J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
7(2000), no. 1, 1–6

A RADO TYPE EXTENSION OF HÖLDER'S INEQUALITY

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ABSTRACT. An extension of Hölder's inequality whose discrete form is described as follows is given. Let ν be a positive measure on a space Y, $\nu(Y) \neq 0$, and let f_j $(j = 1, 2, \cdots, n)$ be positive ν-integrable functions on Y. If $\alpha_j > 0$ $(j = 1, 2, \cdots, n)$ and β_j $(j = 1, 2, \cdots, k < n)$ are related to be

$$\sum_{j=1}^{n} \alpha_j = 1, \text{ and } \beta_j = \frac{\alpha_j}{\sum_{j=1}^{k} \alpha_j},$$

then

$$1 - \frac{\int_Y \left(\prod_1^n f_j^{\alpha_j}\right)^q d\nu}{\prod_1^n \left(\int_Y f_j^q \, d\nu\right)^{\alpha_j}} \geq \frac{k}{n} \left\{ 1 - \frac{\int_Y \left(\prod_1^k f_j^{\beta_j}\right)^q d\nu}{\prod_1^k \left(\int_Y f_j^q \, d\nu\right)^{\beta_j}} \right\}.$$

1. Introduction

Throughout the paper, we let $X = (X, \mathcal{S}, \mu)$ and $Y = (Y, \mathcal{T}, \nu)$ be σ-finite measure spaces with positive measures μ and ν. When we call f defined on $X \times Y$ measurable it refers to $(\mathcal{S} \times \mathcal{T})$-measurable. $\mu \times \nu$ denotes the product measure of μ and ν (see [R, Chapter 7]). We discard the obvious case $\nu(Y) = 0$.

If $0 \leq x \leq 1$, then Hölder's inequality says that

$$\int_Y f_1(y)^x f_2(y)^{1-x} d\nu(y) \leq \left(\int_Y f_1(y) d\nu(y)\right)^x \left(\int_Y f_2(y) d\nu(y)\right)^{1-x} \tag{1.1}$$

for all positive functions f_1 and f_2 of $L^1(\nu)$. It is known that (1.1) can be extended to the case of a multiple product of functions [BB]. (1.1) was generalized in [K1] to the following theorem.

Received by the editors December 6, 1999, and in revised form February 17, 2000.
2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 28A35.
Key words and phrases. Hölder's inequality.

Theorem A (Continuous form of Hölder's inequality [K1, Theorem 1]). *Let $\mu(X) = 1$. Let $f(x,y)$ be a positive measurable function defined on $X \times Y$. Then*

$$\int_Y \exp\left(\int_X \log f\, d\mu\right) d\nu \;\le\; \exp\left\{\int_X \log\left(\int_Y f\, d\nu\right) d\mu\right\}. \tag{1.2}$$

Equality holds in (1.2) *as a nonzero finite value if and only if*

$$f(x,y) = g(x)h(y) \quad \textit{almost everywhere}\, \mu \times \nu$$

for a positive μ-measurable function g with $-\infty < \int_X \log g d\mu < \infty$ and a positive ν-measurable h with $\int_Y h\, d\nu = 1$.

As is well known, the arithmetic-geometric mean inequality (abbreviated as AM-GM hereafter) is of the form

$$G_n \;=\; \prod_{j=1}^{n} {x_j}^{1/n} \;\le\; \frac{1}{n}\sum_{j=1}^{n} x_j \;=\; A_n$$

for n positive real numbers $x_1, x_2, \cdots, x_n$. It has played a central role in the development of the theory of inequalities. We refer to the book "Inequalities" by Beckenbach and Bellman [BB], where one can find interesting collections of different proofs for what is so called "probably the most interesting and certainly a keystone of the theory of inequalities" [BB, p3]. Also, we refer to the classical book "Inequalities" of Hardy, Littlewood and Polya [HLP]. Among various generalizations and extensions of AM-GM, the arithmetic-geometric mean inequality, there are the inequality of Rado and the inequality of Popoviciu. The inequality of Rado (see [BB, p.12, Eq. (4)] or [HLP, p.61]) extends AM-GM to

$$(n-1)(A_{n-1} - G_{n-1}) \le n(A_n - G_n),$$

and the inequality of Popoviciu [P] extends AM-GM to

$$\left(\frac{A_{n-1}}{G_{n-1}}\right)^{n-1} \le \left(\frac{A_n}{G_n}\right)^{n}.$$

If $0 < \mu(X_1) < \infty$, $0 < \nu(Y_1) < \infty$, $X_1 \subset X$, $Y_1 \subset Y$ and f is a positive function of $L^1(\mu \times \nu)$, then we denote $G_{X_1} f$ and $A_{Y_1} f$ respectively by

$$G_{X_1} f(y) = \exp \int_{X_1} \log\, f(x,y)\, \frac{d\mu(x)}{\mu(X_1)}, \quad y \in Y, \tag{1.3}$$

and

$$A_{Y_1} f(x) = \int_{Y_1} f(x,y) \frac{d\nu(y)}{\nu(Y_1)}, \quad x \in X. \tag{1.4}$$

(1.3) and (1.4) are defined almost everywhere because

$$f_x(y) = f(x,y), \quad y \in Y, \quad \text{and} \quad f_y(x) = f(x,y), \quad x \in X,$$

are functions respectively of $L^1(\mu)$ and $L^1(\nu)$ almost everywhere (see [R, Theorem 7.8]). These are the geometric mean (with respect to X_1) of f at y and the arithmetic mean (with respect to Y_1) of f at x. When f is a function of single variable, we use $G_{X_1} f$, $A_{Y_1} f$, etc. Mixed means such as $G_X(A_Y f)$ and $A_Y(G_X f)$ are defined by the obvious meanings. If $0 < \nu(Y) < \infty$, (1.2) can be expressed as

$$A_Y(G_X f) \le G_X(A_Y f). \tag{1.5}$$

It is worth while to see that Hölder's inequality is expressible in terms of the arithmetic inequality and the geometric inequality.

If f is a μ integrable function on X, we denote

$$M_X^q f = M_q(f, X) = \int_X f^q(x) d\mu(x)$$

for $0 < q < \infty$ and

$$M_X^0 f = M_0(f, X) = \exp\left(\int_X \log f(x) d\mu(x)\right).$$

Then as the same manner Rado's and Popoviciu's inequalities extend AM-GM, the following extends (1.5).

Theorem 1. *If X_1 is a μ-measurable subset of X, then we have*

$$1 - \frac{M_Y^q(M_X^0 f)}{M_X^0(M_Y^q f)} \ge \frac{\mu(X_1)}{\mu(X)} \left(1 - \frac{M_Y^q(M_{X_1}^0 f)}{M_{X_1}^0(M_Y^q f)}\right) \tag{1.6}$$

for all $q : 0 < q < \infty$ and for all positive function f of $L^1(\mu \times \nu)$.

Corollary. *If $\alpha_j > 0$ $(j = 1, 2, \cdots, n)$ and β_j $(j = 1, 2, \cdots, k < n)$ are related to be*

$$\sum_{j=1}^n \alpha_j = 1 \quad \text{and} \quad \beta_j = \frac{\alpha_j}{\sum_{j=1}^k \alpha_j},$$

then

$$1-\frac{\int_Y\left(\prod_1^n f_j(y)^{\alpha_j}\right)^q d\nu(y)}{\prod_1^n\left(\int_Y f_j^q(y)\,d\nu(y)\right)^{\alpha_j}}\geq\frac{k}{n}\left\{1-\frac{\int_Y\left(\prod_1^k f_j(y)^{\beta_j}\right)^q d\nu(y)}{\prod_1^p\left(\int_Y f_j^q(y)\,g\nu(y)\right)^{\beta_j}}\right\} \tag{1.7}$$

for all $q: 0<q<\infty$.

2. Jensen's inequality

We call $\{X_1,X_2\}$ a measurable partition of X if X_1,X_2 are measurable subsets of X, $X_1\cap X_2=\emptyset$, and $X_1\cup X_2=X$. For the proof of Theorem 1, we make use of the following variant of Jensen's inequality.

Lemma B [K2]. *Let f be a real function of $L^1(\mu)$. Let $\{X_1,X_2\}$ be a measurable partition of X with $\mu(X_1)=a$ and $\mu(X_2)=b-a, 0<a<b<\infty$. If ϕ is a convex function defined on an open (possibly infinite) interval containing $f(X)$, we have*

$$\int_{X_2}\phi\circ f\,d\mu\;\geq\;b\,\phi\left(\int_X f\,\frac{d\mu}{b}\right)-a\,\phi\left(\int_{X_1} f\,\frac{d\mu}{a}\right). \tag{2.1}$$

If ψ is a concave function defined on an (possibly infinite) open interval containing $f(X)$, we have

$$\int_{X_2}\psi\circ f\,d\mu\;\leq\;b\,\psi\left(\int_X f\,\frac{d\mu}{b}\right)-a\,\psi\left(\int_{X_1} f\,\frac{d\mu}{a}\right). \tag{2.2}$$

Note that the limiting case $\mu(X_1)=a=0$ of (2.1) is nothing but Jensen's inequality.

Remark. If we take $\phi(t)=e^t$, and $f(x)=\log g(x)$, then (2.1) becomes

$$\int_X g\,d\mu-\int_{X_1}g\,d\mu\;\geq\;b\,\exp\left(\int_X\log g\,\frac{d\mu}{b}\right)-a\,\exp\left(\int_{X_1}\log g\,\frac{d\mu}{a}\right), \tag{2.5}$$

that is,

$$b\,A_X g-a\,A_{X_1}g\;\geq\;b\,G_X g-a\,G_{X_1}g$$

or, equivalently,

$$A_X g-G_X g\;\geq\;\frac{\mu(X_1)}{\mu(X)}\left\{A_{X_1}g-G_{X_1}g\right\}.$$

This is a continuous form of the classical Rado's inequality on AM-GM discussed in Section 1.

3. Proof of the results

Proof of Theorem 1. Let $E = \{x \in X : M_Y^q f(x) \text{ is undefined or } M_Y^q f(x) = \infty\}$. Then $\nu(E) = 0$. Thus we may assume $E = \emptyset$ in proving (1.6) because $G_X f = G_{X-E} f$, $G_{X_1} f = G_{X_1 - E} f$, $G_X(M_Y^q f) = G_{X-E}(M_Y^q f)$ and $G_{X_1}(M_Y^q f) = G_{X_1 - E}(M_Y^q f)$. Let

$$g(x) = \frac{f(x,y)}{A_Y f(x)}, \quad x \in X.$$

Then (2.5) becomes

$$\begin{aligned} &\int_{X_2} \frac{f(x,y)}{M_Y^q f(x)}\, d\mu(x) \\ &\geq b \exp\left(\int_X \log \frac{f(x,y)}{M_Y^q f(x)} \frac{d\mu(x)}{b}\right) - a \exp\left(\int_{X_1} \log \frac{f(x,y)}{M_Y^q f(x)} \frac{d\mu(x)}{a}\right). \end{aligned} \tag{3.1}$$

Taking integration with respect to $d\nu(y)$ over Y on both sides of (3.1), we have, by changing the order of integration, that

$$\begin{aligned} \mu(X_2) &= \int_{X_2} \frac{\int_Y f^q\, d\nu}{\int_Y f^q\, d\nu}\, d\mu \\ &\geq b \int_Y \exp \int_X \log \frac{f^q}{M_Y^q f} \frac{d\mu}{b} d\nu - a \int_Y \exp \int_{X_1} \log \frac{f^q}{M_Y^q f} \frac{d\mu}{a} d\nu \\ &= b \int_Y \exp\left(\int_X (\log f^q - \log M_Y^q f) \frac{d\mu}{b}\right) d\nu \\ &\qquad - a \int_Y \exp\left(\int_{X_1} (\log f^q - \log M_Y^q f) \frac{d\mu}{b}\right) d\nu \\ &= b \int_Y \frac{\exp \int_X \log f^q \frac{d\mu}{b}}{\exp \int_X \log M_Y^q f \frac{d\mu}{b}} d\nu - a \int_Y \frac{\exp \int_{X_1} \log f^q \frac{d\mu}{b}}{\exp \int_{X_1} \log M_Y^q f \frac{d\mu}{b}} d\nu \\ &= b \frac{\int_Y (M_0(f,X))^q \frac{d\mu}{b}}{M_0(M_q(f,Y),X)} - a \frac{\int_Y (M_0(f,X_1))^q \frac{d\mu}{a}}{M_0(M_q(f,Y),X)}. \end{aligned}$$

Therefore

$$\mu(X_2) \geq \mu(X) \frac{M_Y^q(M_X^0 f)}{M_X^0(M_Y^q f)} - \mu(X_1) \frac{M_Y^q(M_{X_1}^0 f)}{M_{X_1}^0(M_Y^q f)},$$

which is equivalent to

$$\mu(X)\left(1-\frac{M_Y^q(M_X^0 f)}{M_X^0(M_Y^q f)}\right) \geq \mu(X_1)\left(1-\frac{M_Y^q(M_{X_1}^0 f)}{M_{X_1}^0(M_Y^q, f)}\right)$$

This verifies (1.6). □

Proof of Corollary. By taking $X = \{1, 2, \cdots, n\}$, $X_1 = \{1, 2, \cdots, k\}$ (where $k < n$), $f(x, y) = f_x(y)$, $x \in X$, and $d\mu = \sum_j \alpha_j d\mu_j$, with $d\mu_j$ the unit mass concentrated at j, (1.6) reduces to (1.7). □

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