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REGULAR CLOSED BOOLEAN ALGEBRA IN THE SPACE WITH EXTENSION TOPOLOGY

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ABSTRACT. Any Hausdorff space on a set which has at least two points has a regular closed Boolean algebra different from the indiscrete regular closed Boolean algebra as indiscrete space. The Sierpinski space and the space with finite complement topology on any infinite set etc. do the same. However, there is T_0 space which does the same with Hausdorff space as above. The regular closed Boolean algebra in a topological space is isomorphic to that algebra in the space with its open extension topology.

1. The space with closed extension topology

Let (X, τ) be any non-empty space and $p \notin X$. Suppose $X^* = X \cup \{p\}$ and $\tau^* = \{U \cup \{p\} : U \in \tau\} \cup \{\varnothing\}$. (X^*, τ^*) is called the space with *closed extension topology* of (X, τ). Assume that Ω and Ω^* denote the families of closed sets in (X, τ) and in (X^*, τ^*) respectively. It is clear that $\Omega^* = \Omega \cup X^*$. If Ξ_x and Ξ_x^* are the neighborhood systems of x in (X, τ) and in (X^*, τ^*) respectively, then $U \in \Xi_x$ if and only if $U^* = U \cup \{p\} \in \Xi_x^*$ for any $x \in X$, and $U^* \in \Xi_p^*$ if and only if there is a set $U \in \tau$ such that $U^* - \{p\} \supset U$. Let c (resp. c^*) denote the closure and i (resp. i^*) the interior operations in (X, τ) (resp. in (X^*, τ^*)).

Proposition 1.1. *For any $A \subset X^*$,*

$$c^*(A) = \begin{cases} c(A) & \text{if } p \notin A, \\ X^* & \text{if } p \in A. \end{cases}$$

Proof. For any $x \in X$,

$$U_x^* \in \Xi_x^*$$

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if and only if

$$\text{there is } U_x \in \Xi_x \text{ such that } U_x^* = U_x \cup \{p\},\ U_x \in \Xi_x$$

if and only if

$$U_x^* = U_x \cup \{p\} \in \Xi_x^*.$$

It follows that $x \in c^*(A)$ if and only if $x \in c(A)$, that is, $c^*(A) = c(A)$ whenever $p \notin A$. Furthermore, the equalities

$$c^*(A \cup \{p\}) = c^*(A) \cup c^*(\{p\}) = X^*$$

hold since $c^*(\{p\}) = X^*$. □

Proposition 1.2. *For any $A \subset X^*$,*

$$i^*(A) = \begin{cases} \varnothing & \text{if } p \notin A, \\ i(A - \{p\}) \cup \{p\} & \text{if } p \in A. \end{cases}$$

Proof. For any $x \in X$ and any $U_x^* \in \Xi_x^*$, $p \in U^*$ so U^* is not in A when $A \subset X$. It follows that $x \notin i^*(A)$, that is, $i^*(A) = \varnothing$. If $p \in A$, then $i^*(A) - \{p\} \in \tau$ so $i^*(A) - \{p\} \subset i(A - \{p\})$. Finally, we have $i^*(A) = i(A - \{p\}) \cup \{p\}$. □

Corollary *For any $A \subset X^*$,*

$$c^*(i^*(A)) = \begin{cases} \varnothing & \text{if } p \notin A, \\ X^* & \text{if } p \in A. \end{cases}$$

It is known that $K(X) = RC(X)$ which is the family of all regular closed sets in (X, τ) is the regular closed Boolean algebra (see Kuratowski and Mostowski [2]). From above, it is clear that $RC^*(X^*) = \{\varnothing, X^*\}$ which is the family of all regular closed sets in (X^*, τ^*).

We call the Boolean algebra in the space with closed extension topology *indiscrete regular closed Boolean algebra* because it has only two trivial elements and is same with the Boolean algebra of indiscrete topological space. The Sierpinski space and the space with finite complement topology on any infinite set have also the indiscrete regular closed Boolean algebra. According to these facts we can assert that different topological spaces can have same regular closed Boolean algebra though they are not homeomorphic each other. However, it is easy to see that the regular closed Boolean algebra in topological spaces which are homeomorphic each other are isomorphic each other, that is, regular closed Boolean algebra is a topological invariant property (see

Maunder [3]), since all of the operations interior, closure, union, intersection and complement of sets can be changed their position with a homeomorphism function (see Cao [1]).

Let (X,τ) be the discrete topological space. The family of regular closed sets in (X,τ) is $K(X) = 2^X$. The regular closed Boolean algebra in (X,τ) is called a *discrete regular closed Boolean algebra* in which $A \bullet B = A \cap B$, $A' = A^c$ and $A \circ B = (A - B) \cup (B - A)$; so $\circ$ is just the symmetric difference.

Theorem 1.1. *Any Hausdorff space on a set which has at least two points has a regular closed Boolean algebra different from the indiscrete regular closed Boolean algebra.*

Proof. Let (X,τ) be a Hausdorff space. If X is a finite set, then the conclusion is correct since any Hausdorff space on a finite set must be a discrete topological space so it has the discrete regular closed Boolean algebra.

If X is an infinite set, for any $x \neq y \in X$, there are $U \in \Xi_x$ and $V \in \Xi_y$ satisfying $U \cap V = \varnothing$ so $X \neq X - U \neq \varnothing$ and $V \subset i(X - U)$. And we have

$$\varnothing \neq c(V) \subset c(i(X-U)) \subset c(X-U) = X - U \neq X.$$

Finally, we get $c(i(X - U)) \in K(X)$ and that the conclusion holds. □

Now we give a counterexample below to prove that the idea which asserts that the regular closed Boolean algebra in any non-Hausdorff space must be an indiscrete regular closed Boolean algebra is wrong.

Example. Let (Y,τ_1) be a Sierpinski space, (Z,τ_2) a discrete space on a non-empty set, and $Y \cap Z = \varnothing$. We construct a new topological space (X,τ) satisfying $X = Y \cup Z$ and $\tau = \{A \cup B : A \in \tau_1, B \in \tau_2\}$. Through calculating, it is not difficult to see that $\varnothing \neq \{Z - B : B \in \tau_2\} \subset K(X)$, so that $K(X) \neq \{\varnothing, X\}$.

Remark. The regular closed Boolean algebra in the space with closed extension topology has only one atom (see Kuratowski and Mostowski [2]), that is, X^*. The regular closed Boolean algebra in a Sierpinski space and that in indiscrete topological space has also only one atom respectively.

2. The space with open extension topology

Let (X,τ) be any non-empty space and $p \notin X$. Suppose $X^* = X \cup \{p\}$ and describe a topology τ^* on X^* by calling a set in X^* open if and only if it is X^*

or in τ. We call (X^*, τ^*) the open extension of (X, τ) since other than X^* itself the open sets of τ^* are just the open sets of τ. That is, $\tau^* = \tau \cup \{X^*\}$ and $\Omega^* = \{A \cup \{p\} : A \in \Omega\} \cup \{\varnothing\}$. From above, it is obvious that the following propositions hold.

Proposition 2.1. *For any $A \subset X^*$,*

$$i^*(A) = \begin{cases} i(A) & \text{if } p \notin A, \\ i(A - \{p\}) & \text{if } p \in A \neq X^*, \\ X^* & \text{if } A = X^*. \end{cases}$$

Proposition 2.2. *For any $A \subset X^*$,*

$$c^*(A) = \begin{cases} \varnothing & \text{if } A = \varnothing, \\ c(A) \cup \{p\} & \text{if } \varnothing \neq A \subset X, \\ c(A - \{p\}) \cup \{p\} & \text{if } p \in A. \end{cases}$$

Proof. If $\varnothing \neq A \subset X$, then

$$c^*(A) = X^* - i^*(X^* - A) = [X - i(X - A)] \cup \{p\} = c(A) \cup \{p\}.$$

If $p \in A$, we have $c^*(A) = X \cup \{p\} - i^*(X \cup \{p\} - A) = c(A - \{p\}) \cup \{p\}$. □

Corollary. *For any $A \subset X^*$,*

$$c^*(i^*(A)) = \begin{cases} \varnothing & \text{if } A = \varnothing, \\ c(i(A)) \cup \{p\} & \text{if } \varnothing \neq A \subset X, \\ c(i(A - \{p\})) \cup \{p\} & \text{if } p \in A. \end{cases}$$

Proof. If $\varnothing \neq A \subset X$, then $c^*(i^*(A)) = c^*(i(A)) = c(i(A)) \cup \{p\}$. If $p \in A \neq X^*$, we have

$$c^*(i^*(A)) = c^*(i(A - \{p\})) = c(i(A - \{p\})) \cup \{p\}$$

and, when $A = X^*$, $c^*(i^*(A)) = X^* = c(i(A - \{p\})) \cup \{p\}$. □

From the corollary above, we can get the following propositions easily (see Propositions 2.3~2.6).

Proposition 2.3. *If $p \in A$, then $A \in RC^*(X^*)$ if and only if $A - \{p\} \in RC(X)$.*

Proposition 2.4. *If $p \notin A \neq \varnothing$, then $A \in RC(X)$ if and only if*

$$A \cup \{p\} \in RC^*(X^*).$$

Proposition 2.5. *If $A \in RC^*(X^*)$, then $p \in A$ or $A = \varnothing$.*

Proposition 2.6. $K^*(X^*) = RC^*(X^*) = \{A \cup \{p\} : A \in K(X)\} \cup \{\varnothing\}$.

Definition 1. The retraction regular closed function $f : K^*(X^*) \to K(X)$ is defined by

$$f(A^*) = \begin{cases} A^* - \{p\} & \text{if } A^* \neq \varnothing, \\ \varnothing & \text{if } A^* = \varnothing \end{cases}$$

for any $A^* \in K^*(X^*)$.

Proposition 2.7. *Any retraction regular closed function is a bijection.*

Proof. For any $A \in K(X)$, if $A = \varnothing$ then there is $A^* = \varnothing \in K^*(X^*)$ and if $A \neq \varnothing$ then $A^* = A \cup \{p\} \in K^*(X^*)$ such that $f(A^*) = A$. So f is a surjection. For any $A^* \neq B^* \in K^*(X^*)$, we have $A^* \neq \varnothing$ or $B^* \neq \varnothing$.

Without lose of generality, we suppose that $A^* \neq \varnothing$. According to Proposition 2.5, we have $p \in A^*$. It follows that

$$f(A^*) = A^* - \{p\} \neq B^* - \{p\} = f(B^*)$$

since $c^*(i^*(\{p\})) = \varnothing \neq \{p\}$ and $\{p\} \notin K^*(X^*)$. □

Proposition 2.8. *For any $A^* \in K^*(X^*)$, $[A^*]'_{X^*} = [A^* - \{p\}]'_X \cup \{p\}$.*

Proof. The situation can be divided into the following three cases.

Case 1: $A^* = \varnothing$. We have $[A^*]'_{X^*} = X^* = [\varnothing - \{p\}]'_X \cup \{p\}$.

Case 2: $\varnothing \neq A^* \subset X$. It is clear that $[A^*]'_{X^*} = c(i(X^* - A^* - \{p\})) \cup \{p\}$ since $p \in X^* - A^*$. On the other hand, it is easy to see that

$$[A^* - \{p\}]'_X \cup \{p\} = [A^*]'_X \cup \{p\} = c(i(X - A^*) \cup \{p\}.$$

So the conclusion holds.

Case 3: $p \in A^*$. Since $p \notin X^* - A^*$,

$$[A^*]'_{X^*} = c(i(X^* - A^*)) \cup \{p\} = [A^* - \{p\}]'_X \cup \{p\}.$$

□

Proposition 2.9. *For any $A^* \in K^*(X^*)$, $f([A^*]'_{X^*}) = [f(A^*)]'_X$.*

Proof. This situation can be divided into the following four cases.

Case 1: $A^* = \varnothing$. It is clear that $f([A^*]'_{X^*}) = f(X^*) = X = [\varnothing]'_X = [f(A^*)]'_X$.

Case 2: $A^* = X^*$. We have $f([A^*]'_{X^*}) = f(\varnothing) = \varnothing = [X]'_X = [f(X^*)]'_X$.
Case 3: $p \notin A^* \neq \varnothing$. Because $p \in X^* - A^*$,

$$f([A^*]'_{X^*}) = f(c(i(X^* - A^* - \{p\}))) \cup \{p\} = [A^*]'_X = [f(A^*)]'_X.$$

Case 4: $p \in A^* \neq X^*$. From $p \notin X^* - A^*$, we can get that

$$f([A^*]'_{X^*}) = f(c(i(X^* - A^*)) \cup \{p\}) = c(i(X^* - A^*)) = [A^* - \{p\}]'_X = [f(A^*)]'_X.$$

□

Proposition 2.10. *For any $A^* \neq \varnothing \neq B^* \in K^*(X^*)$, there are $A, B \in K(X)$ such that $A^* \bullet_{X^*} B^* = (A \bullet_{X^*} B) \cup \{p\}$.*

Proof. From Proposition 2.6, for any $A^* \neq \varnothing \neq B^* \in K^*(X^*)$, there are $A, B \in K(X)$ satisfying $A^* = A \cup \{p\}$ and $B^* = B \cup \{p\}$ so that

$$A^* \bullet_{X^*} B^* = (A \cup \{p\}) \bullet_{X^*} (B \cup \{p\}) = (A \bullet_{X^*} B) \cup \{p\}.$$ □

Proposition 2.11. *For any $A \neq \varnothing \neq B \in K(X)$, $A \bullet_{X^*} B = (A \bullet_X B) \cup \{p\}$.*

Proof. Since $p \notin A \cap B$, we have $A \bullet_{X^*} B = c(i(A \cap B)) \cup \{p\} = (A \bullet_X B) \cup \{p\}$. □

Proposition 2.12. *For any $A^*, B^* \in K^*(X^*)$, $(A^* \bullet_{X^*} B^*) - \{p\} = f(A^*) \bullet_X f(B^*)$*

Proof. It is clear that the equality holds if $A^* = \varnothing$ or $B^* = \varnothing$. If $A^* \neq \varnothing$ and $B^* \neq \varnothing$, then there are $A, B \in K(X)$ such that $A = A^* - \{p\} = f(A^*)$ and $B = B^* - \{p\} = f(B^*)$. From these equalities we can get $(A^* \bullet_{X^*} B^*) - \{p\} = A \bullet_{X^*} B$, using Proposition 2.10. According to Proposition 2.11, we have $A \bullet_{X^*} B = A \bullet_X B$ so $(A^* \bullet_{X^*} B^*) - \{p\} = A \bullet_X B = f(A^*) \bullet_X f(B^*)$. □

Proposition 2.13. *For any $A^*, B^* \in K^*(X^*)$, $f(A^* \bullet_{X^*} B^*) = f(A^*) \bullet_X f(B^*)$.*

Proof. It is obvious from Proposition 2.12 and Definition 1. □

Proposition 2.14. *For any $A^*, B^* \in K^*(X^*)$, $f(A^* \cup B^*) = f(A^*) \cup f(B^*)$.*

Proof. It is clear that the equality holds if $A^* = \varnothing$ or $B^* = \varnothing$. Suppose that $A^* \neq \varnothing$ and $B^* \neq \varnothing$. We have

$$f(A^* \cup B^*) = (A^* \cup B^*) - \{p\} = (A^* - \{p\}) \cup (B^* - \{p\}) = f(A^*) \cup f(B^*).$$ □

Proposition 2.15. *For any* $A^*, B^* \in K^*(X^*)$, $f(A^* \circ_{X^*} B^*) = f(A^*) \circ_X f(B^*)$

Proof. Through calculating we can get the equality from Propositions 2.14, 2.13 and 2.9. □

From above, we can easily prove the following theorem:

Theorem 2.1. *The regular closed Boolean algebra in a topological space is isomorphic to the Boolean algebra in the space with its open extension topology, that is,* $K^*(X^*) \cong K(X)$.

The retraction regular closed function is an isomorphism from the regular closed Boolean algebra in a topological space into that in the space with its open extension topology.

Definition 2. An element a of a Boolean algebra is called an *atom* if $a \neq 0$ and for each x the relation $x \leq a$ implies $x = 0$ or $x = a$.

Proposition 2.16. *A set* $A \subset X$ *is an atom of the regular closed Boolean algebra* $K(X)$ *in topological space* (X, τ) *if and only if* $A \cup \{p\}$ *is an atom of the regular closed Boolean algebra* $K^*(X^*)$ *in the space* (X^*, τ^*) *which has the open extension topology of* (X, τ) *where* p *belongs to* X^* *but to* X.

Proof. Since $(A \cup \{p\}) \bullet_{X^*} (B \cup \{p\}) = (A \bullet_X B) \cup \{p\}$,

$$(A \cup \{p\}) \bullet_{X^*} (B \cup \{p\}) = B \cup \{p\}$$

if and only if $A \bullet_X B = B$. So that $A \cup \{p\}$ is an atom in $K^*(X^*)$ if and only if A is an atom in $K(X)$. □

It is easy to see that the number of atoms in regular closed Boolean algebra $K(X)$ is just that in $K^*(X^*)$. Any atom in $K^*(X^*)$ must not be a singleton, in fact, at least two points.

References

1. S. Cao, On product of α (s, pre)-homeomorphisms, *J. Liaocheng Teachers Univ.* **2** (1998), 30–35.
2. K. Kuratowski and A. Mostowski, *Set Theory*, With an introduction to descriptive set theory. Translated from the 1966 Polish original. Second, completely revised edition. Studies in Logic and the Foundations of Mathematics, Vol. 86, PWN-Polish Scientific Publishers, Warszawa, 1976. MR **58**#5230

3. C. R. F. Maunder, *Algebraic Topology*, Cambridge University Press, London, 1980. MR **84k**:55001

4. L. A. Steen and J. A. Seebach, Jr., *Counterexamples in Topology*, Second edition, Springer-Verlag, New York, Heidelberg, Berlin, 1978. MR **80a**:54001

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