOREM. *If a function $g : (0,\infty) \longrightarrow R$ satisfies the following inequality*

$$|g(x+p) - \varphi(x)g(x)| \le \phi(x) \tag{4}$$

for all $x > 0$, then there exists a unique solution $f : (0,\infty) \longrightarrow R$ of the equation (2) *with*

$$|g(x) - f(x)| \le \Phi(x) \tag{5}$$

for all $x > 0$. If the range of g is $(0,\infty)$, then the range of f is $(0,\infty)$.

2. Proof of Theorem

For any $x > 0$ and for every positive integer n we define

$$P_n(x) = g(x+np)\prod_{i=0}^{n-1}\frac{1}{\varphi(x+ip)}.$$

By (4), we have

$$\begin{aligned}
&|P_{n+1}(x) - P_n(x)| \\
&= |g(x+(n+1)p) - \varphi(x+np)\, g(x+np)|\prod_{i=0}^{n}\frac{1}{\varphi(x+ip)} \\
&\le \phi(x+np)\prod_{i=0}^{n}\frac{1}{\varphi(x+ip)}.
\end{aligned}$$

Now we use an induction on n to prove

$$|P_n(x) - g(x)| \le \sum_{j=0}^{n-1}\phi(x+jp)\prod_{i=0}^{j}\frac{1}{\varphi(x+ip)}$$

for the fixed $x > 0$ and for all positive integers n. For the case $n = 1$, the above inequality is an immediate consequence of (4). Assume that it holds true for some n. Then

$$\begin{aligned}
|P_{n+1}(x) - g(x)| &\le |P_{n+1}(x) - P_n(x)| + |P_n(x) - g(x)| \\
&\le \sum_{j=0}^{n}\phi(x+jp)\prod_{i=0}^{j}\frac{1}{\varphi(x+ip)}.
\end{aligned}$$

Now let m, n be positive integers with $n \geq m$. Suppose $x > 0$ are given By (3), we have

$$\begin{aligned}|P_n(x) - P_m(x)| &\leq \sum_{j=m}^{n-1} |P_{j+1}(x) - P_j(x)| \\ &\leq \sum_{j=m}^{n-1} \phi(x + jp) \prod_{i=0}^{j} \frac{1}{\varphi(x + ip)} \longrightarrow 0\end{aligned}$$

as $m \longrightarrow \infty$. This implies that $\{P_n(x)\}$ is a Cauchy sequence for $x > 0$ and hence we can define a function $f : (0, \infty) \longrightarrow R$ by

$$f(x) = \lim_{n \to \infty} P_n(x).$$

Since $P_n(x + p) = \varphi(x) P_{n+1}(x)$, we have

$$f(x + p) = \varphi(x) f(x)$$

for any $x > 0$. Also we have

$$\begin{aligned}|f(x) - g(x)| &= \lim_{n \to \infty} |P_n(x) - g(x)| \\ &\leq \sum_{j=0}^{\infty} \phi(x + jp) \prod_{i=0}^{j} \frac{1}{\varphi(x + ip)} = \Phi(x)\end{aligned}$$

for all $x > 0$. If $h : (0, \infty) \longrightarrow R$ is an another function which satisfies

$$h(x + p) = \varphi(x) h(x)$$

and $|h(x) - g(x)| \leq \Phi(x)$ for all $x > 0$, then

$$
\begin{aligned}
|f(x)-h(x)| & \\
&= \prod_{i=0}^{n-1} \frac{1}{\varphi(x+ip)} |f(x+np) - h(x+np)| \\
&\leq 2\Phi(x+np) \prod_{i=0}^{n-1} \frac{1}{\varphi(x+ip)} \\
&= 2\sum_{j=0}^{\infty} \phi(x+(n+j)p) \prod_{i=0}^{n+j} \frac{1}{\varphi(x+ip)} \\
&= 2\sum_{j=n}^{\infty} \phi(x+jp) \prod_{i=0}^{j} \frac{1}{\varphi(x+ip)} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.
\end{aligned}
$$

Hence, $f(x) = h(x)$ holds.

3. Applications to Gamma Functional Equation

The following functional equation

$$g(x+1) = xg(x) \quad \text{for all } x > 0 \tag{6}$$

is called the gamma functional equation. It is well-known that the gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \qquad (x > 0)$$

is a solution of the gamma functional equation (6). Jung ([3], [2]) obtained the stability theorems of the gamma functional equation. We can obtain them from our Theorem as follows:

COROLLARY 2. *If a mapping $g : (0, \infty) \longrightarrow R$ satisfies the inequality*

$$|g(x+1) - xg(x)| \leq \delta$$

for all $x > 0$, then there exists a unique solution $f : (0, \infty) \longrightarrow R$ of the gamma functional equation (6) with

$$|g(x) - f(x)| \leq \frac{3\delta}{x}$$

for all $x > 0$.

PROOF. Apply Theorem and condition (3) with $p = 1$, $\varphi(x) = x$, $\phi(x) = \delta$. □

COROLLARY 3. *If a mapping* $g : (0, \infty) \longrightarrow R$ *satisfies the inequality*

$$|g(x+1) - xg(x)| \le \phi(x)$$

for all $x > 0$, *then there exists a unique solution* $f : (0, \infty) \longrightarrow R$ *of the gamma functional equation* (6) *with*

$$|g(x) - f(x)| \le \Phi(x)$$

for all $x > 0$.

PROOF. Apply Theorem and condition (3) with $p = 1$, $\varphi(x) = x$. □

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