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FUNDAMENTAL THEOREM OF NULL CURVES

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Abstract. The purpose of this paper is to prove the fundamental existence and uniqueness theorems of null curves in semi-Riemannian manifolds M of index 2.

1. Introduction

Theory of space curves of a Riemannian manifold M is fully developed and its local and global geometry is well-known. In case M is proper semi-Riemannian, there are three categories of curves, namely, spacelike, timelike and null, depending on their causal character.

We know, from O'Neill [8], that the study of timelike curves has many similarities with the spacelike curves. However, since the induced metric of a null curve is degenerate, this case is much more complicated and also different from the non-degenerate case.

Motivated by the growing importance of null curves and surfaces (in particular, null congruences) in mathematical physics, and very limited information available on its geometry, Duggal-Bejancu [3] published their work on "*general theory of null curves in Lorentz manifolds*" (cf. [3, Chapter 3, pp. 52–76]). They constructed a Frenet frame and proved the fundamental existence and uniqueness theorem for this class of null curves. But their study was restricted to Lorentz manifolds M.

Recently Duggal-Jin [4] studied the geometry of null curves in semi-Riemannian manifolds M of index 2. We showed that it is possible to construct two types of Frenet frames suitable for M, each invariant under any causal change and derived several results needed to discuss null curves in M.

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The objective of this paper is to further study on null curves in a semi-Riemannian manifold M of index 2 and to prove the fundamental existence and uniqueness theorems of null curves in M, with a variety of Frenet frames of each type.

2. Frenet Equations of Null Curves

Let (M, g) be a real $(m+2)$-dimensional semi-Riemannian manifold of index $q \geq 1$ and C be a smooth null curve in M locally given by

$$x^i = x^i(t), \qquad t \in I \subset \mathbb{R}, \qquad i \in \{0, 1, \cdots, m+1\}$$

for a coordinate neighborhood $\mathcal{U}$ on C. Then, the tangent vector field $\frac{d}{dt}$ on $\mathcal{U}$ satisfies $g(\frac{d}{dt}, \frac{d}{dt}) = 0$. We denote by TC the "tangent bundle" of C and $TC^{\perp}$ the "TC perp" (see O'Neill [8]). This means that

$$TC^{\perp} = \bigcup_{x \in C} T_x C^{\perp} \text{ and } T_x C^{\perp} = \{V_x \in T_x M : g(V_x, \xi_x) = 0\}$$

where ξ_x is null vector tangent over C at any $x \in C$. Clearly, $TC^{\perp}$ is a vector bundle over C of rank $m+1$. Since ξ_x is null, it follows that the tangent bundle TC is a vector subbundle of $TC^{\perp}$, of rank 1. This implies that $TC^{\perp}$ is not complementary to TC in $TM_{|C}$. Thus we must find complementary vector bundle to TC in TM which will play the role of the normal bundle $TC^{\perp}$ consistent with the classical non-degenerate theory.

A few researchers have done research on this matter dealing with only specified problems (see [2], [3], [5], [6]). Recently, Duggal-Bejancu [3] developed a general mathematical theory to deal with the null case, which we brief as follows:

Summary 1 (Duggal-Bejancu [3]). Suppose $S(TC^{\perp})$ denotes the complementary vector subbundle to TC in $TC^{\perp}$, i.e.,

$$TC^{\perp} = TC \perp S(TC^{\perp})$$

where $\perp$ means the *orthogonal direct sum.* Then it follows that $S(TC^{\perp})$ is a non-degenerate m-dimensional vector subbundle of TM. We call $S(TC^{\perp})$ a *screen vector bundle* of C, which is non-degenerate. In this case, we have

$$TM_{|C} = S(TC^{\perp}) \perp S(TC^{\perp})^{\perp},$$

where $S(TC^{\perp})^{\perp}$ is a 2-dimensional complementary orthogonal vector subbundle to $S(TC^{\perp})$ in $TM_{|C}$.

Throughout this paper we denote by $F(C)$ the algebra of smooth functions on C and by $\Gamma(E)$ the $F(C)$ module of smooth sections of a vector bundle E over C. We use the same notation for any other vector bundle. The reader may find basic information on differential geometric structures (such as vector bundles, differential operators and distributions on manifolds) in standard books such as Kobayashi-Nomizu [7] and Spivak [9].

Theorm 2.1 (Duggal-Bejancu [3]). *Let C be a null curve of a semi-Riemannian manifold (M, g) and $S(TC^{\perp})$ a screen vector bundle of C. Then, there exists a unique vector bundle E over C, of rank 1, such that on each coordinate neighborhood $\mathcal{U} \subset C$ there is a unique section $N \in \Gamma(E_{|\mathcal{U}})$ satisfying*

$$g(\frac{d}{dt}, N) = 1, \qquad g(N, N) = g(N, X) = 0, \tag{1}$$

for every $X \in \Gamma(S(TC^{\perp})_{|\mathcal{U}})$.

We denote the vector bundle E by $\text{ntr}\,(C)$ and call it the *null transversal bundle* of C with respect to $S(TC^{\perp})$. Next consider the vector bundle

$$\text{tr}(C) = \text{ntr}(C) \perp S(TC^{\perp}),$$

which according to (1) is complementary but not orthogonal to TC in $TM_{|C}$. More precisely, we have

$$TM_{|C} = TC \oplus \text{tr}\,(C) = (TC \oplus \text{ntr}\,(C)) \perp S(TC^{\perp}).$$

We call $\text{tr}(C)$ the *transversal vector bundle* of C with respect to $S(TC^{\perp})$. The vector field N in Theorem 2.1 is called the *null transversal vector field* of C with respect to $\frac{d}{dt}$. As $\{\frac{d}{dt}, N\}$ is a null basis of $\Gamma((TC \oplus \text{ntr}(C))_{|\mathcal{U}})$, satisfying (1), we obtain

Proposition 2.1 (Duggal-Bejancu [3]). *Let C be a null curve of a semi-Riemannian manifold (M, g) of index $q \geq 1$. Then, any screen vector bundle of C is semi-Riemannian of index $q - 1$.*

In this paper, we assume that M is of index 2, then we obtain two types (labeled Types **I** and **II**) of Frenet equations of their respective null curve (cf. [4]).

Denote $\frac{d}{dt} = \lambda$ and ∇ the "Levi-Civita connection" on M. Since any screen distribution of C will be Lorentzian from the Proposition 2.1, there are three cases (spacelike, timelike and null) by the causality of the vector field $\nabla_{\lambda}\,\lambda$.

Type I. First, we obtain a class of null curves C by general method whose Frenet frame is made up of two null vectors λ and N, one timelike vector and $m-1$ spacelike vectors (cf. Duggal-Jin [4]).

We denote the Frenet equations of this particular class of C by Type **I**:

$$\begin{cases} \nabla_\lambda \lambda &= h\,\lambda + k_1 W_1 \\ \nabla_\lambda N &= -h\,N + k_2 W_1 + k_3 W_2 \\ \epsilon_1 \nabla_\lambda W_1 &= -k_2\,\lambda - k_1 N + k_4 W_2 + k_5 W_3 \\ \epsilon_2 \nabla_\lambda W_2 &= -k_3\,\lambda - k_4 W_1 + k_6 W_3 + k_7 W_4 \\ \epsilon_3 \nabla_\lambda W_3 &= -k_5 W_1 - k_6 W_2 + k_8 W_4 + k_9 W_5 \\ & \vdots \\ \epsilon_{m-1} \nabla_\lambda W_{m-1} &= -k_{2m-3} W_{m-3} - k_{2m-2} W_{m-2} + k_{2m} W_m \\ \epsilon_m \nabla_\lambda W_m &= -k_{2m-1} W_{m-2} - k_{2m} W_{m-1} \end{cases} \tag{2}$$

where h and $\{k_1, \cdots, k_{2m}\}$ are smooth functions on $\mathcal{U}$, $\{W_1, \cdots, W_m\}$ is a certain orthonormal basis of $\Gamma(S(TC^\perp)_{|\mathcal{U}})$ and, for each i, $\epsilon_i = g(W_i, W_i)$ is the signature of W_i such that $\epsilon_i = +1$ or -1. We call

$$F_{\rm I} = \{\lambda, N, W_1, \cdots, W_m\}$$

a *Frenet frame* of Type **I** on M along C, with respect to a given screen vector bundle $S(TC^\perp)$ and the equations (2) are called its *Frenet equations* of Type **I**.

Finally, the functions $\{k_1, \cdots, k_{2m}\}$ are called *curvature functions* of C with respect to $F_{\rm I}$.

Type II. Next, we obtain a class of null curves C whose Frenet frame is generated by a pseudo-orthonormal basis consisting of the two null vectors λ and N and additional two null vectors L_i and L_{i+1} defined by

$$L_i = \frac{W_i + W_{i+1}}{\sqrt{2}}, \qquad L_{i+1} = \frac{W_{i+1} - W_i}{\sqrt{2}}. \tag{3}$$

where W_i and W_{i+1} are timelike and spacelike vector fields respectively. We denote the Frenet equations of this particular class of C by Type **II**.

There are $(m-1)$ choices for L_i. To choose L_1, we let the vector field $\nabla_\lambda \lambda - h\,\lambda$ be null and define the curvature function K_1 by $\nabla_\lambda \lambda - h\,\lambda = K_1 L_1$, where L_1 is a null vector field along C which is everywhere perpendicular to λ and N. Since $S(TC^\perp)$ is Lorentz, there exists another unique null vector L_2 along C such that $g(L_1, L_2) = 1$. Set this case so that the equation (3) holds for $i = 1$.

By the method of Type **I** we have the following system of another Frenet equations for C (cf. Duggal-Jin [4]):

$$\begin{cases}\nabla_\lambda \lambda &= h\,\lambda + K_1\,L_1\\ \nabla_\lambda N &= -h\,N + K_2\,L_1 + K_3\,L_2 + \ell_2\,W_3\\ \nabla_\lambda L_1 &= -K_3\,\lambda + k_4\,L_1 + K_5\,W_3 + K_6\,W_4\\ \nabla_\lambda L_2 &= -K_2\,\lambda - K_1\,N - k_4\,L_2 + K_7\,W_3 + K_8\,W_4\\ \nabla_\lambda W_3 &= -\ell_2\,\lambda - K_7\,L_1 - K_5\,L_2 + k_8\,W_4 + k_9\,W_5\\ &\vdots\\ \nabla_\lambda W_{m-1} &= -k_{2m-3}\,W_{m-3} - k_{2m-2}\,W_{m-2} + k_{2m}\,W_m\\ \nabla_\lambda W_m &= -k_{2m-1}\,W_{m-2} - k_{2m}\,W_{m-1}\end{cases} \tag{4}$$

In the above case, we call

$$F_{\mathrm{II}} = \{\lambda,\, N,\, L_1,\, L_2,\, W_3, \cdots,\, W_m\}$$

a *Frenet frame* of Type **II** on M along C with respect to a given screen vector bundle $S(TC^\perp)$ and the equations (4) its *Frenet equations* of Type **II**.

It is straightforward to show that the system of equations (4) are equivalent to the following set of equations:

$$\begin{cases}\nabla_\lambda \lambda &= h\,\lambda + k_1\,(W_1 + W_2)\\ \nabla_\lambda N &= -h\,N + k_2\,W_1 + k_3\,W_2 + \ell_2\,W_3\\ \nabla_\lambda W_1 &= +k_2\,\lambda + k_1\,N - k_4\,W_2 - k_5\,W_3 - \ell_3\,W_4\\ \nabla_\lambda W_2 &= -k_3\,\lambda - k_1\,N + k_4\,W_1 + k_6\,W_3 + k_7\,W_4\\ &\vdots\\ \nabla_\lambda W_{m-1} &= -k_{2m-3}\,W_{m-3} - k_{2m-2}\,W_{m-2} + k_{2m}\,W_m\\ \nabla_\lambda W_m &= -k_{2m-1}\,W_{m-2} - k_{2m}\,W_{m-1}\end{cases} \tag{5}$$

In the next cases, if the vector fields

$$\nabla_\lambda N + h\,N - k_2\,W_1,$$

$$\nabla_\lambda W_1 + k_2\lambda + k_1\,N - k_4\,W_2,$$

$$\cdots,$$

$$\nabla_\lambda W_{m-3} + k_{2m-7}\,W_{m-5} + k_{2m-6}\,W_{m-4} - k_{2m-4}\,W_{m-2}$$

are null in turn, then using the same procedure as above for each such case, we finally obtain the following system of general equations:

$$\begin{cases} \nabla_\lambda \lambda &= h\,\lambda + k_1\,W_1 \\ \nabla_\lambda N &= -\,h\,N + k_2\,W_1 + k_3\,W_2 \\ \nabla_\lambda W_1 &= -\,k_2\,\lambda - k_1\,N + k_4\,W_2 + k_5\,W_3 \\ &\vdots \\ \nabla_\lambda L_i &= -\,K_{2i+1}\,W_{i-1} + K_{2i+3}\,L_i + K_{2i+4}\,W_{i+2} + K_{2i+5}\,W_{i+3} \\ \nabla_\lambda L_{i+1} &= -\,K_{2i-1}\,W_{i-2} - K_{2i}\,W_{i-1} - K_{2i+3}\,L_{i+1} - K_{2i+6}\,W_{i+2} + K_{2i+7}\,W_{i+3} \\ &\vdots \\ \nabla_\lambda W_{m-1} &= -\,k_{2m-3}W_{m-3} - k_{2m-2}\,W_{m-2} + k_{2m}W_m \\ \nabla_\lambda W_m &= -\,k_{2m-1}\,W_{m-2} - k_{2m}\,W_{m-1} \end{cases} \tag{6}$$

where $2 \leq i \leq m$. In the above case also the equations (6) are the Frenet equations of Type **II** for $i > 1$ with the Frenet frame

$$F_{\mathrm{II}} = \{\lambda,\, N,\, W_1, \cdots, L_i,\, L_{i+1}, \cdots, W_m\}.$$

Also, by replacing L_i and L_{i+1} in (6) with its values in terms of W_i and W_{i+1} (see relations (3)), we can get another set of Frenet equations in terms of one timelike and all others spacelike basis of its Frenet frame.

Finally, proceeding as before, one can show that the Frenet equations of Type **II** are equivalent to the following set of general equations:

$$\begin{cases} \nabla_\lambda \lambda &= h\,\lambda + k_1\,W_1 + \ell_1\,W_2 \\ \nabla_\lambda N &= -\,h\,N + k_3\,W_2 + \ell_2\,W_3 \\ \epsilon_1\,\nabla_\lambda W_1 &= -\,k_2\,\lambda - k_1\,N + k_4\,W_2 + k_5\,W_3 + \ell_3\,W_4 \\ \epsilon_2\,\nabla_\lambda W_2 &= -\,k_3\,\lambda - \ell_1\,N - k_4\,W_1 + k_6\,W_3 + k_7\,W_4 + \ell_4\,W_5 \\ &\vdots \\ \epsilon_{m-1}\,\nabla_\lambda W_{m-1} &= -\,\ell_{m-2}\,W_{m-4} - k_{2m-3}\,W_{m-3} - k_{2m-2}\,W_{m-2} + k_{2m}\,W_m \\ \epsilon_m\,\nabla_\lambda W_m &= -\,\ell_{m-1}\,W_{m-3} - k_{2m-1}\,W_{m-2} - k_{2m}\,W_{m-1} \end{cases} \tag{7}$$

We call the functions $\{\,k_1, \cdots, k_{2m}\,\}$ and the only surviving successive three fuctions $\{\,\ell_i = k_{2i-1},\, \ell_{i+1},\, \ell_{i+2}\,\}$, $i \in \{\,1, 2, \cdots, m-3\,\}$, as the *curvature* and the *torsion functions* respectively of C for the general Frenet equations (7).

Remark 1. One can verify that the general Frenet equations (7) include all m-th different Frenet equations of Type **I** and all $(m-1)$-th different Frenet equations of Type **II**. Hence we call the equations (7) the *compound Frenet equations* of the null curve C and $F = \{\frac{d}{dt}, N, W_1, \cdots, W_m\}$ the *compound Frenet frame* on M along C.

We consider two Frenet frames F and F^* along two neighborhoods $\mathcal{U}$ and $\mathcal{U}^*$ respectively with non-null intersection, with respect to a given screen vector bundle $S(TC^{\perp})$. Then we have

$$\lambda^* = \frac{dt}{dt^*}\,\lambda, \quad N^* = \frac{dt^*}{dt}\,N$$

and

$$W_\alpha^* = \sum_{\beta=1}^{m} A_\alpha^\beta\, W_\beta; \quad \alpha, \beta \in \{1, \cdots, m\}$$

where A_α^β are smooth functions on $\mathcal{U} \cap \mathcal{U}^*$ and the matrix $[A_\alpha^\beta(x)]$ is an element of the Lorentz group $O(1, m+1)$ for any $x \in \mathcal{U} \cap \mathcal{U}^*$. If we write the first equation of the compound Frenet equations (7) for both F and F^* and use the last relationships, we obtain

$$\frac{d^2 t}{dt^*} + h\left(\frac{dt}{dt^*}\right)^2 = h^*\,\frac{dt}{dt^*}.$$

Now, we consider the following differential equation

$$\frac{d^2 t}{dt^{*2}} - h^*\,\frac{dt}{dt^*} = 0$$

whose general solution comes from

$$t = a\int_{t_0^*}^{t^*} \exp\left(\int_{s_0}^{s} h^*(t^*)\,dt^*\right) ds + b; \quad a, b \in \mathbb{R}.$$

It follows that any of these solutions, with $a \neq 0$, might be taken as special parameter on C, such that $h = 0$. Denote one such solution by $p = \frac{t-b}{a}$, where t is the general parameter as defined in above equation. We call p a *distinguished parameter* of C, in terms for which $h = 0$. It is important to note that when t is replaced by p in the compound Frenet equations (7), the first two equations become

$$\nabla_{\frac{d}{dp}}\frac{d}{dp} = k_1\,W_1 + \ell_1\,W_2$$

$$\nabla_{\frac{d}{dp}} N = k_2\,W_1 + k_3\,W_2 + \ell_2\,W_3$$

and the other equations remain unchanged.

3. Fundamental Theorem for Null Curves

Let $\mathbb{R}_2^{m+2}$ be the $(m+2)$-dimensional semi-Euclidean space of index 2 with the semi-Euclidean metric

$$g(x,\, y) \;=\; -x^0y^0 - x^1y^1 + \sum_{a=2}^{m+1} x^a y^a.$$

Suppose C is a null curve in $\mathbb{R}_2^{m+2}$ locally given by the equations

$$x^i = x^i(t), \quad t \in I \subset \mathbb{R}, \quad \text{for} \quad i \in \{0, 1, \cdots, m+1\}.$$

First, we define in $\mathbb{R}_2^{m+2}$ the natural orthonormal basis

$$\begin{cases} N_0^o &= \left(\frac{1}{\sqrt{2}},\, 0,\, \frac{1}{\sqrt{2}},\, 0, \cdots, 0\right) \\ N_1^o &= \left(-\frac{1}{\sqrt{2}},\, 0,\, \frac{1}{\sqrt{2}},\, 0, \cdots, 0\right) \\ W_1^o &= (0,\, 1,\, 0,\, 0, \cdots, 0) \\ W_2^o &= (0,\, 0,\, 0,\, 1,\, 0, \cdots, 0) \\ & \vdots \\ W_{m-1}^o &= (0, \cdots, 0,\, 1,\, 0) \\ W_m^o &= (0, \cdots, 0,\, 0,\, 1) \end{cases} \tag{8}$$

where $\{N_0^o,\, N_1^o\}$ are null vectors such that $g\,(N_0^o,\, N_1^o) \,=\, 1,$ W_1^o is a timelike vector and $\{W_2^o, \cdots, W_m^o\}$ are orthonormal spacelike vectors. It is easy to see that

$$N_0^{oi}\, N_1^{oj} \,+\, N_0^{oj}\, N_1^{oi} \,-\, W_1^{oi}\, W_1^{oj} \,+\, \sum_{\alpha=2}^{m} W_\alpha^{oi}\, W_\alpha^{oj} \,=\, h^{ij}, \tag{9}$$

for any $i,\, j \,\in\, \{0, \cdots, m+1\}$, where

$$h^{ij} \,=\, \begin{cases} -1, & i \,=\, j \,\in \{0,\, 1\}, \\ 1, & i \,=\, j \,\notin \{0,\, 1\}, \\ 0, & i \,\neq\, j. \end{cases}$$

We are now in a position to state the fundamental existence and uniqueness theorem for null curves of semi-Euclidean space $\mathbb{R}_2^{m+2}$.

Theorm 3.1. *Let $k_1, \cdots, k_{2m} \,:\, [-\,\varepsilon,\, \varepsilon] \,\to\, \mathbb{R}$ be everywhere continuous functions, $x_o \,=\, (x_o^i)$ be a fixed point of $\mathbb{R}_2^{m+2}$ and let $N_0^o,\, N_1^o,\, W_1^o, \cdots, W_m^o$ be an orthonormal basis in (8). Then there exists a unique null curve $C \,:\, [-\,\varepsilon,\, \varepsilon] \,\to\, \mathbb{R}_2^{m+2}$ given by the equations $x^i \,=\, x^i(p)$, where p is a distinguished parameter on C, such that*

$C(0) = x_o$ *whose curvature functions are* $\{k_1, \cdots, k_{2m}\}$ *and whose Frenet frames of* Type **I**

$$\left\{\frac{d}{dp}, N, W_1, \cdots, W_m\right\}$$

satisfies $\frac{d}{dp}(0) = N_0^o$, $N(0) = N_1^o$ *and* $W_i(0) = W_i^o$ *for* $1 \leq i \leq m$.

Proof. Using the distinguished parameter p and the equations (2) with $\epsilon_1 = -1$ and $\epsilon_\alpha = 1$, $\alpha \in \{2, \cdots, m\}$ and note that $\nabla_{\frac{d}{dp}} X$ is just X' for any vector field X defined on $\mathcal{U}$ we consider the system of differential equation:

$$\begin{cases} N_0' &= k_1 W_1 \\ N_1' &= k_2 W_1 + k_3 W_2 \\ W_1' &= k_2 N_0 + k_1 N_1 - k_4 W_2 - k_5 W_3 \\ W_2' &= -k_3 N_0 - k_4 W_4 + k_6 W_3 + k_7 W_4 \\ W_{m-1}' &= -k_{2m-3} W_{m-3} - k_{2m-2} W_{m-2} + k_{2m} W_m \\ &\vdots \\ W_m' &= -k_{2m-1} W_{m-2} - k_{2m} W_{m-1} \end{cases} \tag{10}$$

Then there exists a unique solution $\{N_0, N_1, W_1, \cdots, W_m\}$ satisfying the initial conditions

$$N_i(0) = N_i^o, \quad W_\alpha(0) = W_\alpha^o; \quad i \in \{1, 2\}, \ \alpha \in \{1, \cdots, m\}.$$

Now we claim that $\{N_i(p), W_\alpha(p)\}$ is a pseudo-orthonormal basis such that $\{N_0, N_1\}$, $\{W_1\}$ and $\{W_2, \cdots, W_m\}$ are null, timelike and spacelike vectors respectively, for $p \in [-\varepsilon, \varepsilon]$. To this end, by direct calculations using (10), we obtain

$$\frac{d}{dp}\left\{N_0^i N_1^j + N_0^j N_1^i - W_1^i W_1^j + \sum_{\alpha=2}^{m} W_\alpha^i W_\alpha^j\right\} = 0. \tag{11}$$

As for $p = 0$ we have (9), from (11) it follows that

$$N_0^i N_1^j + N_0^j N_1^i - W_1^i W_1^j + \sum_{\alpha=2}^{m} W_\alpha^i W_\alpha^j = h^{ij}. \tag{12}$$

Further on, construct the field of frames

$$W_{m+1} = \frac{1}{\sqrt{2}}(N_0 - N_1), \qquad W_{m+2} = \frac{1}{\sqrt{2}}(N_0 + N_1). \tag{13}$$

Then (12) becomes

$$-W_1^i W_1^j - W_{m+1}^i W_{m+1}^j + W_{m+2}^i W_{m+2}^j + \sum_{\alpha=2}^{m} W_\alpha^i W_\alpha^j = h^{ij}. \tag{14}$$

We define for each $p \in [-\varepsilon, \varepsilon]$ the matrix $D(p) = [d^{ij}(p)]$ such that

$$d^{ab} = W_b^a, \;\; d^{aB} = \sqrt{-1}\, W_B^a, \;\; d^{Ab} = -\sqrt{-1}\, W_b^A, \;\; d^{AB} = W_B^A$$

for any $a, b \in \{1, m+1\}$; $A, B \in \{2, 3, \cdots, m-1, m, m+2\}$. By using (14) it is easy to check that $D(p)D(p)^t = I_{m+2}$, which implies that $\{W_1, \cdots, W_{m+2}\}$ is an orthonormal basis for any $p \in [-\varepsilon, \varepsilon]$. Then from (13) we conclude that $\{N_0, N_1, W_1, W_2, \cdots, W_m\}$ is a pseudo-orthonormal basis for any $p \in [-\varepsilon, \varepsilon]$. The null curve is obtained by integrating the system

$$\frac{dx^i}{dp} = N_0^i(t), \qquad x^i(0) = x_o^i.$$

Taking into account of (10) we see

$$F = \left\{ \frac{d}{dp} = W_0, W_1, \cdots, W_m \right\}$$

is a Frenet frame of Type **I** for C with curvature functions $\{k_1, \cdots, k_{2m}\}$. This completes the proof of theorem. □

Next, we define in $\mathbb{R}_2^{m+2}$ the pseudo-orthonormal basis

$$\begin{cases} N_0^o &= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, \cdots, 0\right) \\ N_1^o &= \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, \cdots, 0\right) \\ N_2^o &= \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, \cdots, 0\right) \\ N_3^o &= \left(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, \cdots, 0\right) \\ W_3^o &= (0, 0, 0, 0, 1, \cdots, 0) \\ W_4^o &= (0, 0, 0, 0, 0, 1, \cdots, 0) \\ & \vdots \\ W_{m-1}^o &= (0, \cdots, 0, 1, 0) \\ W_m^o &= (0, \cdots, 0, 0, 1) \end{cases} \tag{15}$$

where $\{N_0^o, N_1^o, N_2^o, N_3^o\}$ are null vectors such that

$$g(N_0^o, N_1^o) = 1, \qquad g(N_2^o, N_3^o) = 1$$

and $\{W_3^o, \cdots, W_m^o\}$ are orthonormal spacelike vectors. In this case also we find

$$N_0^{oi} N_1^{oj} + N_0^{oj} N_1^{oi} + N_2^{oi} N_3^{oj} + N_2^{oj} N_3^{oi} + \sum_{\alpha=3}^{m} W_\alpha^{oi} W_\alpha^{oj} = h^{ij}, \tag{16}$$

for any $i,\, j \in \{0, \cdots, m+1\}$.

Theorm 3.2. *Let $k_1, \cdots, k_{2m}\,;\, \ell_1, \cdots, \ell_{m-1} : [-\varepsilon,\, \varepsilon] \to \mathbb{R}$ be everywhere continuous functions, $x_0 = (x_0^i)$ be a fixed point of $\mathbb{R}_2^{m+2}$ and let*

$$\{N_i^0,\, W_\alpha^0\};\;\; 0 \le i \le 3,\; 3 \le \alpha \le m$$

be a pseudo-orthonormal basis in (15). Then there exists a unique null curve $C : [-\varepsilon,\, \varepsilon] \to \mathbb{R}_2^{m+2}$ such that $x^i = x^i(p)$, $C(0) = x_0$, and $\{k_1, \cdots, k_{2m}\}$ are the curvature functions and $\{\ell_1, \cdots, \ell_{m-1}\}$ the torsion functions with respect to a Frenet frame of Type **II**

$$F = \left\{\frac{d}{dp},\, N,\, L_1,\, L_2,\, W_3, \cdots, W_m\right\}$$

that satisfies

$$\frac{d}{dp}(0) = N_0^o,\, N(0) = N_1^o,\, L_1(0) = N_2^o,\, L_2(0) = N_3^o$$

and

$$W_\alpha(0) = W_\alpha^o,\;\; \alpha \in \{3, \cdots, m\}.$$

Proof. Using the distinguished parameter p and the equations (4), or equivalently (5), we consider the system of differential equations:

$$\begin{cases} N_0' &= K_1\, L_1 \\ N_1' &= K_2\, N_2 + K_3\, N_3 + \ell_2\, W_3 \\ N_2' &= -\, K_3\, N_0 + k_4\, N_2 + K_5\, W_3 + K_6\, W_4 \\ N_3' &= -\, K_2\, N_0 - K_1\, N_1 - k_4\, N_3 + K_7\, W_3 + K_8\, W_4 \\ W_3' &= -\, \ell_2\, N_0 - K_7\, N_2 - K_5\, N_3 + k_8\, W_4 + k_9\, W_5 \\ W_{m-1}' &= -\, k_{2m-3}\, W_{m-3} - k_{2m-2}\, W_{m-2} + k_{2m}\, W_m \\ & \vdots \\ W_m' &= -\, k_{2m-1}\, W_{m-2} - k_{2m}\, W_{m-1}. \end{cases} \tag{17}$$

Then, there exists a unique solution $\{N_i(p),\, W_\alpha(p)\}$ satisfying the initial conditions

$$N_i(0) = N_i^o,\; W_\alpha(0) = W_\alpha^o,\; i \in \{0,\, 1,\, 2,\, 3\},\; \alpha \in \{3,\, 4, \cdots, m\}$$

such that

$$N_0^i\, N_1^j + N_0^j\, N_1^i + N_2^i\, N_3^j + N_2^j\, N_3^i + \sum_{\alpha=3}^{m} W_\alpha^i\, W_\alpha^j = h^{ij}. \tag{18}$$

Now we claim that $\{N_0(p), N_1(p), N_2(p), N_3(p)\}$ are null vectors such that

$$g\,(N_0(p),\, N_1(p)) = 1,\;\; g\,(N_2(p),\, N_3(p)) = 1$$

and $\{W_\alpha(p)\}$ are spacelike vectors, for $p \in [-\varepsilon, \varepsilon]$. To this end, construct the field of frames

$$\begin{cases} W_0 &= \frac{1}{\sqrt{2}}\,(N_0 - N_1\,), \\ W_1 &= \frac{1}{\sqrt{2}}\,(N_0 + N_1\,), \\ & \vdots \\ W_{m+1} &= \frac{1}{\sqrt{2}}\,(N_2 - N_3\,), \\ W_{m+2} &= \frac{1}{\sqrt{2}}\,(N_2 + N_3\,). \end{cases} \tag{19}$$

Then (18) becomes

$$-W_0^i W_0^j - W_{m+1}^i W_{m+1}^j + W_1^i W_1^j - W_{m+2}^i W_{m+2}^j + \sum_{\alpha=3}^{m} W_\alpha^i W_\alpha^j = h^{ij} \tag{20}$$

and it is easy to check that the matrix $D(p) = [d^{ij}(p)]$ satisfies $D(p)D(p)^t = I_{m+2}$, which implies also that $\{W_0, W_1, \cdots, W_{m+2}\}$ is an orthonormal basis for any $p \in [-\varepsilon, \varepsilon]$. Then from (19) we conclude that $\{N_i, W_\alpha\}$ is a pseudo-orthonormal basis for any $p \in [-\varepsilon, \varepsilon]$. Thus there is a null C such that $C(0) = x_0$ and

$$F = \left\{ \frac{d}{dp} = N_0,\, N = N_1,\, L_1 = N_2,\, L_2 = N_3,\, W_3, \cdots, W_m \right\}$$

is a Frenet frame of Type **II** for C with curvature functions $\{k_1, \cdots, k_{2m}\}$ and torsion functions $\{\ell_1, \cdots, \ell_{m-1}\}$. □

4. Concluding Remark

In this paper, we proved the fundamental existence and uniqueness theorems of a null curve in a semi-Riemannian manifold of index 2, with a variety of Frenet frames of Types **I** and **II**. This is only a step further of the earlier work of Duggal-Bejancu [3] on null curves of Lorentzian manifolds. However, the general case of null curves in semi-Riemannian manifolds of arbitrary index q is still an open problem. We hope that the publication of this paper will help in solving the general case.

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