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CONTROLLABILITY OF NONLINEAR DELAY PARABOLIC EQUATIONS WITH NONLOCAL INITIAL CONDITION UNDER BOUNDARY INPUT

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ABSTRACT. In this paper, we study controllability of nonlinear delay parabolic equations with nonlocal initial condition under boundary input.

1. Introduction

Let $A(\xi, \partial)$ be a second order uniformly elliptic operator

$$A(\xi,\partial)u = -\sum_{j,k=1}^{n} \frac{\partial}{\partial \xi_i}(a_{jk}(\xi)\frac{\partial u}{\partial \xi_k}) + \sum_{j=1}^{n} b_j(\xi)\frac{\partial u}{\partial \xi_j} + c(\xi)u$$

with real, smooth coefficients a_{jk}, b_j, c defined on $\xi \in \Omega$, Ω a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary Γ.

In this paper, we consider the following parabolic equation:

$$(1) \quad \begin{cases} \dfrac{\partial(t,\xi)}{\partial t} = A(\xi,\partial)u(t,\xi) + F(t,\xi,u_t) \quad \text{in } (0,T] \times \Omega, \\ u(t,\xi) + g(t_1+\theta, t_2+\theta, \cdots, t_N+\theta, \xi, u(\cdot,\xi)) = \phi(t,\xi) \quad \text{in } [-r,0] \times \Omega, \\ Bu|_\Gamma = f(t,\xi) \quad \text{on } (0,T] \times \Gamma. \end{cases}$$

Here

$$u : [-r,\infty] \times \bar{\Omega} \to \mathbb{R},$$

$$u_t(\theta,\xi) = u(t+\theta,\xi) \text{ for all } \theta \in [-r,0],$$

$$F : \mathbb{R} \times \bar{\Omega} \times C([-r,0];\ \mathbb{R}) \to \mathbb{R},$$

and

$$g : [-r,T]^N \times \bar{\Omega} \times C([-r,0];\ \mathbb{R}) \to \mathbb{R}$$

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are given nonlinear functions and B is an associated boundary operator of the usual form

$$B : u \to \alpha u + \beta u_\mu \quad \text{at} \quad \Gamma$$

where u_μ is the exterior conormal derivative

$$u_\mu = \frac{\partial u}{\partial \mu} = \sum_{j,k} a_{jk} \frac{\partial u}{\partial \xi_k} \eta_k,$$

where $\eta = (\eta_k)$ is the unit exterior vector normal to Γ, with "sufficiently smooth" (real) coefficients, normalized so that $\alpha^2 + \beta^2 \equiv 1$. We distinguish between (and admit) the two cases;

$$\text{Dirichlet}: \quad \alpha = 1, \ \beta = 0;$$

$$\text{Neumann}: \quad \alpha = 0, \ \beta = 1.$$

In this paper, we prove controllability of nonlinear delay parabolic equations with nonlocal initial condition under boundary input by using a "semigroup approach".

In Section 2, we give some notations used throughout this paper and formulate the problem.

In Section 3, we will attempt to solve controllability for the equation (1) using the method by Nussbaum's fixed point theorem.

2. Notations and Formulations

Let $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ be the operator $Af = A(\xi, \partial)f$ for $f \in D(A)$ where

$$D(A) = \{u \in L^2(\Omega) : Au \in L^2(\Omega), Bu|_\Gamma = 0\}.$$

It is well known that A generates an analytic semigroup $S(t)$ on $L^2(\Omega)$ under the assumption that $A(\xi, \partial)$ is strongly elliptic. For simplicity we assume that the spectrum of A is on the right of the complex plane and $0 \in \rho(A)$, so that the fractional powers of A are well defined (see Pazy [10]).

Let $X = L^2(\Omega)$ and $Y = L^2(\Gamma)$. The space $X^{2\alpha}$ is defined by

$$D(A^\alpha) = X^{2\alpha} = X_\alpha,$$

and the norm of $X^{2\alpha}$ is given by

$$\|u\|_{2\alpha} = \|A^\alpha u\|_X = \|u\|_{D(A^\alpha)}, \quad u \in X_\alpha. \tag{2}$$

For a real non-negative index α, let $\{X^\alpha\}$ and $\{Y^\alpha\}$ be continuous Hilbert spaces such that

$$\left.\begin{aligned} X^{\alpha_1} \subset X^{\alpha_2} \subset \cdots \subset X^0 \equiv X \\ Y^{\alpha_1} \subset Y^{\alpha_2} \subset \cdots \subset Y^0 \equiv Y \end{aligned}\right\}, \quad \alpha_1 > \alpha_2 > \cdots > 0$$

where the injections are continuous.

We next extend the definitions of $\{X^\alpha\}$ and $\{Y^\alpha\}$ by setting

$$\begin{aligned} X^{-2\alpha} &= [D(A^{*\alpha})]' \\ Y^{-2\alpha} &= [Y^{2\alpha}]' \end{aligned}$$

for all $\alpha \geq 0$, so that the inclusions $X \subset \cdots \subset X^{-\alpha_2} \subset X^{-\alpha_1}$ and $Y \subset \cdots \subset Y^{-\alpha_2} \subset Y^{-\alpha_1}$ also hold.

Finally, for the purpose of uniformity of notation, we find it convenient to introduce the symbol

$$D(A^{-\alpha}) = [D(A^{*\alpha})]'$$

for all $\alpha \geq 0$.

Park [8] formulated the basic assumptions concerning $D(A^\alpha)$ and the so called Green operator G. Also, by [8], we have

$$u_t(\phi)(0) = S(t)\phi(0) + \int_0^t [S(t-s)F(s, u_s(\phi)) + A^{1-\sigma}S(t-s)A^\sigma Df(s)]ds. \tag{3}$$

Put $X_\alpha = D(A^\alpha) = X^{2\alpha}$ with norm

$$\|u\|_\alpha = \|u\|_{D(A^\alpha)} = \|A^\alpha u\|, \qquad u \in X_\alpha.$$

Let $Y_0 = L^\infty(0,T;Y)$. We shall denote by C_α the Banach space of continuous functions $C([-r,0];X_\alpha)$ with the norm

$$\|\phi\|_{C_\alpha} = \sup_{-r\leq\theta\leq 0} \|A^\alpha \phi(\theta)\|.$$

Let $\phi \in C([-r,0];X_\alpha)$. If $u_t(\phi)$ is an element in $C([-r,0];X_\alpha)$, then it has pointwise definition

$$u_t(\phi)(\theta) = u(\phi)(t+\theta), \quad \text{for} \quad \theta \in [-r,0].$$

Next section we consider the mild solution of (3) under following assumptions:

(C1) The nonlinear function

$$F(\cdot,\cdot) : \mathbb{R} \times C([-r,0];X_\alpha) \to X$$

is continuous, $F(t,0) = 0$ for all $t > 0$, there exists an $L > 0$ and $\omega > 0$ so that

$$\|F(t,\phi) - F(t,\psi)\| \leq e^{-\omega t} L \|\phi - \psi\|_{C_\alpha}.$$

(C2) The nonlinear function

$$g : [-r, T]^N \times C_\alpha \to C_\alpha$$

is bounded, $g(t_1+\theta, \cdots, t_N+\theta, u(\cdot)) \in D(A)$, there exists $M(t) > 0$ such that $\|g(t_1+\theta, \cdots, t_N+\theta, u(\cdot)) - g(t_1+\theta, \cdots, t_N+\theta, v(\cdot))\|_X \leq M(t)\|u_t - v_t\|_{C_\alpha}$ and $g(t_1+\theta, \cdots, t_N+\theta, 0) \equiv 0$.

We know that the following inequality holds for some constant $M_\alpha > 0$:

$$\|A^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\delta t}.$$

3. Controllability

Existence, uniqueness and norm estimate of mild solution are showed by [8] under satisfying (C1) and (C2).

Now we consider the controllability for the nonlinear delay parabolic systems with nonlocal initial condition under boundary input by using Nussbaum's fixed point theorem.

$$(4) \quad \begin{cases} \dfrac{\partial u}{\partial t} = A(\xi, \partial) u(t,\xi) + F(t,\xi,u_t) \quad \text{in} \quad (0,T] \times X; \\ u(t,\xi) + g(t_1+\theta, t_2+\theta, \cdots, t_N+\theta, u(\cdot,\xi)) = \phi(t,\xi) \quad \text{in} \quad [-r,0] \times \Omega; \\ Bu|_\Gamma = f(t,\xi) \quad \text{on} \quad (0,T] \times \Gamma. \end{cases}$$

where A is an analytic generator of semigroup $\{S(t)\}_{t\geq 0}$ on a Banach space X, $0 \leq t_1 \leq \cdots \leq t_N \leq T$, and nonlinear functions

$$F(\cdot,\cdot) : \mathbb{R} \times C([-r,0]; X_\alpha) \to X, \quad \text{and}$$
$$g : [-r,T]^N \times C_\alpha \to C_\alpha$$

are given functions. The system (4) is related to the following integral equation

$$(5) \quad \begin{cases} u_t(0) = u(t) = S(t)\{\phi(0) - g(t_1, \cdots, t_N, u)\} \\ \qquad\qquad + \int_0^t \big[S(t-s)F(s, u_s(\phi)) + A^{1-\sigma} S(t-s) A^\sigma D f(s)\big] ds, \quad t \in [0,T]; \\ u(t,\xi) = \phi(t,\xi) - g(t_1+\theta, \cdots, t_N+\theta, u(\cdot,\xi)), \quad t \in [-r,0]. \end{cases}$$

The continuous solution (5) is called *mild solution* of (4).

Definition 3.1. The equation (5) is said to be *controllable* if given any $\phi(0) \in C([-r,0];X_\alpha)$, there exists a control $f:[0,T]\to V$ such that the solution $u(t)$ of (5) with

$$u(0)=\phi(0)-g(t_1,\cdots,t_N,u)$$

satisfies $u(T)=u^1$, where u^1 is the target.

We assume that system (5) is controllable. Then

$$\begin{aligned} u(T) &= S(T)\{\phi(0)-g(t_1,\cdots,t_N,u)\} \\ &\qquad + \int_0^T \Big[S(T-s)F(s,u_s(\phi)) + A^{1-\sigma}S(T-s)A^{\sigma}Df(s)\Big]ds \\ &= u^1 \end{aligned}$$

where u^1 is target.

Assume that the following inequality holds for some constant $M_0>0$ such that

$$\|A^\alpha S(t)\| \le M_0.$$

The result depends on the exact controllability of the linear system,

$$u_t(\phi)(0)=\int_0^t A^{1-\sigma}S(t-s)A^\sigma Df(s)ds. \tag{6}$$

We assume that it can be steered to the subspace V then Range$(G)\supset V$, where

$$Gf=\int_0^T A^{1-\sigma}S(t-s)A^\sigma Df(s)ds.$$

Actually, without loss generality, we can assume that Range$(G)=V$, and we can construct an invertible operator $\tilde{G}$ defined on $L^2(0,T;Y)/\ker G$ (cf. [2]). Then, the control can be introduced

$$f(s)=\tilde{G}^{-1}\Big[u^1-S(T)\{\phi(0)-g(t_1,\cdots,t_N,u)\}-\int_0^T S(T-s)F(s,u_s(\phi))ds\Big](s)$$

This control is substituted into (5) in order to provide the operator

$$\begin{aligned} &\Phi u_t(\phi)(0) \\ &= S(t)\{\phi(0)-g(t_1,\cdots,t_N,u)\} + \int_0^t S(t-s)F(s,u_s(\phi))ds \\ &\quad + \int_0^t A^{1-\sigma}S(t-s)A^\sigma D\tilde{G}^{-1} \\ &\qquad \times \Big[u^1 - S(T)\{\phi(0)-g(t_1,\cdots,t_N,u)\} - \int_0^T S(T-s)F(s,u_s(\phi))ds\Big](s)ds. \end{aligned}$$

Notice that $\Phi u_T(\phi)(0) = u^1$, which means that the control f steers the nonlinear system from the origin to u^1 in time T, provided that we can obtain a fixed point of the nonlinear operator Φ. We assume the following hypotheses;

(C3) The nonlinear function

$$F(\cdot,\cdot) : [0,T] \times C([-r,0]; X_\alpha) \to X$$

is continuous and satisfies a Lipschitz type condition

$$\|F(t,\phi) - F(t,\psi)\| \leq r(t)\|\phi - \psi\|_{C_\alpha}$$

where $r(\|\phi\|, \|\psi\|) = r(t)$ is continuous on $[0,T]$, $r(t) \to 0$ as $t \to 0$ and $F(t,0) \equiv 0,\ \ 0 \leq t \leq T$.

(C4) The linear system (6) is exactly controllable to the subspace V.

(C5) The nonlinear function

$$g : [-r,T]^N \times C([-r,0] : X_\alpha) \to C([-r,0]; X_\alpha)$$

is continuous and satisfies following inequality

$$\|g(t_1,\cdots,t_N,u) - g(t_1,\cdots,t_N,v)\|_{C_\alpha} \leq K\|u_t - v_t\|_{C_\alpha}$$

where K is constant and $g(t_1,\cdots,t_N,0) \equiv 0$.

(C6) $S(t)x \in X \cap V$ for all $x \in X,\ \ 0 \leq t \leq T$;

$$\|S(t)x\|_X \leq p(t)\|x\|_X, \quad \|p\|_{L^2(0,T;X)} = c < \infty,$$
$$\|S(t)x\|_V \leq q(t)\|x\|_X, \quad \|q\|_{L^2(0,T;X)} = d < \infty;$$

also $S(t)$ is compact on X for each $t > 0$, and $\|S(t)\| \leq M$.

(C7) There exists a positive $L > 0$ for the constant α and σ satisfying $1 - 2(\alpha + 1 - \sigma) > 0$ such that

$$\left\| \int_0^{\cdot} A^{\alpha+1-\sigma} S(\cdot - s) A^{\sigma} Df(s)ds \right\| \leq L(\cdot)\|f\|_{L^2(0,T;V)}.$$

(C8) There exist c_1 for the constant α satisfying $1 - \alpha > 0$ such that

$$\left\| \int_0^t A^{\alpha} S(t-s) f(s)ds \right\| \leq \|f\|_{L^2(0,T;X)}.$$

(C9) γ is chosen so that the following conditions hold

$$\sup_{\|\phi(t)\| \leq \gamma} \{(1+K)(M_0 + LM) + (c_1 + Ld)r\} \leq h < 1, \text{ and}$$

$$\sup_{0 \leq \phi(t),\psi(t) \leq \gamma} (M_0 K + c_1 r) \leq h < 1.$$

Lemma 3.1 (Nusbaum [6]). *Suppose that S is a closed, bounded and convex subset of a Banach space X. If Φ_1 and Φ_2 are continuous mappings from S into X such that*

(i) $(\Phi_1 + \Phi_2)S \subset S$,

(ii) $\|\Phi_1 x - \Phi_2 x'\| \leq k\|x - x'\|$ *for all* $x, x' \in S$, *where k is constant,* $0 \leq k \leq 1$, *and*

(iii) $\overline{\Phi_2(S)}$ *is compact.*

Then the operator $\Phi_1 + \Phi_2$ has a fixed point in S.

Theorm 3.1. *Suppose that the hypotheses* (C3)–(C9) *are satisfied. Then the state of the system* (5) *can be steered from the initial state $\phi(0) - g(t_1, \cdots, t_N, u)$ to any final state u^1, satisfying*

$$\|u^1\|_V \leq \frac{(1-h)\gamma}{L}$$

in the time interval $[0, T]$.

Proof. We define

$$\Phi_1 u_t(\phi)(0) = S(t)\{\phi(0) - g(t_1, \cdots, t_N, \hat{u})\} + \int_0^t S(t-s)F(s, u_s(\phi))ds, \text{ and}$$

$$\Phi_2 u_t(\phi)(0) = \int_0^t A^{1-\sigma}S(t-s)A^\sigma D\tilde{G}^{-1} \times \Big[u^1 - S(T)\{\phi(0) - g(t_1, \cdots, t_N, u)\} - \int_0^T S(T-s)F(s, u_s(\phi))ds\Big](s)ds.$$

We can now employ Lemma 3.1 with

$$S = \{u_t(\phi)(\cdot) \in C([-r, T]; X) : \|u_t(\phi)\|_{C_\alpha} \leq \gamma\}.$$

Then the set S is closed, bounded and convex. From the definition,

$$\begin{aligned}
&\Phi u_t(\phi)(0) \\
&= \Phi_1 u_t(\phi)(0) + \Phi_2(\phi)(0) \\
&= S(t)\{\phi(0) - g(t_1, \cdots, t_N, \hat{u})\} + \int_0^t S(t-s)F(s, u_s(\phi))ds + \int_0^t A^{1-\sigma}S(t-s)A^\sigma D\tilde{G}^{-1} \\
&\qquad \times \Big[u^1 - S(T)\{\phi(0) - g(t_1, \cdots, t_N, u)\} - \int_0^T S(T-s)F(s, u_s(\phi))ds\Big](s)ds.
\end{aligned}$$

Thus, for any $u_t(\phi)(\cdot)$, $\hat{u}(\phi)(\cdot) \in S$, we obtain

$$
\begin{aligned}
&\|\Phi u_t(\phi)\|_\alpha \\
&= \|A^\alpha \Phi u_t(\phi)(\theta)\| \\
&= \|A^\alpha \Phi u_{t+\theta}(\phi)(0)\| \\
&= \|A^\alpha S(t+\theta)\{\phi(0) - g(t_1, \cdots, t_N, \hat{u})\}\| + \left\| \int_0^{t+\theta} A^\alpha S(t+\theta-s) F(s, u_s(\phi)) ds \right\| \\
&\quad + \left\| \int_0^{t+\theta} A^{\alpha+1-\sigma} S(t+\theta-s) A^\sigma D \tilde{G}^{-1} \right. \\
&\qquad\qquad \left. \times \left[u^1 - S(T)\{\phi(0) - g(t_1, \cdots, t_N, u)\} - \int_0^T S(T-s) F(s, u_s(\phi)) ds \right](s) ds \right\| \\
&\le \|A^\alpha S(t+\theta)\| (\|\phi(0)\|_{C_\alpha} + \|g(t_1, \cdots, t_N, \hat{u})\|_{C_\alpha}) + c_1 \|F(s, u_s(\phi))\|_{L^2(0,T;X)} \\
&\quad + L\left\{ \|u^1\|_V + \|S(T)\| (\|\phi(0)\|_{C_\alpha} + \|g(t_1, \cdots, t_N, u)\|_{C_\alpha}) + \int_0^T q(T) \|F(s, u_s(\phi))\| ds \right\} \\
&\le M_0 (\|\phi(0)\|_{C_\alpha} + K \|\hat{u}_t(\phi)\|_{C_\alpha}) + c_1 r \|u_s(\phi)\|_{C_\alpha} \\
&\qquad\qquad + L\{\|u^1\|_V + M(\|\phi(0)\|_{C_\alpha} + K\|u_t(\phi)\|_{C_\alpha}) + dr\|u_s(\phi)\|\} \\
&\le M_0(\gamma + K\gamma) + c_1 r \gamma + L\|u^1\|_V + LM\gamma + LMK\gamma + Ldr\gamma \\
&\le \{(1+K)(M_0 + LM) + (c_1 + Ld)r\}\gamma + (1-h)\gamma \\
&\le h\gamma + (1-h)\gamma \\
&= \gamma,
\end{aligned}
$$

where $-r \le \theta \le 0$. Hence

$$\sup_{-r \le \theta \le 0} \|\Phi u_t(\phi)\|_\alpha = \|\Phi u_t(\phi)\|_{C_\alpha} \le \gamma.$$

Therefore

$$\Phi u_t = \Phi_1 u_t + \Phi_2 u_t \in S \quad \text{for all} \quad u_t \in S,$$

which means that part (i) of Lemma 3.1 is satisfied.

To show that Φ_1 and Φ_2 are completely continuous, we consider

$$
\begin{aligned}
&\|\Phi_1(u_t(\phi) + \eta) - \Phi_1 u_t(\phi)\|_\alpha \\
&\quad = \|A^\alpha \Phi_1 (u_{t+\theta}(\phi) + \eta)(0) - A^\alpha \Phi_1 u_{t+\theta}(\phi)(0)\|_{C_\alpha} \\
&\quad = \left\| \int_0^{t+\theta} A^\alpha S(t+\theta-s)\{F(s, u_s(\phi) + \eta) - F(s, u_s(\phi))\} ds \right\|_{C_\alpha} \\
&\quad \le c_1 \|F(s, u_s(\phi) + \eta) - F(s, u_s(\phi))\|_{C_\alpha} \\
&\quad \le c_1 r(t) \|\eta\|_{C_\alpha},
\end{aligned}
$$

where $-r \le \theta \le 0,\ 0 \le t \le T$. Hence, we obtain

$$\sup_{-r\le\theta\le 0} \|\Phi_1(u_t(\phi)+\eta) - \Phi_1 u_t(\phi)\|_\alpha = \|\Phi_1(u_t(\phi)+\eta) - \Phi_1 u_t(\phi)\|_{C_\alpha} \le c_1 r(t)\|\eta\|_{C_\alpha} \to 0$$

as $\eta \to 0$. Thus we have

$$\begin{aligned}
&\|\Phi_2(u_t(\phi)+\eta') - \Phi_2 u_t(\phi)\|_\alpha \\
&\quad = \|A^\alpha \Phi_2(u_{t+\theta}(\phi)+\eta)(0) - A^\alpha \Phi_2 u_{t+\theta}(\phi)(0)\|_{C_\alpha} \\
&\quad = \Big\| \int_0^{t+\theta} \Big[A^{\alpha+1-\sigma} S(t+\theta-s) A^\sigma D\tilde{G}^{-1} \\
&\qquad\qquad \times \int_0^T S(T-s)\{F(s,u_s(\phi)+\eta') - F(s,u_s(\phi))\}(s)ds\Big] ds \Big\|_{C_\alpha} \\
&\quad \le Ld\|F(s,u_s(\phi)+\eta') - F(s,u_s(\phi))\|_{C_\alpha} \\
&\quad \le Ldr(s)\|\eta'\|_{C_\alpha}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\sup_{-r\le\theta\le 0} \|\Phi_2(u_t(\phi)+\eta') - \Phi_2 u_t(\phi)\|_\alpha \\
&\quad = \|\Phi_2(u_t(\phi)+\eta') - \Phi_2 u_t(\phi)\|_{C_\alpha} \\
&\quad \le Ldr\|\eta'\|_{C_\alpha} \to 0
\end{aligned}$$

as $\eta' \to 0$. Thus Φ_1 and Φ_2 are continuous.

Using the Arzela-Ascoli Theorem, we show that Φ_2 maps S into a precompact subset of S. We consider

$$\begin{aligned}
\Phi_2 u_t(\phi)(\theta) = &\int_0^{t+\theta} A^{1-\sigma} S(t+\theta-s) A^\sigma D\tilde{G}^{-1} \\
&\times \Big[u^1 - S(T)\{\phi(0) - g(t_1,\cdots,t_N,u)\} - \int_0^T S(T-s)F(s,u_s(\phi))ds\Big](s)ds.
\end{aligned}$$

Now we define

$$\begin{aligned}
\Phi_{2-\epsilon} u_t(\phi)(\theta) = &\int_0^{t+\theta-\epsilon} A^{1-\sigma} S(t+\theta-s) A^\sigma D\tilde{G}^{-1} \\
&\times \Big[u^1 - S(T)\{\phi(0) - g(t_1,\cdots,t_N,u)\} - \int_0^T S(T-s)F(s,u_s(\phi))ds\Big](s)ds.
\end{aligned}$$

Then

$$\begin{aligned}
&\Phi_{2-\epsilon}u_t(\phi)(\theta)\\
&= S(\epsilon)\int_0^{t+\theta-\epsilon} A^{1-\sigma}S(t+\theta-s)A^{\sigma}D\tilde{G}^{-1}\\
&\qquad \times \Big[u^1 - S(T)\{\phi(0)-g(t_1,\cdots,t_N,u)\} - \int_0^T S(T-s)F(s,u_s(\phi))ds\Big](s)ds.
\end{aligned}$$

By the hypothesis (C6), $S(\epsilon)$ is compact operator. Thus the set

$$K_{2-\epsilon}[u_t(\phi)(\theta)] = \{\Phi_{2-\epsilon}u_t(\phi)(\theta) : u_t(\phi) \in S\}$$

is precompact. Also

$$\begin{aligned}
&\|\Phi_2 u_t(\phi) - \Phi_{2-\epsilon}u_t(\phi)\|_\alpha\\
&= \|A^\alpha\Phi_2 u_{t+\theta}(\phi)(0) - A^\alpha\Phi_{2-\epsilon}u_{t+\theta}(\phi)(0)\|\\
&= \Big\|\int_{t+\theta-\epsilon}^{t+\theta} A^{\alpha+1-\sigma}S(t+\theta-s)A^{\sigma}D\tilde{G}^{-1}\\
&\qquad \times \Big[u^1 - S(T)\{\phi(0)-g(t_1,\cdots,t_N,u)\} - \int_0^T S(T-s)F(s,u_s(\phi))ds\Big](s)ds\Big\|\\
&\le L(\epsilon)\{\|u^1\|_V + \|S(T)\|(\|\phi(0)\|_{C_\alpha} + \|g(t_1,\cdots,t_N,u)\|_{C_\alpha}) + \int_0^T q(T)\|F(s,u_s(\phi))\|ds\}\\
&\le L(\epsilon)\{\|u^1\|_V + M\|\phi(0)\|_{C_\alpha} + MK\|u_t(\phi)\|_{C_\alpha} + dr(s)\|u_s(\phi)\|_{C_\alpha}\}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\sup_{-r\le\theta\le 0}\|\Phi_2 u_t(\phi) - \Phi_{2-\epsilon}u_t(\phi)\|_\alpha\\
&\quad = \|\Phi_2 u_t(\phi) - \Phi_{2-\epsilon}u_t(\phi)\|_{C_\alpha}\\
&\quad \le L(\epsilon)\{\|u^1\|_V + M\|\phi(0)\|_{C_\alpha} + MK\|u_t(\phi)\|_{C_\alpha} + dr\|u_s(\phi)\|_{C_\alpha}\} \to 0
\end{aligned}$$

as $\epsilon \to 0$. Thus $K_2[u_t(\phi)]$ is the closed set of

$$K_{2-\epsilon}[u_t(\phi)] = \{\Phi_{2-\epsilon}u_t(\phi) : u_t(\phi) \in S\}$$

and therefore $K_2[u_t(\phi)]$ is precompact.

We next show that Φ_2 maps the function in S into an equicontinuous family of functions.

For equicontinuity from the left, we take $t > \epsilon > t' > 0$, then we have

$$\begin{aligned}
&\|\Phi_2 u_t(\phi) - \Phi_2 u_{t-t'}(\phi)\|_\alpha\\
&= \|A^\alpha\Phi_2 u_{t+\theta}(\phi)(0) - A^\alpha\Phi_2 u_{t-t'+\theta}(\phi)(0)\|
\end{aligned}$$

$$
\begin{aligned}
= \Big\| \int_0^{t+\theta} & A^{\alpha+1-\sigma} S(t+\theta-s) A^{\sigma} D\tilde{G}^{-1} \\
& \times \Big[u^1 - S(T)\{\phi(0) - g(t_1, \cdots, t_N, u)\} - \int_0^T S(T-s) F(s, u_s(\phi)) ds \Big](s) ds \\
- \int_0^{t-t'+\theta} & A^{\alpha+1-\sigma} S(t-t'+\theta-s) A^{\sigma} D\tilde{G}^{-1} \\
& \times \Big[u^1 - S(T)\{\phi(0) - g(t_1, \cdots, t_N, u)\} - \int_0^T S(T-s) F(s, u_s(\phi)) ds \Big](s) ds \Big\| \\
\le \Big\| \int_0^{t+\theta-\epsilon} & A^{\alpha+1-\sigma} S(t+\theta-s) A^{\sigma} D\tilde{G}^{-1} \\
& \times \Big[u^1 - S(T)\{\phi(0) - g(t_1, \cdots, t_N, u)\} - \int_0^T S(T-s) F(s, u_s(\phi)) ds \Big](s) ds \\
- \int_0^{t+\theta-\epsilon} & A^{\alpha+1-\sigma} S(t-t'+\theta-s) A^{\sigma} D\tilde{G}^{-1} \\
& \times \Big[u^1 - S(T)\{\phi(0) - g(t_1, \cdots, t_N, u)\} - \int_0^T S(T-s) F(s, u_s(\phi)) ds \Big](s) ds \Big\| \\
+ \Big\| \int_{t+\theta-\epsilon}^{t+\theta} & A^{\alpha+1-\sigma} S(t+\theta-s) A^{\sigma} D\tilde{G}^{-1} \\
& \times \Big[u^1 - S(T)\{\phi(0) - g(t_1, \cdots, t_N, u)\} - \int_0^T S(T-s) F(s, u_s(\phi)) ds \Big](s) ds \Big\| \\
+ \Big\| \int_{t+\theta-\epsilon}^{t-t'+\theta} & A^{\alpha+1-\sigma} S(t-t'+\theta-s) A^{\sigma} D\tilde{G}^{-1} \\
& \times \Big[u^1 - S(T)\{\phi(0) - g(t_1, \cdots, t_N, u)\} - \int_0^T S(T-s) F(s, u_s(\phi)) ds \Big](s) ds \Big\| \\
\le \| S(t'+\epsilon) - S(\epsilon) \| \int_0^{t+\theta-\epsilon} & \Big\| A^{\alpha+1-\sigma} (t-t'+\theta-s-\epsilon) A^{\sigma} D\tilde{G}^{-1} \\
& \times \Big[u^1 - S(T)\{\phi(0) - g(t_1, \cdots, t_N, u)\} - \int_0^T S(T-s) F(s, u_s(\phi)) ds \Big](s) \Big\| ds \\
+ \int_{t+\theta-\epsilon}^{t+\theta} & \Big\| A^{\alpha+1-\sigma} S(t+\theta-s) A^{\sigma} D\tilde{G}^{-1} \\
& \times \Big[u^1 - S(T)\{\phi(0) - g(t_1, \cdots, t_N, u)\} - \int_0^T S(T-s) F(s, u_s(\phi)) ds \Big](s) \Big\| ds \\
+ \int_{t+\theta-\epsilon}^{t-t'+\theta} & \Big\| A^{\alpha+1-\sigma} S(t-t'+\theta-s) A^{\sigma} D\tilde{G}^{-1} \\
& \times \Big[u^1 - S(T)\{\phi(0) - g(t_1, \cdots, t_N, u)\} - \int_0^T S(T-s) F(s, u_s(\phi)) ds \Big](s) \Big\| ds
\end{aligned}
$$

$$\leq \|S(t'+\epsilon)-S(\epsilon)\|\, L(t+\theta-\epsilon)(\|u^1\|_V + M\|\phi\|_{C_\alpha} + MK\|tu_t(\phi)\|_{C_\alpha} + dr\|u_s(\phi)\|_{C_\alpha})$$
$$+ L(\epsilon)(\|u^1\|_V + M\|\phi\|_{C_\alpha} + MK\|u_t(\phi)\|_{C_\alpha} + dr\|u_s(\phi)\|_{C_\alpha})$$
$$+ L(\epsilon - t')(\|u^1\|_V + M\|\phi\|_{C_\alpha} + MK\|u_t(\phi)\|_{C_\alpha} + dr\|u_s(\phi)\|_{C_\alpha})$$
$$= \{\|S(t'+\epsilon) - S(\epsilon)\|\, L(t+\theta-\epsilon) + L(\epsilon) + L(\epsilon - t')\}$$
$$\times \{\|u^1\|_V + M\|\phi\|_{C_\alpha} + MK\|u_t(\phi)\|_{C_\alpha} + dr\|u_s(\phi)\|_{C_\alpha}\} \to 0$$

as $\epsilon \to 0$, by $L(t) \to 0$ as $t \to 0$, and $S(t)$ is continuous. Thus we have

$$\sup_{-r\leq\theta\leq 0} \|\Phi_2 u_t(\phi) - \Phi_2 u_{t-t'}(\phi)\|_\alpha = \|\Phi_2 u_t(\phi) - \Phi_2 u_{t-t'}(\phi)\|_{C_\alpha} \to 0$$

as $t' \to 0$. The case of equicontinuity from the right is similar.

Finally, we must have a Lipschitz condition for the operator Φ_1. For $u_t(\phi) \in S$ and $\hat{u}_t(\phi) \in S$,

$$\|\Phi_1 u_t(\phi) - \Phi_1 \hat{u}_t(\phi)\|_\alpha$$
$$= \Big\|A^\alpha S(t+\theta)\{\phi(0) - g(t_1,\cdots,t_N,u)\} + \int_0^{t+\theta} A^\alpha S(t-s)F(s,u_s(\phi))ds$$
$$- A^\alpha S(t+\theta)\{\phi(0) - g(t_1,\cdots,t_N,\hat{u})\} + \int_0^{t+\theta} A^\alpha S(t-s)F(s,v_s(\phi))ds\Big\|$$
$$\leq \|A^\alpha S(t+\theta)\{g(t_1,\cdots,t_N,u) - g(t_1,\cdots,t_N,\hat{u})\}\|$$
$$+ \|\int_0^{t+\theta} A^\alpha S(t-s)\{F(s,u_s(\phi) - F(s,\hat{u}_s(\phi))\}ds\|$$
$$\leq M_0 K\|u_t(\phi) - \hat{u}_t(\phi)\|_{C_\alpha} + c_1\|F(s,u_s(\phi) - F(s,\hat{u}_s(\phi)\|_{C_\alpha}$$
$$\leq M_0 K\|u_t(\phi) - \hat{u}_t(\phi)\|_{C_\alpha} + c_1 r\|u_s(\phi) - \hat{u}_s(\phi)\|_{C_\alpha}$$
$$= (M_0 K + c_1 r)\|u_t(\phi) - \hat{u}_t(\phi)\|_{C_\alpha}.$$

Consequently,

$$\sup_{-r\leq\theta\leq 0} \|\Phi_1 u_t(\phi) - \Phi_1 \hat{u}_t(\phi)\|_\alpha = \|\Phi_1 u_t(\phi) - \Phi_1 \hat{u}_t(\phi)\|_{C_\alpha} \leq h\|u_t(\phi) - \hat{u}_t(\phi)\|_{C_\alpha}.$$

Therefore the proof of Theorem 3.1 is complete by Lemma 3.1. □

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