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p-ADIC HEIGHTS

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ABSTRACT. In this paper, for a given p-adic quasicharacter $c_v : k_v^* \to \mathbb{Q}_p$ satisfying a special condition, we will explicitly construct an admissible pairing corresponding to c_v. We define a p-adic height on the arbitrary abelian varieties associated to divisors and c_v by using admissible pairings at every nonarchimedean places. We also show that our p-adic height satisfies similar properties of Néron-Tate's canonical p-adic height.

1. p-adic quasicharacter and admissible pairing

Let l be a prime. Let k_v be a finite extension of $\mathbb{Q}_l$ which is locally compact with respect to $| \ |_v$. Here $| \ |_v = | \ |_l \circ N_{k_v/\mathbb{Q}_l} : k_v^* \to \mathbb{Q}^*$ where $| \ |_l$ is canonical norm defined by the condition $|l|_l = l^{-1}$. The kernel U_v is called the group of units. Let p be a fixed prime number.

DEFINITION 1.1. We call any continuous homomorphism $c_v : k_v^* \to \mathbb{Q}_p$ a p-adic (additive) quasicharacter of the field k_v. A p-adic quasicharacter c_v is said to be unramified if it is trivial on U_v.

For example, $\log_p \circ | \ |_v$ is a p-adic quasicharacter where $\log_p : \mathbb{Q}_p^* \to \mathbb{Z}_p$ denotes the p-adic logarithm extended by the usual convention $\log_p p = 0$. All quasicharacters of the form $\log_p \circ | \ |_v^n$, $n \in \mathbb{Z}$ are unramified.

Suppose k_v is a finite extention of $\mathbb{Q}_l$. We select a fixed element π of uniformizer. We can write an element $\alpha \in k_v^*$ uniquely in the form $\alpha = \tilde{\alpha} \cdot \rho$ where $\tilde{\alpha} \in U_v$ and ρ is power of π. In this case the map $\alpha \to \tilde{\alpha}$ is continuous homomorphism of k_v^* onto U_v which is identity

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on U_v. Let c be a quasicharacter and $\tilde{c}$ be its restriction to U_v. Then $z \mapsto \log_p(\tilde{c}(z) \cdot |z|_v^n)$ is again a quasicharacter.

Now let A be an abelian variety over k_v. We let $D(A)_{k_v}$ be the group of divisors on A whose components are k_v-rational and $D_a(A)_{k_v}$ the subgroup of $D(A)_{k_v}$ which consists of algebraically equivalent to 0 and $Z_0(A)_{k_v}$ the group of 0-cycles of degree 0 whose components are k_v-rational. If $\Delta = (f)$ is principal, for any cycle $\mathfrak{a}$ such that $|\Delta| \cap |\mathfrak{a}| = \varnothing$ we let

$$f(\mathfrak{a}) = \prod_{i=1}^{r} f(a_i)^{n_i}, \text{ for } \mathfrak{a} = \sum_{i=1}^{r} n_i(a_i) \in Z_0(A)_{k_v}.$$

This value depends only on Δ since the constant disappears when we take the product over the points of a 0-cycle of degree 0.

DEFINITION 1.2. Let A be an abelian variety over a local field k_v. A p-adic quasicharacter $c_v : k_v^* \longrightarrow \mathbb{Q}_p$ is said to be admissible if there is a pairing

$$D_a(A)_{k_v} \times Z_0(A)_{k_v} \longrightarrow \mathbb{Q}_p,$$

$$(\Delta, \mathfrak{a}) \mapsto \langle \Delta, \mathfrak{a} \rangle_{c_v} \text{such that } \ |\Delta| \cap |\mathfrak{a}| = \varnothing$$

which satisfies the following properties:

(1) It is bilinear.
(2) If $\Delta = (f)$ is principal then $\langle \Delta, \mathfrak{a} \rangle_{c_v} = c_v(f(\mathfrak{a}))$.
(3) $\langle \Delta_a, \mathfrak{a}_a \rangle_{c_v} = \langle \Delta, \mathfrak{a} \rangle_{c_v}$ for $a \in A(k_v)$, i.e., it is invariant under translation.

The fundamental result of the paper of Néron [3] consists in the construction of a certain canonical $|\ |_v$- pairing $\langle , \rangle_v$. Thus we know that every unramified quasicharacter is admissible. Manin [1] showed that a certain ramified quasicharacter is admissible. Zarhin [5] proved that for any quasi character $c_v : k_v^* \longrightarrow M$ (where M is an injective group) is admissible if and only if it is trivial on the values of the Weil pairing between the torsions of $A(k_v)$ and $A'(k_v)$ where A' is the Picard variety for A. We will explicitly construct an admissible pairing for a p-adic quasicharacter c_v satisfying the above conditions.

Let A be an abelian variety defined over a locally compact field k_v of characteristic 0 with a ring of integers $\mathfrak{O}_v$. Let $\mathcal{A}$ be the Néron model of

A and let $\mathcal{A}^0$ be the identity component of $\mathcal{A}$. The Néron model of the dual variety $A' = \mathrm{Ext}^1(A, \mathbb{G}_m)$ is then $\mathcal{A}' = \mathrm{Ext}^1_{\mathfrak{O}_v}(\mathcal{A}^0, \mathbb{G}_m)$. Thus given a divisor Δ on A defined over k_v and algebraically equivalent to 0, we get a corresponding extension

$$1 \to \mathbb{G}_m \to \mathfrak{X}_\Delta \to \mathcal{A}^0 \to 1.$$

Restricting to $\mathrm{Spec}(k_v)$, we have an exact sequence

$$1 \longrightarrow k_v^* \longrightarrow \mathfrak{X}_\Delta(k_v) \xrightarrow{p} A(k_v) \longrightarrow 1,$$

which splits over $A \setminus |\Delta|$.

Oesterlé showed that the homomorphism $c_v : k_v^* \longrightarrow \mathbb{Q}_p$ can be extended to $\tilde{c}_v : \mathfrak{X}_\Delta(k_v) \to \mathbb{Q}_p$.

THEOREM 1.3 (Oesterlé [4]). *There exists a continuous homomorphism $\tilde{c}_v : \mathfrak{X}_c(k_v) \longrightarrow \mathbb{Q}_p$ extending c_v. If k_v is not a ultrametric field with characteristic p residue field then $\tilde{c}_v$ is unique. If k_v is a finite extension of $\mathbb{Q}_p$ then $\tilde{c}_v$ is unique up to addition of the form $\varepsilon \circ p$, where ε is a continuous homomorphism of $A(k_v)$ into $\mathbb{Q}_p$.*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & k_v^* & \longrightarrow & \mathfrak{X}_\Delta(k_v) & \longrightarrow & A(k_v) & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle c_v} & & \big\downarrow{\scriptstyle \tilde{c}_v} & & \big\uparrow & & \\
 & & \mathbb{Q}_p & & \mathbb{Q}_p & & A - |\Delta| & &
\end{array}$$

We have a section $\sigma_\Delta : A \setminus |\Delta| \to \mathfrak{X}_\Delta(k_v)$ which is unique up to constants in k_v^*. We obviously get a canonical homomorphism

$$\sigma_\Delta : \{\mathfrak{a} \in Z_0(A)_{k_v}; |\mathfrak{a}| \cap |\Delta| = \varnothing\} \to \mathfrak{X}_\Delta(k_v).$$

Thus we have the map

$$\langle\ ,\ \rangle_{c_v} : \{\mathfrak{a} \in Z_0(A)_{k_v}; |\mathfrak{a}| \cap |\Delta| = \varnothing\} \to \mathbb{Q}_p$$

defined by $\langle \Delta, \mathfrak{a} \rangle_{c_v} = \tilde{c}_v \circ \sigma_\Delta(\mathfrak{a})$.

THEOREM 1.5. *Let $c_v : k_v^* \to \mathbb{Q}_p$ be a p-adic quasicharacter which is trivial on the values of Weil pairing between the torsions of $A(k_v)$ and $A'(k_v)$. Then the pairing*

$$\langle , \rangle_{c_v} : \{(\Delta, \mathfrak{a}) \in D_a(A)_{k_v} \times Z_0(A)_{k_v}; |\Delta| \cap |\mathfrak{a}| = \varnothing\} \to \mathbb{Q}_p$$

defined by $\langle \Delta, \mathfrak{a} \rangle_{c_v} = \tilde{c}_v \circ \sigma_\Delta(\mathfrak{a})$ *is admissible.*

PROOF. We need to check conditions of Definition 1.2. First, suppose that $\Delta = (f)$ is principal then $\sigma_\Delta = f$ up to constant. Thus

$$\langle \Delta, \mathfrak{a} \rangle_{c_v} = \tilde{c_v}(f(\mathfrak{a})) = c_v(f(\mathfrak{a})) \quad \text{since} \quad f(\mathfrak{a}) \in k_v^*.$$

Second, since our section σ_Δ is additive in Δ and p-adic quasicharacter is a homomorphism, our pairing is biadditive. Third, to prove invariance under translation, let $a \in A(k_v)$ and $\mathfrak{a} = \sum n_i x_i$. Let a' be a point of $\mathfrak{X}_\Delta$ above a. Let τ_a and $\tau_{a'}$ be the translation by a and a' respectively. Then we can take

$$\sigma_{\Delta_a} = \tau_{a'} \circ \sigma_\Delta \circ \tau_{-a}.$$

Hence for $a \notin |\Delta|$, we get $\sigma_{\Delta_a}(x_i + a) = a'\sigma_\Delta(x_i)$. Then

$$\langle \Delta_a, \mathfrak{a}_a \rangle_{c_v} = \tilde{c}_v \circ \sigma_{\Delta_a}(\mathfrak{a}_a) = \sum n_i\{\tilde{c}_v(a') + \tilde{c}_v(\sigma_\Delta(x_i))\} = \tilde{c}_v \circ \sigma_\Delta(\mathfrak{a})$$

since $\sum n_i = 0$. This proves invariance under translation. □

According to Oesterlé, for v dividing p the extension $\tilde{c}_v$ of c_v is not unique. Thus the above pairing may not be unique. Let c_v be a p-adic quasicharacter, and $\langle\ ,\ \rangle'_{c_v}, \langle\ ,\ \rangle''_{c_v}$ two admissible c_v-pairings. Then their difference $\langle\ ,\ \rangle'_{c_v} - \langle\ ,\ \rangle''_{c_v} = \tau$ is trivial if Δ is linearly equivalent to 0. Such pairings may well be nontrivial.

2. Global p-adic height over a number field

Let K be a number field, $\mathcal{V}$ be the set of places of K and $\mathcal{V}_0$ be the set of finite places of K. Let I_K be the idèle group of K. A continuous homomorphism $c = \sum_{v \in \mathcal{V}_0} c_v : I_K \to \mathbb{Q}_p$ which is trivial on K^* and unramified at all the places v but a finite set of (finite) places will be called a *global p-adic quasicharacter* .

Let A be an abelian variety over a number field K. A global p-adic quasicharacter c will be said to be *admissible* if each c_v is admissible in the sense of Definition 1.2. (Note this definition differs from that given in [2] p. 215.)

Choose a c_v-pairing $\langle\ ,\ \rangle_{c_v}$ for each v. For a disjoint pair $(\Delta,\ \mathfrak{a}) \in D_a(A)\times Z_0(A)$, consider $\sum_{v\in\mathcal{V}_0}\langle\Delta,\mathfrak{a}\rangle_{c_v}$ which we often denote by $\langle\Delta,\mathfrak{a}\rangle_c$. Then triviality of c on K^* implies that $\langle\Delta,\mathfrak{a}\rangle_c$ depends only on the linear equivalence class of Δ.

Let $D \in D(A)$ be a divisor which is not necessarily algebraically equivalent to 0. Then for $a \in A(K), D_a - D$ is algebraically equivalent to 0. Hence $\langle D_a - D, (b) - (0)\rangle_c$ is well defined for all $a, b \in A(K)$. In fact if $D_a - D$ and $(b) - (0)$ intersect then we can choose D' such that $D' \sim D_a - D$ and D' and $(b) - (0)$ have disjoint support. But $\langle D_a - D, (b) - (0)\rangle_c$ depends on the linear equivalence class of $D_a - D$. On the other hand if D is algebraically equivalent to 0 then $D_a - D \sim 0$. Hence $\langle D_a - D, (b) - (0)\rangle_c = 0$.

DEFINITION 2.1. Let c be a global p-adic quasicharacter. Let δ be a element of a Néron -Severei group NS(A)$(=\frac{D(A)}{D_a(A)})$ of A which is represented by D. We define a pairing

$$\langle\ ,\ \rangle_\delta : A(K) \times A(K) \to \mathbb{Q}_p$$

by $\langle a, b\rangle_\delta \equiv \langle D_a - D, (b) - (0)\rangle_c$.

PROPOSITION 2.2. *Using the above notation the pairing* $\langle\ ,\ \rangle_\delta$ *is bilinear.*

PROOF. We compute,

$$\begin{aligned}\langle a+c, b\rangle_\delta &= \langle D_{a+c} - D, (b) - (0)\rangle_c \\ &= \langle D_{a+c} - \dot{D}_a + D_a - D, (b) - (0)\rangle_c \\ &= \langle D_{a+c} - D_a, (b) - (0)\rangle_c + \langle D_a - D, (b) - (0)\rangle_c \\ &= \langle D_c - D, (b) - (0)\rangle_c + \langle D_a - D, (b) - (0)\rangle_c \\ &= \langle c, b\rangle_\delta + \langle a, b\rangle_\delta.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle a, b+c\rangle_\delta &= \langle D_a - D, (b+c) - (0)\rangle_c \\ &= \langle D_a - D, (b+c) - (b) + (b) - (0)\rangle_c \\ &= \langle D_a - D, (b+c) - (b)\rangle_c + \langle D_a - D, (b) - (0)\rangle_c \\ &= \langle T_b^{-1}(D_a - D), (c) - (0)\rangle_c + \langle D_a - D, (b) - (0)\rangle_c \\ &= \langle a, c\rangle_\delta + \langle a, b\rangle_\delta .\end{aligned}$$

The last equality is true because $T_b^{-1}(D_a - D) \sim D_a - D$. Therefore our pairing is bilinear. □

Let D be an ample divisor on A and $\delta \in NS(A)$ be represented by D and $a \in A(K)$. Then $D_a - D$ is algebraically equivalent to 0. We define

$$h_{D,c}(a) = \langle a, a\rangle_c$$

which we call a *p-adic height* . Then $h_{D,c}$ satisfies similar properties of Néron-Tate's canonical heights on abelian varieties.

PROPOSITION 2.3. *Let A be an abelian variety over a number field. The $h_{D,c}$ satisfies the properties*

(1) (*Parallelogram Law*) *For all $a, b \in A(K)$,*

$$h_{D,c}(a+b) + h_{D,c}(a-b) = 2h_{D,c}(a) + 2h_{D,c}(b)$$

(2) *For all $a \in A(K)$ and $m \in \mathbb{Z}$, $h_{D,c}(ma) = m^2 h_{D,c}(a)$.*
(3) *If a is a torsion point then $h_{D,c}(a) = 0$.*

PROOF. (1) and (2) follow from bilinearity of $\langle\ ,\ \rangle_\delta$. For (3), suppose $ma = 0$. Then we have $0 = h_{D,c}(ma) = m^2 h_{D,c}(a)$. Now our assertion follows from divisibility of $\mathbb{Q}_p$. □

Our bilinear form $\langle\ ,\ \rangle_\delta$ induces p-adic height and conversely a p-adic height $h_{D,c}$ induces a bilinear form which can be easily deduced by using linear algebra.

LEMMA 2.4. $\langle a, b\rangle_\delta = \langle b, a\rangle_\delta$, *i.e., this bilinear form is symmetric.*

PROOF. Let $\mathfrak{a} = (a) - (0), \mathfrak{b} = (b) - (0), D_{\mathfrak{a}} = D_a - D, D_{\mathfrak{b}} = D_b - D$,

$$\begin{aligned}\langle a, b\rangle_\delta &= \langle D_a - D, (b) - (0)\rangle_c \\ &= \langle D_{\mathfrak{a}}, \mathfrak{b}\rangle_c = \langle D_{\mathfrak{b}^-}, \mathfrak{a}^-\rangle_c \\ &= \langle \bar{D}_{\mathfrak{b}}, \mathfrak{a}\rangle_c \quad \text{(by using reciprocity law)} \\ &= \langle \bar{D}_b - \bar{D}, \mathfrak{a}\rangle_c \text{ Since } D - \bar{D} \equiv 0, (D_b - D) \sim (\bar{D}_b - \bar{D}) \\ &= \langle D_b - D, (a) - (0)\rangle_c \\ &= \langle b, a\rangle_\delta. \end{aligned}$$

□

THEOREM 2.5. *Let D be an ample divisor and δ be the element of $NS(A)$ which is represented by D. Then we have*

$$\langle a, b\rangle_\delta = \frac{1}{2}(h_{D,c}(a) + h_{D,c}(b) - h_{D,c}(a-b)).$$

PROOF. We have

$$\begin{aligned}\langle a, b\rangle_\delta &= \langle D_a - D, (b) - (0)\rangle_c \\ &= \langle D_a - D, (b) - (a) + (a) - (0)\rangle_c \\ &= h_{D,c}(a) + \langle D_a - D, (b) - (a)\rangle_c \\ &= h_{D,c}(a) + \langle D_a - D_b, (b) - (a)\rangle_c + \langle D_a - D, (b) - (0)\rangle_c \\ &= h_{D,c}(a) + \langle D_{a-b} - D, (0) - (a-b)\rangle_c + \langle D_b - D, (0) - (a)\rangle_c \\ &= h_{D,c}(a) + h_{D,c}(b) - h_{D,c}(a-b) - \langle b, a\rangle_\delta.\end{aligned}$$

Since $\langle\ \rangle_\delta$ is symmetric, $\langle a, b\rangle_\delta = \langle b, a\rangle_\delta$. Hence

$$2\langle a, b\rangle_\delta = h_{D,c}(a) + h_{D,c}(b) - h_{D,c}(a-b).$$

Therefore we have

$$\langle a, b\rangle_\delta = \frac{1}{2}(h_{D,c}(a) + h_{D,c}(b) - h_{D,c}(a-b)).$$

□

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