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SYMMETRIC DUALITY FOR NONLINEAR MIXED INTEGER PROGRAMS WITH A SQUARE ROOT TERM

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Abstract. We formulate a pair of symmetric dual mixed integer programs with a square root term and establish the weak, strong and converse duality theorems under suitable invexity conditions. Moreover, the self duality theorem for our pair is obtained by assuming the kernel function to be skew symmetric.

1. Introduction

The study of symmetric duality in nonlinear programs was initiated by Dantzig et al. [3]. Balas [1] considered the symmetric duality results of Dantzig et al. [3] for the case that some primal and dual variables were constrained to belong to some arbitrary set, for example, the set of integers. While Balas [1] used concave /convex functions and nonnegative orthants as the cone, Mishra and Das [9] generalized this to any arbitrary cone. And then Kim et al.[5] established the symmetric duality theorems for multiobjective nonlinear programs with arbitrary cones. The duality results of Balas [1] of the problems with convex cone domains and the pseudo-convex /pseudo-concave kernel function were extended by Mishra et al.[8]. Recently Kumar et al. [6] presented a modified pair of symmetric dual minimax integer programs which was in the spirit of Mond and Weir [11,12] and did not require any of several assumptions of the type given by Mishra et al. [8].

On the other hand, Mond et al.[7,10] gave symmetric duality theorems for certain nondifferentiable programs with square roots of quadratic forms in the objective functions. Since then Chandra and Husain

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[2] gave symmetric and self duality theorems for nondifferentiable programs with the convexity/concavity conditions of kernel function. The following pair of programs was considered by Chandra and Husain [2]:

$$
\begin{aligned}
(P) \qquad \text{Min} \quad & F(x,y,w) = f(x,y) - y^T\nabla_y f(x,y) + (x^TBx)^{\frac{1}{2}} \\
\text{subject to} \quad & -\nabla_y f(x,y) + Cw \geqq 0, \\
& w^TCw \leqq 1, \\
& x \geqq 0, \quad y \geqq 0, \\
(D) \qquad \text{Max} \quad & G(u,v,z) = f(u,v) - u^T\nabla_x f(u,v) - (v^TCv)^{\frac{1}{2}} \\
\text{subject to} \quad & -\nabla_{x_2} f(u,v) - Bz \leqq 0, \\
& z^TBz \leqq 1, \\
& u \geqq 0, \quad v \geqq 0.
\end{aligned}
$$

In this paper, we formulate a pair of symmetric dual mixed integer programs with a square root term and establish the weak, strong and converse duality theorems under the condition that $f(x,y) + {x_2}^TBz$ is invex in x_2 for each (x_1, y) and $-f(x,y) + {y_2}^TCw$ is invex in y_2 for each (x, y_1), where $(x,y) = (x_1, x_2, y_1, y_2)$. Moreover, the self duality theorem for our pair is obtained by assuming $f(x,y)$ to be skew symmetric.

2. Preliminaries and notations

We constrain some of the components of x and y to arbitrary sets of integers. Suppose the first n_1 components of x and the first m_1 components of y $(0 \leq n_1 \leq n, 0 \leq m_1 \leq m)$ are arbitrarily constrained to be integers and the following notations are introduced:

$$(x,y) = (x_1, x_2, y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^m, x_1 \in \mathbb{R}^{n_1} \text{and} y_1 \in \mathbb{R}^{m_1},$$

where $n = n_1 + n_2,\ m = m_1 + m_2$. Let U and V be two arbitrary sets of integers in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{m_1}$, respectively. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a twice differentiable function. Let $\nabla_{x_2} f(x,y)$ denote the gradient of $f(x,y)$ with respect to x_2 and $\nabla_{y_2} f(x,y)$ be defined similarly. Also let $\nabla_{x_2x_2} f(x,y)$ denote the Hessian matrix of $f(x,y)$ with respect to x_2. $\nabla_{y_2y_2} f(x,y)$, $\nabla_{y_2x_2} f(x,y)$ and $\nabla_{x_2y_2} f(x,y)$ are defined similarly.

DEFINITION 1 [2]. A differentiable function $F : \mathbb{R}^n \to \mathbb{R}$ is said to be invex with respect to the $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$F(x) - F(u) \geqq \eta(x, u)^T \nabla F(u).$$

DEFINITION 2. Let $(S_1, S_2, \cdots, S_p)$ be elements of an arbitrary vector space. A real valued function $G(S_1, S_2, \cdots, S_p)$ is said to be separable with respect to S_1 if there exist real valued functions $H(S_1)$ and $K(S_2, S_3, \cdots, S_p)$ such that

$$G(S_1, S_2, \cdots, S_p) \equiv H(S_1) + K(S_2, S_3, \cdots, S_p).$$

We shall make use of the following generalized Schwarz inequality:

$$x^T Ay \leqq (x^T Ax)^{\frac{1}{2}} (y^T Ay)^{\frac{1}{2}}.$$

where $x, y \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is symmetric and positive semidefinite.

Now we consider the following minimization problem with a square root term:

$$(P) \qquad \begin{array}{ll} \text{minimize} & f(x) + (x^T Bx)^{\frac{1}{2}} \\ \text{subject to} & g(x) \geqq 0, \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^m$ and B is symmetric positive semidefinite $n \times n$ matrix.

In order to obtain the symmetric duality results, we introduce Fritz John type necessary optimality theorem for (P).

LEMMA 1[4]. *If $\overline{x}$ is optimal to (P), then there exist $r \in \mathbb{R}, \rho \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$ such that*

$$\begin{aligned} \rho^T g(\overline{x}) &= 0, \\ r(\nabla f(\overline{x}) + Bz) &= \nabla \rho^T g(\overline{x}), \\ z^T Bz &\leqq 1, \\ (\overline{x}^T B\overline{x})^{\frac{1}{2}} &= \overline{x}^T Bz, \\ (r, \rho) &\geqq 0, \\ (r, \rho) &\neq 0. \end{aligned}$$

We formulate the following symmetric dual mixed integer problems with a square root term:
Primal (SP)

$$\begin{array}{lll} \text{Max}_{x_1}\ \text{Min}_{x_2,y} & F(x,y,w) = f(x,y) - y_2^T\nabla_{y_2}f(x,y) + (x_2^TBx_2)^{\frac{1}{2}} \\ (1)\quad \text{subject to} & -\nabla_{y_2}f(x,y) + Cw \geqq 0, \\ (2) & w^TCw \leqq 1, \\ & x_1 \in U, \quad y_1 \in V, \end{array}$$

Dual (SD)

$$\begin{array}{lll} \text{Min}_{y_1}\ \text{Max}_{x,y_2} & G(u,v,z) = f(u,v) - u_2^T\nabla_{x_2}f(u,v) - (v_2^TCv_2)^{\frac{1}{2}} \\ (3)\quad \text{subject to} & -\nabla_{x_2}f(u,v) - Bz \leqq 0, \\ (4) & z^TBz \leqq 1, \\ & u_1 \in U, \quad v_1 \in V, \end{array}$$

where (i) $B \in \mathbb{R}^{n_2\times n_2}$ and $C \in \mathbb{R}^{m_2\times m_2}$ are symmetric positive semi-definite, (ii) z and w are vectors in $\mathbb{R}^{n_2}$ and $\mathbb{R}^{m_2}$ respectively, (iii) $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is twice differentiable in x_2 and y_2 and separable with respect to x_1(or y_1) and (iv) $\eta_1(x_2,u_2) + u_2 \geqq 0$ and $\eta_2(v_2,y_2) + y_2 \geqq 0$.

For notational convenience, the sets of feasible solutions of primal X and dual Y are denoted by

$$X = \left\{(x,y,w)|x_1 \in U, y_1 \in V, -\nabla_{y_2}f(x,y) + Cw \geqq 0, w^TCw \leqq 1\right\},$$
$$Y = \left\{(u,v,z)|u_1 \in U, v_1 \in V, -\nabla_{x_2}f(u,v) - Bz \leqq 0, z^TBz \leqq 1\right\}.$$

3. Symmetric duality

In this section, we establish weak, strong, converse and self duality theorems between (SP) and (SD).

THEOREM 1 (Weak Duality). *Let $f(x,y) + x_2^TBz$ be invex in x_2 for every (x_1,y) with respect to η_1 and $-f(x,y) + y_2^TCw$ be invex in y_2 for every (x,y_1) with respect to η_2. Then, for any $(x,y,w) \in X$ and $(u,v,z) \in Y$,*

$$F(x,y,w) \geqq G(u,v,z).$$

Proof. Denote

$$Z = \max_{x_1} \min_{x_2, y} \{F(x, y, w) | (x, y, w) \in X\}, \text{ and}$$
$$W = \min_{y_1} \max_{x, y_2} \{G(u, v, z) | (u, v, z) \in Y\}.$$

We give the proof only for the case $f(x, y)$ is separable with respect to x_1. (The case $f(x, y)$ is separable with respect to y_1 can be handled in a similar way.) Since $f(x, y)$ is separable with respect to x_1, it follows that $f(x, y) = f_1(x_1) + f_2(x_2, y)$. Therefore $\nabla_{y_2} f(x, y) = \nabla_{y_2} f_2(x_2, y)$ and Z can be written as

$$Z = \max_{x_1} \min_{x_2, y} \{f_1(x_1) + f_2(x_2, y) - y_2^T \nabla_{y_2} f_2(x_2, y) + (x_2^T B x_2)^{\frac{1}{2}} | \ x_1 \in U,$$
$$y_1 \in V, \ -\nabla_{y_2} f(x, y) + Cw \geqq 0, \ w^T C w \leqq 1\}.$$

or $Z = \max_{x_1} \min_{y_1} \{f_1(x_1) + \phi(y_1) : x_1 \in U, \ \ y_1 \in V\}$, where

$$(5) \qquad \phi(y_1) = \min_{x_2, y_2} \{f_2(x_2, y) - y_2^T \nabla_{y_2} f_2(x_2, y) + (x_2^T B x_2)^{\frac{1}{2}}$$
$$| - \nabla_{y_2} f_2(x_2, y) + Cw \geqq 0, \ w^T C w \leqq 1\}.$$

Similarly, $W = \min_{y_1} \max_{x_1} [f_1(u_1) + \psi(v_1) : u_1 \in U \ \ v_1 \in V]$, where

$$(6) \qquad \psi(v_1) = \max_{x_2, y_2} \{f_2(u_2, v) - u_2^T \nabla_{x_2} f_2(u_2, v) - (v_2^T C v_2)^{\frac{1}{2}}$$
$$| - \nabla_{x_2} f_2(u_2, v) - Bz \leqq 0, \ z^T B z \leqq 1\}.$$

Let $(x, y, w) \in X$ and $(u, v, z) \in Y$. In order to prove the theorem, it is sufficient to show that $\phi(y_1) \geqq \psi(v_1)$.

$$\begin{aligned}
&\phi(y_1) - \psi(v_1) \\
&\quad \geqq f_2(x_2, y) - y_2^T \nabla_{y_2} f_2(x_2, y) + (x_2^T B x_2)^{\frac{1}{2}} \\
&\qquad - f_2(u_2, v) + u_2^T \nabla_{x_2} f_2(u_2, v) + (v_2^T C v_2)^{\frac{1}{2}} \\
&\quad \geqq f_2(x_2, y) - y_2^T \nabla_{y_2} f_2(x_2, y) + (x_2^T B x_2)^{\frac{1}{2}} (z^T B z)^{\frac{1}{2}} \\
&\qquad - f_2(u_2, v) + u_2^T \nabla_{x_2} f_2(u_2, v) + (v_2^T C v_2)^{\frac{1}{2}} (w^T C w)^{\frac{1}{2}} \\
&\qquad \text{(by (4), (2) and } (i)) \\
&\quad \geqq f_2(x_2, y) - y_2^T \nabla_{y_2} f_2(x_2, y) + x_2^T B z - f_2(u_2, v) + u_2^T \nabla_{x_2} f_2(u_2, v) \\
&\qquad + v_2^T C w \ \text{ (by the generalized Schwarz inequality)}
\end{aligned}$$

$$\geqq \eta_1(x_2,u_2)^T(\nabla_{x_2}f_2(u_2,v)+Bz)-\eta_2(v_2,y_2)^T(\nabla_{y_2}f_2(x_2,y)-Cw)$$
$$+y_2^TCw+u_2^TBz-y_2^T\nabla_{y_2}f_2(x_2,y)+u_2^T\nabla_{x_2}f_2(u_2,v)$$
(by the invexity of $f(x,y)+{x_2}^TBz$, and $-f(x,y)+{y_2}^TCw$)
$$=\{\eta_1(x_2,u_2)+u_2\}^T(\nabla_{x_2}f_2(u_2,v)+Bz)$$
$$-\{\eta_2(v_2,y_2)+y_2\}^T(\nabla_{y_2}f_2(x_2,y)-Cw)$$
$$\geqq 0$$
(by (1), (3) and (iv)).

Therefore $F(x,y,w)\geqq G(u,v,z)$. □

Using the Lemma 1, we can prove the following strong duality theorem.

THEOREM 2 (Strong Duality). *If $(\overline{x},\ \overline{y},\ \overline{w})\in X$ solves (SP) and the matrix $\nabla_{y_2y_2}f_2(\overline{x}_2,\ \overline{y})$ is positive or negative definite, then there exists $\overline{z}\in\mathbb{R}^{n_2}$ such that $(\overline{x},\ \overline{y},\ \overline{z})\in Y$ with $F(\overline{x},\overline{y},\overline{w})=G(\overline{x},\overline{y},\overline{z})$. If, in addition, $f(x,y)+x_2^TBz$ be invex in x_2 for every (x_1,y) with respect to η_1 and $-f(x,y)+y_2^TCw$ be invex in y_2 for every (x,y_1) with respect to η_2, then $(\overline{x},\overline{y},\overline{z})$ is optimal for (SD) and objective value of the dual is equal to that of the primal.*

$$F(\overline{x},\overline{y},\overline{w})=G(\overline{x},\overline{y},\overline{z}).$$

Proof. For given $y_1=v_1=\overline{y}_1$, (5) and (6) are a pair of symmetric dual nondifferentiable programs. Since $(\overline{x},\ \overline{y},\ \overline{w})$ solves (SP), by Lemma 1 there exist $r\in\mathbb{R},\ \alpha\in\mathbb{R}^{m_2}$ and $\beta\in\mathbb{R}$ such that

(7) $$\alpha\nabla_{y_2}f_2(\overline{x}_2,\overline{y})=\alpha^TC\overline{w},$$

(8) $$\beta(1-\overline{w}^TC\overline{w})=0,$$

(9) $$r\nabla_{x_2}f_2(\overline{x}_2,\overline{y})+(\alpha-r\overline{y}_2)^T\nabla_{x_2y_2}f_2(\overline{x}_2,\overline{y})+rB\overline{z}=0,$$

(10) $$(\alpha-r\overline{y}_2)^T\nabla_{y_2y_2}f_2(\overline{x}_2,\overline{y})=0,$$

(11) $$C\alpha=2\beta C\overline{w},$$

(12) $$\overline{z}^TB\overline{z}\leqq 1,$$

(13) $$(\overline{x}_2^TB\overline{x}_2)^{\frac{1}{2}}=\overline{x}_2^TB\overline{z},$$

(14) $$(r,\alpha,\beta)\geqq 0,$$

(15) $$(r,\alpha,\beta)\neq 0.$$

Now multiplying (10) by $(\alpha - r\overline{y}_2)$, we have

$$(\alpha - r\overline{y}_2)^T \nabla_{y_2 y_2} f_2(\overline{x}_2, \overline{y})(\alpha - r\overline{y}_2) = 0.$$

Since $\nabla_{y_2 y_2} f_2(\overline{x}_2, \overline{y})$ is positive or negative definite, we obtain

$$\alpha = r\overline{y}_2. \tag{16}$$

Now multiplying (11) by $\overline{w}^T$, we get

$$\overline{w}^T C\alpha = 2\beta \overline{w}^T C\overline{w}. \tag{17}$$

It is to be observed here that $r > 0$, for otherwise $\alpha = r\overline{y}_2 = 0$, and (17) together with (8) imply $\beta = 0$, a contradiction to (15). Now equation (11) with the aid of (16) and the fact $r > 0$, gives

$$\overline{y}_2^T C\overline{w} = (\overline{y}_2^T C\overline{y}_2)^{\frac{1}{2}} (\overline{w}^T C\overline{w})^{\frac{1}{2}}. \tag{18}$$

Also from (8), either $\beta = 0$, and hence $C\overline{y}_2 = 2(\beta/r)C\overline{w} = 0$ or $\overline{w}^T C\overline{w} = 1$. In either case (18) give

$$\overline{y}_2^T C\overline{w} = (\overline{y}_2^T C\overline{y}_2)^{\frac{1}{2}}. \tag{19}$$

From (9) and (16) together with $r > 0$ we get

$$-\nabla_{x_2} f_2(\overline{x}_2, \overline{y}) - B\overline{z} = 0, \tag{20}$$

and from (12) and (20), $(\overline{x}_2, \overline{y}, \overline{z})$ is feasible for (SD). Multiplying (20) by $\overline{x}_2$, we get

$$-\overline{x}_2^T \nabla_{x_2} f_2(\overline{x}_2, \overline{y}) = \overline{x}_2^T B\overline{z}. \tag{21}$$

Hence

$$\begin{aligned} F(\overline{x}, \overline{y}, \overline{w}) =& f_1(\overline{x}_1) + f_2(\overline{x}_2, \overline{y}) - \overline{y}_2^T \nabla_{y_2} f_2(\overline{x}, \overline{y}) + (\overline{x}_2^T B\overline{x}_2)^{\frac{1}{2}} \\ =& f_1(\overline{x}_1) + f_2(\overline{x}_2, \overline{y}) - \overline{y}_2^T C\overline{w} + (\overline{x}_2^T B\overline{z}) \\ & \text{(using (7), (16) with } r > 0 \text{ and then (13))} \\ =& f(\overline{x}, \overline{y}) - \overline{x}_2^T \nabla_{x_2} f(\overline{x}, \overline{y}) - (\overline{y}_2^T C\overline{y}_2)^{\frac{1}{2}} \\ & \text{(using (19), (20), } f = f_1 + f_2, \text{and } \nabla_{x_2} f = \nabla_{x_2} f_2) \\ =& G(\overline{x}, \overline{y}, \overline{z}). \end{aligned}$$

Thus $(\overline{x}, \overline{y}, \overline{z})$ is optimal for (SD) by Theorem 1. □

A converse duality theorem may be stated as follows:

THEOREM 3 (Converse Duality). *If* $(\overline{x},\ \overline{y},\ \overline{z}) \in Y$ *solves* (SD) *and the matrix* $\nabla_{x_2x_2} f_2(\overline{x}_2,\ \overline{y})$ *is positive or negative definite, then there exists* $\overline{w} \in \mathbb{R}^{n_2}$ *such that* $(\overline{x},\ \overline{y},\ \overline{w}) \in X$ *with* $F(\overline{x}, \overline{y}, \overline{w}) = G(\overline{x}, \overline{y}, \overline{z})$. *If, in addition,* $f(x, y) + x_2^T Bz$ *be invex in* x_2 *for every* (x_1, y) *with respect to* η_1 *and* $-f(x, y) + y_2^T Cw$ *be invex in* y_2 *for every* (x, y_1) *with respect to* η_2*, then* $(\overline{x}, \overline{y}, \overline{w})$ *is optimal for* (SP) *and the objective value of the primal is equal to that of the dual.*

$$G(\overline{x}, \overline{y}, \overline{z}) = F(\overline{x}, \overline{y}, \overline{w}).$$

We now establish the self duality of (SP).

Assume that $n_1 = m_1,\ n_2 = m_2,\ C = B,\ z = w$ and $f(x, y) = -f(y, x)$.

It follows that (SD) may be rewritten as follows:

$$\begin{array}{lll} (SD') & \text{Max}_{y_1}\ \text{Min}_{x,y_2} & f(v, u) - u_2^T \nabla_{y_2} f(v, u) + (v_2^T C v_2)^{\frac{1}{2}} \\ & \text{subject to} & -\nabla_{y_2} f(v, u) + Bz \geqq 0, \\ & & z^T B z \leqq 1, \\ & & u_1 \in U, \quad v_1 \in V. \end{array}$$

(SD') is formally identical to (SP); that is, the objective and constraint functions of (SP) and (SD') are identical. This problem is said to be self dual.

It is easily seen that whenever (x, y, z) is feasible for (SP), then (y, x, z) is feasible for (SD), and vice versa.

THEOREM 4 (Self Duality). *Assume that* (SP) *is self dual and that the invexity conditions of Theorem 1 are satisfied. If* $(\overline{x}, \overline{y}, \overline{z})$ *is an optimal solution for* (SP)*, and the matrix* $\nabla_{y_2y_2} f_2(\overline{x}_2,\ \overline{y})$ *is positive or negative definite, then* $(\overline{y}, \overline{x}, \overline{z})$*is an optimal solution for both* (SP) *and* (SD)*, and the common optimal value is* 0.

Proof. By Theorem 2, $(\overline{x}, \overline{y}, \overline{z})$ is an optimal solution for (SD), and the optimal values of (SP) and (SD) are equal to $F(\overline{x}, \overline{y}, \overline{z})$. From the