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CENTER SYMMETRY OF INCIDENCE MATRICES

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ABSTRACT. The T-ideal of $F\langle X\rangle$ generated by x^n for all $x \in X$, is generated also by the symmetric polynomials. For each symmetric polynomial, there corresponds one row of the incidence matrix. Finding the nilpotency of nil-algebra of nil-index n is equivalent to determining the smallest integer $\mathcal{N}$ such that the $\langle n, \mathcal{N}\rangle$-incidence matrix has rank equal to $\mathcal{N}!$. In this work, we show that the $\langle n, \frac{n(n+1)}{2}\rangle^{(1,\dots,n)}$-incidence matrix is center-symmetric.

1. Introduction

Let F be a field of characteristic 0 and A be an F-algebra. If there exists $n \in \mathbb{N}$ such that $a^n = 0$ for all $a \in A$, then A is called a *nil-algebra* and the natural number n is called the *nil-index* of A. And A is *nilpotent of index* m or A has *nilpotency* m if $A^m = 0$, but $A^{m-1} \neq 0$.

THEOREM 1.1. [1, 3] (Nagata-Higman Theorem) *Any nil-algebra of finite nil-index is nilpotent of finite index.*

We denote by $\mathcal{N}(n)$ or simply $\mathcal{N}$ the nilpotency of a nil-algebra of nil-index n. This theorem was proved by Nagata ([3]) in 1952 and Higman showed $\mathcal{N}(n) \leq 2^n - 1$ and $\frac{n^2}{c^2} < \mathcal{N}(n)$ for sufficiently large n. In 1975, Kuzmin ([2]) improved the lower bound of $\mathcal{N}(n)$ to $\frac{n(n+1)}{2}$ and conjectured $\mathcal{N}(n) = \frac{n(n+1)}{2}$. Meanwhile, using the fact that a Young diagram with n^2+1 boxes has either $n+1$ boxes or more in the first row, or $n+1$ boxes or more in the first column, Razmyslov ([4]) showed that $\mathcal{N}(n) \leq n^2$.

Let $F\langle X\rangle = F\langle x_1, x_2, \dots\rangle$ be a free associative algebra in countably many variables. If $a_1, a_2, \dots, a_n \in F\langle X\rangle$, we denote by $S_n(a_1, a_2, \dots, a_n)$

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or simply S_n the sum of the $n!$ products of $a_1, a_2, \dots, a_n$ in every possible order, so called the *symmetric polynomial* of $a_1, a_2, \dots, a_n$, i.e.,

$$S_n(a_1, a_2, \dots, a_n) = \sum_{\sigma \in \mathrm{Sym}(n)} a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} ,$$

where $\mathrm{Sym}(n)$ is the symmetric group on n letters.

An ideal I of $F\langle X\rangle$ is called a *T-ideal* if $\phi(I) \subseteq I$ for every algebra endomorphism ϕ of $F\langle X\rangle$. If we let I_n be the T-ideal generated by x^n for all $x \in X$. then the Nagata-Higman Theorem can be rephrased as following; there exists $\mathcal{N}$ such that $x_1 x_2 \cdots x_{\mathcal{N}} \in I_n$ for all $x_1, x_2, \dots, x_{\mathcal{N}} \in X$ if $x_i^n \in I_n$, $1 \le i \le \mathcal{N}$.

LEMMA 1.2. *The ideal I_n is generated by S_n.*

Throughout this paper, N will denote $\frac{n(n+1)}{2}$.

Let's consider a matrix whose rows and columns are indexed by the various $S_n(*, \dots, *)$ where the total degree of $*$'s is m, and all the multilinear monomials of degree m in $F\langle x_1, x_2, \dots x_m\rangle$. First of all, one labels the columns by the (multilinear) monomials of degree m lexicographically. In other words, the first column is labeled by $x_1\ x_2 \cdots\ x_{m-2}\ x_{m-1}\ x_m$, the second by $x_1\ x_2 \cdots\ x_{m-2}\ x_m\ x_{m-1}$ and so on. Thus the last column is indexed by $x_m\ x_{m-1}\ x_{m-2} \cdots\ x_2\ x_1$. We use $1, 2, \cdots$, for $x_1, x_2, \cdots$, if there is no risk of confusion. Suppose that j-th column is indexed by $i_1^j \cdots i_m^j$ or simply $i_1 \cdots i_m$. Fix a partition $P = (p_1, \dots, p_n)$ of m with n parts where $p_i \le p_{i+1}$, $1 \le i \le n-1$. Then j-th row of the matrix corresponding to the partition $P = (p_1, \dots, p_n)$ is labeled by

$$S_n(i_1 \cdots i_{p_1}, i_{p_1+1} \cdots i_{p_1+p_2}, \dots, i_{p_1+\cdots+p_{n-1}+1} \cdots i_{p_1+\cdots+p_n}),$$

and the matrix is called $\langle n, m\rangle^P$-incidence matrix. In the j-th row one places 1 for the columns labeled by the monomials that appear in that row index, and 0 elsewhere. In other words, if

$$\begin{aligned} &S_n(i_1 \cdots i_{p_1}, i_{p_1+1} \cdots i_{p_1+p_2}, \dots, i_{p_1+\cdots+p_{n-1}+1} \cdots i_{p_1+\cdots+p_n}) \\ &= i_1 \cdots i_{p_1} i_{p_1+1} \cdots i_{p_1+p_2} \cdots i_{p_1+\cdots+p_{n-1}+1} \cdots i_{p_1+\cdots+p_n} \\ &\quad + \cdots + i_{p_1+\cdots+p_{n-1}+1} \cdots i_{p_1+\cdots+p_n} \cdots i_{p_1+1} \cdots i_{p_1+p_2} i_1 \cdots i_{p_1}, \end{aligned}$$

then we put 1 for the columns labeled by

$$i_1 \cdots i_{p_1} i_{p_1+1} \cdots i_{p_1+p_2} \cdots i_{p_1+\cdots+p_{n-1}+1} \cdots i_{p_1+\cdots+p_n},$$

$$\dots,$$

$$i_{p_1+\cdots+p_{n-1}+1} \cdots i_{p_1+\cdots+p_n} \cdots i_{p_1+1} \cdots i_{p_1+p_2} i_1 \cdots i_{p_1}.$$

TABLE 1. The $\langle 2,3\rangle^{(1,2)}$-incidence matrix.

	123	132	213	231	312	321
$S_2(1,23)$	1	0	0	1	0	0
$S_2(1,32)$	0	1	0	0	0	1
$S_2(2,13)$	0	1	1	0	0	0
$S_2(2,31)$	0	0	0	1	1	0
$S_2(3,12)$	1	0	0	0	1	0
$S_2(3,21)$	0	0	1	0	0	1

For instance, the $\langle 2,3\rangle^{(1,2)}$-incidence matrix is in Table 1. By stacking up those $\langle n,m\rangle^{P}$-incidence matrices for all partition P of m with n parts, we may get the $\langle n,m\rangle$-incidence matrix M with $k \cdot m!$ rows and $m!$ columns where k is the number of partitions of m with n parts. To have the nilpotency of a nil-algebra of nil-index n as m, it is sufficient to show that m is the smallest positive integer such that

$$\text{rank of } \langle n,m\rangle\text{-incidence matrix} = m!.$$

2. Incidence Matrix

Let's start with the simple facts.

PROPOSITION 2.1. *Let q be a positive integer. Then the followings hold.*

(I) $q! = 1 + 1\cdot 1! + 2\cdot 2! + \cdots + (q-1)\cdot(q-1)!$.
(II) *For any positive integer s that is less than or equal to $q!$, there exists a unique ordered pair $[s_1,\ldots,s_{q-1}]$ such that*

$$s = 1 + s_1\cdot 1! + s_2\cdot 2! + \cdots + s_{q-1}\cdot(q-1)!,$$

where $0 \le s_i \le i$, $1 \le i \le q-1$.

PROOF. (I) Suppose that the statement is true for any integer less than or equal to $q-1$. Then

$$\begin{aligned} q! &= (q-1)! + (q-1)\cdot(q-1)! \\ &= 1 + 1\cdot 1! + \cdots + (q-2)\cdot(q-2)! + (q-1)\cdot(q-1)!.. \end{aligned}$$

(II) If $s = q!$, then it is clear by (I). Suppose $s < q!$. Then there exists the largest positive integer k such that $(k-1)! \leq s-1 < k!$ for $1 \leq k \leq q$. Let s_{k-1} be the largest integer such that $s - s_{k-1}(k-1)!$ is nonnegative. By repeating this for $s - s_{k-1}(k-1)!$, one can get $[s_1, \ldots, s_{q-1}]$ such that

$$s = 1 + s_1\cdot 1! + s_2\cdot 2! + \cdots + s_{q-1}\cdot(q-1)!.$$

If

$$\begin{aligned} &1 + r_1\cdot 1! + \cdots + r_{q-1}\cdot(q-1)! = s \\ &= 1 + s_1\cdot 1! + \cdots + s_{q-1}\cdot(q-1)!, \end{aligned} \tag{1}$$

where $0 \leq r_i, s_i \leq i$, $1 \leq i \leq q-1$, then we let l be the largest integer such that $r_l \neq s_l$. Without loss of generality, we may assume $r_l < s_l$. Then in Eq (1), one gets

$$l! \leq (s_l - r_l)l! = (r_{l-1} - s_{l-1})(l-1)! + \cdots = \sum_{j=1}^{l-1}(r_j - s_j)j! < l! \tag{2}$$

which is a contradiction so that we conclude $r_i = s_i$ for $i = 1, \ldots, q-1$. □

Suppose that $j \leq \left(\frac{n(n+1)}{2}\right)! = N!$ and

$$j = 1 + j_1\cdot 1! + j_2\cdot 2! + \cdots + j_{N-1}\cdot(N-1)!,$$

where $0 \leq j_k \leq k$, $1 \leq k \leq N-1$. The multilinear monomial $i_1^j \cdots i_N^j$ corresponding to j-th column, denoted by $C(j)$, can be found as following;

(I) Choose the $(j_{N-1}+1)$-st smallest number i_1^j in $\{1, 2, \ldots, N\}$.
(II) The k-th factor of $i_1 \cdots i_N$ is the $(j_{N-k}+1)$-st smallest number in $\{1, 2, \ldots, N\} \setminus \{i_1, i_2, \ldots, i_{k-1}\}$.

From the construction of the $\langle n, N\rangle^{(1,\ldots,n)}$-incidence matrix, the j-th row index, denoted by $R(j)$, is $S_n(i_1, i_2 i_3, \ldots, i_{\frac{(n-1)n}{2}+1} \cdots i_N)$.

REMARK 2.1. There is an 1-1 correspondence between the followings;

$$\begin{aligned} & \text{an integer } j, \text{ where } 1 \le j \le N! \\ \leftrightarrow\ & \text{an ordered pair } [j_1, \dots, j_{N-1}] \\ \leftrightarrow\ & i_1 i_2 i_3 \cdots i_{\frac{(n-1)n}{2}+1} \cdots i_N \end{aligned}$$

If there is no risk of confusion, we use any one of them. So $S_n(i_1, i_2 i_3, \dots, i_{\frac{(n-1)n}{2}+1} \cdots i_N) = [j_1, \dots, j_{N-1}] + \dots + [k_1, \dots, k_{N-1}]$ means that $S_n(i_1, i_2 i_3, \dots, i_{\frac{(n-1)n}{2}+1} \cdots i_N)$ includes the monomials corresponding to $[j_1, \dots, j_{N-1}]$ and $[k_1, \dots, k_{N-1}]$.

The incidence matrix has entries either 0 or 1, i.e., $\{0,1\}$-matrix, with constant row sums and constant column sums. To show $\mathcal{N}(2) = 3$, it is sufficient to show the $\langle 2,3 \rangle^P$-incidence matrix has some square submatrix whose determinant is nonzero for some partition P of 3 with 2 parts. Then P must be $(1,2)$. Each rows and each columns of the submatrix include all 1's in those rows and columns in the incidence matrix. In fact the $\langle 2,3 \rangle^{(1,2)}$-incidence matrix has the following 3×3 submatrix,

	123	231	312
$S_2(1,23)$	1	1	0
$S_2(2,31)$	0	1	1
$S_2(3,12)$	1	0	1

whose determinant is 2. This observation is generalized in the following lemma.

PROPOSITION 2.2. *For a nil-algebra A of finite index n, the nilpotency of A is $\mathcal{N}(n)$ if and only if $\mathcal{N}(n)$ is the smallest integer such that the $\langle n, \mathcal{N}(n) \rangle$-incidence matrix has an invertible submatrix.*

PROOF. It's clear that if the $\langle n, \mathcal{N}(n) \rangle$-incidence matrix has such invertible submatrix, then the nilpotency of A is less than or equal to $\mathcal{N}(n)$.

Conversely, if the nilpotency of A is $\mathcal{N}(n)$, then $12 \cdots \mathcal{N}(n) = \sum c_I S_n(W_I)$ where $W_I = \{w_1^I, \dots, w_n^I\}$. If $S_n(W_K)$ is linearly dependent, then we substitute $S_n(W_K) = \sum c_J S_n(W_J)$. One gets $12 \cdots \mathcal{N}(n) = \sum c_I S_n(W_I)$ in which all of $S_n(W_I)$ are linearly independent. These $S_n(W_I)$ form the rows of the invertible submatrix. □

PROPOSITION 2.3. *Let $(1^{r_1}2^{r_2}\cdots k^{r_k})$ be a partition of m with n parts where the superscript r_i is the multiplicity of part of size i. The $\langle n,m\rangle$-incidence matrix is a $\{0,1\}$-matrix, with constant row sums $n!$, and constant column sums*

$$\sum \frac{n!}{\prod_{i=1}^{k} r_i!},$$

where the summation is taken over all partition types of m with n parts.

PROOF. The $\langle n,m\rangle$-incidence matrix has $m!$ columns and there are $n!$ many 1's in each row if $m \geq n$, i.e., each row has the constant row sum $n!$. For the column sum, we need to count the number of ways dividing the string $12\cdots m$ into n sets, that is, the column sum of the $\langle n,m\rangle$-incidence matrix is equal to the number of way partitioning $12\cdots m$ into n submonomials, each of which has length r_i without changing the order of variables in $12\cdots m$. Hence the column sum of the $\langle n,m\rangle$-incidence matrix is

$$\sum \frac{n!}{\prod_{i=1}^{k} r_i!},$$

where the summation is taken over all partition types of m with n parts. The number of distinct rows in the $\langle n,m\rangle$-incidence matrix is

$$\sum \frac{m!}{\prod_{i=1}^{k} r_i!},$$

where the summation is taken over all partition types of m with n parts. □

For example, in the $\langle 3,6\rangle$-incidence matrix, 123456 appears in the following rows;

partition type	rows including 123456
$(1^2 4) = (4,1,1)$	$S_3(1234,5,6),\ S_3(1,2345,6),\ S_3(1,2,3456)$;
$(1^1 2^1 3^1) = (3,2,1)$	$S_3(123,45,6),\ S_3(123,4,56),\ S_3(12,3,456),$ $S_3(12,345,6),\ S_3(1,23,456),\ S_3(1,234,56)$;
$(2^3) = (2,2,2)$	$S_3(12,34,56)$.

The diagonal entries of $\langle n,N\rangle^{(1,\dots,n)}$-incidence matrix are 1 and the dimension of the $\langle n,N\rangle^{(1,\dots,n)}$-incidence matrix is $N! \times N!$.

DEFINITION 2.4. A square matrix $(b_{ij})_{p\times p}$ is called *center-symmetric* if $b_{ij} = b_{p-i+1,p-j+1}$ for an even positive integer p.

THEOREM 2.5. *The $\langle n,N\rangle^{(1,\dots,n)}$-incidence matrix is center-symmetric.*

PROOF. Suppose that (k,l)-entry of the $\langle n,N\rangle^{(1,\dots,n)}$-incidence matrix is 1 and

$$k = 1 + k_1\cdot 1! + k_2\cdot 2! + \cdots + k_{N-1}\cdot (N-1)!,$$

$$l = 1 + l_1\cdot 1! + l_2\cdot 2! + \cdots + l_{N-1}\cdot (N-1)!,$$

where $0 \le k_m, l_m \le m$, $1 \le m \le N-1$. Then

$$\begin{aligned} R(k) &= S_n(i_1^k, i_2 i_3^k, \dots, i^k_{\frac{(n-1)n}{2}-1}\cdots i^k_{N-1}) \\ &= [k_1,\dots,k_{N-1}] + \cdots + [l_1,\dots,l_{N-1}]. \end{aligned}$$

The monomials $[k_1,\dots,k_{N-1}]$ and $[l_1,\dots,l_{N-1}]$ are the (k,k)- and (k,l)-entries of the incidence matrix. But

$$\begin{aligned} R(N!-k+1) &= S_n(i_1^{N!-k+1},\dots, i^{N!-k+1}_{\frac{(n-1)n}{2}-1}\cdots i^{N!-k+1}_{N-1}) \\ &= [1-k_1,\dots,j-k_j,\dots,(N-1)-k_{N-1}] + \dots \\ &\quad + [1-l_1,\dots,j-l_j,\dots,(N-1)-l_{N-1}], \\ &\text{for } 1 \le j \le N-1 \end{aligned}$$

since $N! - [k_1,\dots,k_{N-1}] = [1-k_1,\dots,j-k_j,\dots,(N-1)-k_{N-1}]$ and $i_j^{N!-k+1} = N+1-i_j^k$. □

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