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ON SCATTERING BY SEVERAL CONVEX BODIES

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ABSTRACT. We consider a zeta function of the classical dynamics in the exterior of several convex bodies. The main result is that the poles of the zeta function cannot converge to the line of absolute convergence if the abscissa of absolute convergence of the zeta function is positive.

1. Introduction

Let $\mathcal{O}_j$ $(j = 1, 2, \ldots, J)$ be open and bounded sets in $\boldsymbol{R}^3$ with smooth boundary $\Gamma_j = \partial\mathcal{O}_j$. We call each $\mathcal{O}_j$ a body, and assume that the number J of bodies is greater than or equal to 3, and also assume about the configuration of bodies the following:

(H.1) Each $\mathcal{O}_j$ is strictly convex, that is, the Gaussian curvature of Γ_j does not vanish.

(H.2) For any $\{j_1, j_2, j_3\} \in \{1, 2, \ldots, J\}^3$ satisfying $j_\ell \neq j_{\ell'}$ if $\ell \neq \ell'$, it holds that

$$(\text{convex hull of } \overline{\mathcal{O}_{j_1}} \text{ and } \overline{\mathcal{O}_{j_2}}) \cap \overline{\mathcal{O}_{j_3}} = \emptyset.$$

We set

$$\mathcal{O} = \cup_{j=1}^{J} \mathcal{O}_j, \quad \Omega = \mathbf{R}^3 \setminus \overline{\mathcal{O}}, \quad \Gamma = \partial\Omega = \cup_{j=1}^{J} \Gamma_j.$$

We consider two mechanics in Ω. The one is the classical mechanics and the other one is the quantum mechanics.

Here, the classical mechanics means the movement of a particle in Ω following the law of geometric optics, and the object of consideration in the classical mechanics is a zeta function $\zeta(\mu)$. It is known that, under

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the assumptions (H.1) and (H.2), a zeta funtion of Ω can be defined as follows:

$$\zeta(\mu) = \prod \left(1 - (-1)^{i_\gamma}(\lambda_{\gamma,1}\lambda_{\gamma,2})^{1/2}e^{-\mu d_\gamma}\right)^{-1} \tag{1.1}$$

where the product is taken over all the primitive, oriented periodic rays in Ω and the notations in (1.1) are as follows:

γ : oriented periodic ray in Ω,

i_γ : the number of the reflection points of γ,

d_γ : the length of γ,

$\lambda_{\gamma,\ell}$:the eigenvalue less than 1 of the Poincaré map of γ.

Let μ_0 be the abscissa of the absolute convergence of $\zeta(\mu)$, that is, μ_0 is the real number such that

$$\text{for } \Re\mu > \mu_0 \quad \sum_\gamma \left|(-1)^{i_\gamma}(\lambda_{\gamma,1}\lambda_{\gamma,2})^{1/2}, e^{-\mu d_\gamma}\right| < \infty,$$

$$\text{for } \Re\mu < \mu_0 \quad \sum_\gamma \left|(-1)^{i_\gamma}(\lambda_{\gamma,1}\lambda_{\gamma,2})^{1/2} e^{-\mu d_\gamma}\right| = \infty.$$

We can see immediately the existence of such μ_0 from the form of each term under the summation. So we have that

$$\zeta(\mu) \text{ is holomorphic in } \Re\mu > \mu_0.$$

Of course, there is a possibility of the analytic continuation of $\zeta(\mu)$ beyond the line $\Re\mu = \mu_0$.

On the other hand, the quantum mechanics in Ω means the propagation phenomena governed by the wave equation with Dirichlet boundary condition, that is,

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \triangle u = 0 & \text{in } \Omega \times \boldsymbol{R}, \\ u(x,t) = 0 & \text{on } \Gamma \times \boldsymbol{R}. \end{cases} \tag{1.2}$$

The object we consider for this problem is the scattering matrix $\mathcal{S}(z)$. The scattering matrix $\mathcal{S}(z)$ is an $\mathcal{L}(L^2(S^2))$-valued function of $z \in \boldsymbol{C}$ which is holomorphic in the lower half plane $\{z;\ \Im z \leq 0\}$ and meromorphic in the whole complex plane $\boldsymbol{C}$. Here, the notation $\mathcal{L}(H)$ denotes the set of all linear bounded operators in the space H.

The problem which we would like to consider is on relationships between the singularities of $\zeta(\mu)$ and $\mathcal{S}(z)$. But it should be remarked that actually it is difficult to show in general even the existence of a singularity of $\zeta(\mu)$. On the other hand, it is known that, if $\zeta(\mu)$ has a singularity, there exists an $\alpha > 0$ such that a slab domain $\{z \in \mathbf{C};\ 0 \leq \Im z \leq \alpha\}$ contains an infinite number of the poles of $\mathcal{S}(z)$. This fact is nothing but the validity of the modified Lax-Phillips conjecture for this obstacle (Ikawa[4]).

In this paper, we shall do an opposite consideration. Our main result is the following theorem:

THEOREM 1.1. *Assume that the abscissa of absolute convergence of $\zeta(\mu)$ is positive. Then, the poles of $\zeta(\mu)$ cannot converge to the line $\Re\mu = \mu_0$.*

The fact that we would like to emphasize here is that the statement of Theorem 1.1 is purely of the classical mechanics in Ω, but the proof we shall do is based on the analysis of solutions of the reduced wave equation and on the informations of the scattering matrix $\mathcal{S}(z)$.

2. Classical mechanics and symbolic dynamics

In this section, we shall explain that, the consideration of periodic rays in Ω can be reduced to the consideration of a symbolic dynamics. A symbolic dynamics is a pair of a set Σ_A of infinite sequences of numbers $\{1, 2, \ldots, J\}$ and a shift operator σ_A that translates sequences to left direction of one step, whose definitions will be given before long. As you have seen, a symbolic dynamics is of structure very simple, but in order to consider the periodic rays in Ω, a reduction to such a simple dynamics works very well.

Let $\mathcal{X}$ be an oriented ray of geometric optics in Ω trapped by $\mathcal{O}$ in the future and in the past, and let $x(t)$ ($t \in (-\infty, \infty)$) be its represention by the length of ray measured from a point on Γ. The ray $\mathcal{X}$ is trapped by $\mathcal{O}$ means that $\{x(t);\ t \in (-\infty, \infty)\}$ is bounded. Since $x(0)$ is on Γ there is $\xi_0 \in \{1, 2, \ldots, J\}$ such that $x(0) \in \Gamma_{\xi_0}$.

Under the assumption (H.1) and (H.2), following the advance of the time, the ray reflects on $\Gamma_{\xi_1}, \Gamma_{\xi_2}, \cdots$ successively and following the time going back to the past, reflects on $\Gamma_{\xi_{-1}}, \Gamma_{\xi_{-2}}, \cdots$ successively. Denote the j-th reflection point by P_j. So, for each ray trapped by $\mathcal{O}$ there corresponds a two sided infinite sequence of the reflection points

$$\cdots, P_{-1}, P_0, P_1, \cdots$$

and a two sided infinite sequence of elements in $\{1, 2, \ldots, J\}$

$$\cdots, \xi_{-1}, \xi_0, \xi_1, \cdots$$

such that $P_j \in \Gamma_{\xi_j}$ for all $j \in \boldsymbol{Z}$. It is clear that the above infinite sequence satisfies $\xi_i \neq \xi_{i+1}$ for all $j \in \boldsymbol{Z}$.

Now, let $A = [A(i,j)]_{i,j=1,2,\ldots,J}$ be a $J \times J$ matrix given by

$$A(i,j) = \begin{cases} 1 & \text{for } \ i \neq j, \\ 0 & \text{for } \ i = j, \end{cases}$$

and we introduce a set of two sided infinite sequence by

$$\Sigma_A = \{\xi = (\ldots, \xi_{-n}, \xi_{-(n-1)}, \ldots, \xi_{-1}, \xi_0, \xi_1, \ldots, \xi_n, \ldots); \\ \xi_i \in \{1, 2, \ldots, J\} \text{ and } A(\xi_i, \xi_{i+1}) = 1 \text{ for all } i\}.$$

We have corresponded for a ray $\mathcal{X}$ trapped by $\mathcal{O}$ a two sided infinite sequence $(\cdots, \xi_{-1}, \xi_0, \xi_1, \cdots)$. It is clear that this sequence belongs to Σ_A.

Conversely, for each element $\xi \in \Sigma_A$, there corresponds uniquely a ray of geometric optics $\mathcal{X}(\xi)$ in Ω trapped by $\mathcal{O}$ such that whose reflections points P_j $(j \in \boldsymbol{Z})$ satisfy

$$P_j \in \Gamma_{\xi_j} \quad \text{for all } j \in \boldsymbol{Z}.$$

Since the all reflection points are determined by $\xi \in \Sigma_A$, we denote the sequence of the reflection points corresponding to ξ as

$$\cdots, P_{-1}(\xi), P_0(\xi), P_1(\xi), \cdots$$

We define a function $f(\xi)$ on Σ_A by

$$f(\xi) = |P_1(\xi) - P_0(\xi)|. \tag{2.1}$$

For $\xi \in \Sigma_A$ we can find a sequence of phase functions $\{\varphi_{\xi,j}(x)\}_{j=-\infty}^{\infty}$ satisfying for all $j \in \boldsymbol{Z}$

$$\begin{cases} |\nabla\varphi_{\xi,j}(x)| = 1 \ \text{ in a neighborhood of } \ P_j(\xi)P_{j+1}(\xi), \\ \nabla\varphi_{\xi,j}(P_j(\xi)) \ \text{ is parallel to } \overrightarrow{P_j(\xi)P_{j+1}(\xi)}, \\ \varphi_{\xi,j}(x) = \varphi_{\xi,j+1}(x) \ \text{ on } \ \Gamma_{\xi_{j-1}} \cap (\text{a neighborhood of } P_{j+1}(\xi)), \\ \triangle\varphi_{\xi,j}(P_j(\xi)) \ \text{ is positive.} \end{cases}$$

Let us set

$$\mathcal{C}_{\xi,j}(x) = \{y;\ \varphi_{\xi,j}(y) = \varphi_{\xi,j}(x)\},$$

and denote by $G_{\xi,j}(x)$ the Gaussian curvature of $\mathcal{C}_{\xi,j}(x)$ at x. We define a function $g(\xi)$ on Σ_A by

$$g(\xi) = \frac{1}{2}\log\Big(G_{\xi,0}\big(P_1(\xi)\big)/G_{\xi,0}\big(P_0(\xi)\big)\Big). \tag{2.2}$$

Denote by σ_A the shift operator in Σ_A, that is, $\eta = \sigma_A\,\xi = (\ldots,\eta_{-1},\eta_0,\eta_1,\ldots)$ means that $\eta_j = \xi_{j+1}$ for all $j \in \boldsymbol{Z}$. So it is easy to see that $\xi \in \Sigma_A$ corresponding to a periodic oriented ray $\mathcal{X}$ in Ω satisfies

$$\sigma_A^m \xi = \xi \quad \text{for some} \ \ m > 0, \tag{2.3}$$

and the trajectry of $\mathcal{X}$ is given by $\cup_{j=0}^{m-1} P_j(\xi)P_{j+1}(\xi)$. The necessary and sufficient condition for this trajectry to be primitive is that the number m in (2.3) is the minimum among the positive integers p satisfying $\sigma_A^p\xi = \xi$.

By using the notations defined up to now, we can represent the zeta function $\zeta(\mu)$ defined in the introduction as follows:

$$\zeta(\mu) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\sigma_A^n\xi=\xi}\exp S_n r(\xi,\mu)\right), \tag{2.4}$$

where

$$r(\xi,\mu) = -\mu f(\xi) + g(\xi) + \sqrt{-1}\pi, \tag{2.5}$$
$$S_n r(\xi,\mu) = r(\xi,\mu) + r(\sigma_A\xi,\mu) + r(\sigma_A^2\xi,\mu) + \cdots + r(\sigma_A^{n-1}\xi,\mu).$$

Then, we have for all oriented periodic ray $\mathcal{X}(\xi)$ in Ω satisfying $\sigma_A^n\xi = \xi$

$$\exp\big(S_n r(\xi,\mu)\big) = (-1)^n(\lambda_{\gamma,1}\lambda_{\gamma,2})^{1/2}\exp(-\mu d_\gamma),$$

where γ denotes the closed trajectory $\cup_{j=0}^{n-1} P_j(\xi)P_{j+1}(\xi)$.

As to the convergence of the right hand side of (2.4), let us remark that

$$\#\{\xi \in \Sigma_A;\ \sigma_A^n\xi = \xi\} \le (J-1)^n,$$
$$S_n f(\xi) \ge n d_{\min},$$

where we set

$$d_{\min} = \min_{i\neq j}\inf\{|x-y|;\ x \in \mathcal{O}_i,\ y \in \mathcal{O}_j\}.$$

Taking account of $0 < \lambda_{\gamma,1}\lambda_{\gamma,2} < 1$, the right hand side of (2.4) converges absolutely in

$$\Re\mu > \frac{J-1}{d_{\min}}. \tag{2.6}$$

Now we shall see that the function defined by the right hand side of (2.4) coincides with $\zeta(\mu)$ given in the introduction. Let $\xi \in \Sigma_A$ satisfy $\sigma_A^m \xi = \xi$, and the trajectory $\gamma = \cup_{j=0}^{m-1} P_j(\xi)P_{j+1}(\xi)$ is primitive. So, all the elements of Σ_A which give the same periodic trajectory are $\{\sigma_A^j \xi;\ j = 0, 1, 2, \dots, m-1\}$, and the number of such elements is just m. On the other hand, for $n = pm$, we have

$$\exp\big(S_n r(\xi,\mu)\big) = \big(\exp(S_m r(\xi,\mu))\big)^p.$$

Therefore, if we set $\Sigma_A(\gamma) = \{\xi \in \Sigma_A;\ \xi \text{ corresponds to } \gamma\}$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\xi\in\Sigma_A(\gamma),\ \sigma_A^n\xi=\xi,} \exp S_n r(\xi,\mu) &= \sum_{p=1}^{\infty} \frac{1}{p}\Big(\exp(S_m r(\xi,\mu))\Big)^p \\ &= -\log\Big(1 - \exp\big(S_m r(\xi,\mu)\big)\Big).\end{aligned} \tag{2.7}$$

Thus,

$$\begin{aligned}&\text{the right hand side of (2.4)} \\ &= \exp\left\{\sum_\gamma \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\xi\in\Sigma_A(\gamma),\ \sigma_A^n\xi=\xi} \exp S_n r(\xi,\mu)\right\} \\ &= \exp\left\{\sum_\gamma \Big(-\log\Big(1 - \exp\big(S_m r(\xi,\mu)\big)\Big)\Big)\right\} \\ &= \prod_\gamma \exp\Big(-\log\Big(1 - \exp\big(S_m r(\xi,\mu)\big)\Big)\Big) \\ &= \prod_\gamma \Big(1 - \exp\big(S_m r(\xi,\mu)\big)\Big)^{-1},\end{aligned}$$

where $\sum_\gamma$ or $\prod_\gamma$ means that the summation or the product is taken over all the oriented primitive periodic rays in Ω. So the function given by the right hand side of (2.4) coincides with the zeta function $\zeta(\mu)$ defined by (1.1).

3. Solutions of the reduced wave equation

We shall consider the following boundary value problem for the reduced wave equation:

$$\begin{cases}(-\triangle - z^2)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma,\end{cases} \tag{3.1}$$

where z is a complex parameter.

For $\Im z < 0$ and $f \in L^2(\Omega)$, the problem has a unique solution in $L^2(\Omega)$. Denote this solution as

$$u(x) = \big(R(z)f\big)(x).$$

Then $R(z) \in \mathcal{L}\big(L^2(\Omega), L^2(\Omega)\big)$ and it depends analytically on $z \in \{z \in \boldsymbol{C};\ \Im z < 0\}$. By the regularity theorem for solutions of elliptic equations, we may regard $R(z)$ as an continuous operator from $C_0^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$. So we have that

$$R(z) \text{ is } \mathcal{L}\big(C_0^\infty(\overline{\Omega}), C^\infty(\overline{\Omega})\big)\text{-valued holomorphic function in } \Im z < 0.$$

Even though the following fact is not needed in this paper, we would like to remark that $R(z)$ as $\mathcal{L}\big(C_0^\infty(\overline{\Omega}), C^\infty(\overline{\Omega})\big)$-valued function of z can be continued meromorphically into the whole complex plane, and

$$\text{the poles of } R(z) \text{ coincide with those of } \mathcal{S}(z).$$

So, the consideration of poles of scattering matrices may be reduced to the consideration of poles of $R(z)$.

The fact which we shall use in this paper is the following estimate which follows immediately from the selfadjointness of $\triangle$ with Dirichlet boundary condition:

$$\|R(z)\|_{\mathcal{L}\left(L^2(\Omega), L^2(\Omega)\right)} \le \frac{1}{2|\Im z|\,|\Re z|} \quad \text{for } \Im z < 0. \tag{3.2}$$

Indeed, we have by integration by parts

$$\begin{aligned}&\int_\Omega f(x)\overline{u(x)}\,dx \\ &= \int_\Omega (-\triangle - z^2)u(x)\overline{u(x)}\,dx \\ &= \int_\Omega |\nabla u(x)|^2\,dx - \big\{(\Re z)^2 - (\Im z)^2 + 2i(\Re z)(\Im z)\big\}\int_\Omega |u(x)|^2\,dx.\end{aligned}$$

Comparing the imaginary parts of the both sides, we have

$$\begin{aligned} 2|\Re z|\,|\Im z|\int_\Omega |u(x)|^2\,dx &\le \left|\int_\Omega f(x)\overline{u(x)}\,dx\right| \\ &\le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}, \end{aligned}$$

from which (3.2) follows.

The argument that we shall do is that, assuming that $\zeta(\mu)$ has a sequence of poles converging to the line $\Re\mu = \mu_0$, we shall construct a sequence of solutions of (3.1) which violates the estimate (3.2). To this end, we shall construct an asymptotic solutions of (3.1) for an oscillatory data $f(x)$, by using the method in [3] and [5].

3.1. Construction of asymptotic solutions

Let f be an oscillatory function in Ω with a parameter $z \in \boldsymbol{C}$ given by

$$f(x,z) = e^{-iz\varphi(x)}g(x), \tag{3.3}$$

where φ is a real valued smooth function satisfying

$$|\nabla\varphi(x)| = 1 \tag{3.4}$$

and $g \in C_0^\infty(\Omega)$. We will construct an asymptotic solution of the problem (3.1). First we look for $u_{(0)}(x,z)$ of the form

$$u_{(0)}(x,z) = e^{-iz\varphi(x)}v(x), \tag{3.5}$$

which satisfies

$$(-\Delta - z^2)u_{(0)}(x,z) = f(x,z) \quad \text{in} \quad \boldsymbol{R}^n. \tag{3.6}$$

If φ satisfies (3.4), in order to satisfy (3.6), it suffices to satisfy

$$2\nabla\varphi(x)\cdot\nabla v(x) + \Delta\varphi(x)v(x) = g(x). \tag{3.7}$$

Next, we construct $u_{(j)}(x,z)$ $(j = 1, 2, \dots, J)$ of the form

$$u_{(j)}(x,z) = e^{-iz\varphi_{(j)}(x)}v_{(j)}(x),$$

which satisfy

$$\begin{cases} (-\Delta - z^2)u_{(j)}(x,z) = e^{-iz\varphi_{(j)}(x)}\Delta v_{(j)}(x) & \text{in} \quad \boldsymbol{R}^n \setminus \overline{\mathcal{O}_j}, \\ u_{(j)}(x,z) + u_{(0)}(x,z) = 0 & \text{on} \quad \Gamma_j. \end{cases} \tag{3.8}$$

To satisfy (3.8), we require the following:

$$\begin{cases} |\nabla\varphi_{(j)}(x)| = 1, \\ \varphi_{(j)}(x) = \varphi(x) \quad \text{on} \quad \Gamma_j, \\ 2\nabla\varphi_{(j)}(x)\cdot\nabla v_{(j)}(x) + \triangle\varphi_{(j)}(x)v_{(j)}(x) = 0, \\ v_{(j)}(x) + v(x) = 0 \quad \text{on} \quad \Gamma_j. \end{cases}$$

Next for each $u_{(j)}(x,z)$, we construct for $k \neq j$ a function $u_{(j,k)}(x,z)$ of the form

$$u_{(j,k)}(x,z) = e^{-iz\varphi_{(j,k)}(x)}v_{(j,k)}(x),$$

satisfying

$$\text{(3.9)} \quad \begin{cases} (-\triangle - z^2)u_{(j,k)}(x,z) = e^{-iz\varphi_{(j,k)}(x)}\triangle v_{(j,k)}(x) \quad \text{in} \quad \boldsymbol{R}^n \setminus \overline{\mathcal{O}_k}, \\ u_{(j,k)}(x,z) + u_{(j)}(x,z) = 0 \quad \text{on} \quad \Gamma_k. \end{cases}$$

To satisfy (3.9), we require

$$\text{(3.10)} \qquad \begin{cases} |\nabla\varphi_{(j,k)}(x)| = 1, \\ \varphi_{(j,k)}(x) = \varphi_{(j)}(x) \quad \text{on} \quad \Gamma_k, \\ 2\nabla\varphi_{(j,k)}(x)\cdot\nabla v_{(j,k)}(x) + \triangle\varphi_{(j,k)}(x)v_{(j,k)}(x) = 0, \\ v_{(j,k)}(x) + v_{(j)}(x) = 0 \quad \text{on} \quad \Gamma_k. \end{cases}$$

In order to repeat this procedure, we prepare some notations. For $n = 1, 2, \ldots$, we set

$$I_n = \Big\{ \boldsymbol{i} = (i_1, i_2, \ldots, i_n);\ i_j \in \{1, 2, \ldots, J\} \\ \text{and } A(i_j, i_{j+1}) = 1 \ \text{ for all } j = 1, 2, \ldots, n-1 \Big\},$$

and

$$I = \cup_{n=1}^{\infty} I_n.$$

Suppose that, for all $\boldsymbol{i} \in I_n$, $u_{\boldsymbol{i}}(x,z) = e^{-iz\varphi_{\boldsymbol{i}}(x)}v_{\boldsymbol{i}}(x)$ is defined. For $\boldsymbol{j} = (\boldsymbol{i}, k) \in I_{n+1}$, we define $u_{\boldsymbol{j}}(x,z) = e^{-iz\varphi_{\boldsymbol{j}}(x)}v_{\boldsymbol{j}}(x)$, in such a way that $u_{\boldsymbol{j}}$ satisfies

$$\text{(3.11)} \qquad \begin{cases} (-\triangle - z^2)u_{\boldsymbol{j}}(x,z) = e^{-iz\varphi_{\boldsymbol{j}}(x)}\triangle v_{\boldsymbol{j}}(x) \quad \text{in} \quad \boldsymbol{R}^n \setminus \overline{\mathcal{O}_k} \\ u_{\boldsymbol{j}}(x,z) + u_{\boldsymbol{i}}(x,z) = 0 \quad \text{on} \quad \Gamma_k. \end{cases}$$

To this end, we choose the phase function φ_j and the amplitude function v_j in this way:

$$\begin{cases} |\nabla\varphi_j(x)| = 1, \\ \varphi_j(x) = \varphi_i(x) \quad \text{on} \quad \Gamma_k, \\ 2\nabla\varphi_j(x)\cdot\nabla v_j(x) + \triangle\varphi_j(x)v_j(x) = 0, \\ v_j(x) + v_i(x) = 0 \quad \text{on} \quad \Gamma_k. \end{cases}$$

By this procedure, we can define successively u_j for all $j \in I$. Now we define $u(x,z)$ by

$$u(x,z) = \sum_{j\in I} u_j(x,z). \tag{3.12}$$

Suppose that there is a constant a_0 such that

for $\Im z < a_0$, the right hand side of (3.12) converges absolutely.

Then, by taking account of the second equality of (3.11), we have for all $\Im z < a_0$

$$u(x,z) = 0 \quad \text{on} \quad \Gamma. \tag{3.13}$$

Similarly by using the first equation of (3.11), we have

$$(-\triangle - z^2)u(x,z) = f(x,z) + \sum_{j\in I} e^{-iz\varphi_j(x)}\triangle v_j(x). \tag{3.14}$$

What we want to emphasize here is that, comparing (3.13) and (3.14) with the problem (3.1) which we want to solve, the function $u(x,z)$ has a property that after operation of $(-\triangle - z^2)$ the error term has the same form as $u(x,z)$. The changing part is only the shape of amplitude functions.

3.2. Ruelle operator

To get informations on analyticity of $u(x,z)$ of (3.12), it is crucial to express u by Ruelle operator, which is a linear operator of a symbolic dynamics. So we shall start with giving some notations and fundamental facts relating to symbolic dynamics(Bowen [1], Pollicott [7]).

For a function $k(\xi)$ on Σ_A, we set

$$\operatorname{var}_n k = \sup\left\{|k(\xi) - k(\xi')|;\ \xi_i = \xi'_i \ \text{ for all } |i| \le n\right\},$$

and for $0 < \theta < 1$,

$$||k||_\theta = \sup_n \frac{\mathrm{var}_n\, k}{\theta^n}.$$

We set

$$||k||_\infty = \sup_{\xi \in \Sigma_A} |k(\xi)|, \quad \text{and} \quad |||k|||_\theta = ||k||_\theta + ||k||_\infty,$$

and introduce a space $\mathcal{F}_\theta(\Sigma_A)$ of functions on Σ_A by

$$\mathcal{F}_\theta(\Sigma_A) = \{k(\xi);\ |||k|||_\theta < \infty\}.$$

When the conditions (H.1) and (H.2) are satisfied, the functions $f(\xi)$ defined by (2.1) and $g(\xi)$ defined by (2.2) belong to $\mathcal{F}_\theta(\Sigma_A)$, where the constant $0 < \theta < 1$ is determined by the configuration of $\mathcal{O}$.

Now introduce a space of one sided infinite sequences

$$\Sigma_A^+ = \big\{\xi = (\xi_0, \xi_1, \xi_2, \cdots);\ A(\xi_i, \xi_{i+1}) = 1 \ \text{ for all } i \geq 0\big\}.$$

For each $k \in \{1, 2, \ldots, J\}$, choose a sequence $(\cdots, \eta_{-2}^{(k)}, \eta_{-1}^{(k)}, \eta_0^{(k)})$ in such a way that $\eta_0^{(k)} = k$ and $A(\eta_{-j-1}^{(k)}, \eta_{-j}^{(k)}) = 1$ for $j = 0, 1, 2, \ldots$. For $\xi = (\cdots, \xi_{-1}, \xi_0, \xi_1, \cdots)$, we define an element $e(\xi)$ of Σ_A by

$$e(\xi) = (\cdots, \eta_{-2}^{(k)}, \eta_{-1}^{(k)}, \eta_0^{(k)}, \xi_1, \xi_2, \cdots)$$

when $\xi_0 = k$. Evidently, we have

$$e(\xi) = e(\xi') \quad \text{if } \ \xi_j = \xi'_j \ \text{ for } j \geq 0.$$

For $r(\xi, \mu)$ defined by (2.5), we shall construct a function $\tilde{r}(\xi, \mu)$ and $\chi(\xi, \mu)$ by

$$\chi(\xi, \mu) = \sum_{j=0}^{\infty} \big\{r(\sigma_A^j \xi, \mu) - r(\sigma_A^j e(\xi), \mu)\big\}, \tag{3.15}$$

$$\tilde{r}(\xi, \mu) = r(\xi, \mu) - \chi(\xi, \mu) + \chi(\sigma_A \xi, \mu). \tag{3.16}$$

Then, it is easy to check

$$\tilde{r}(\xi, \mu) = \tilde{r}(\xi', \mu) \quad \text{if } \ \xi_j = \xi'_j \ \text{ for } j \geq 0.$$

This fact shows that the function $\tilde{r}(\xi, \mu)$ can be regarded as a function on Σ_A^+. And also we see immediately

$$S_n \tilde{r}(\xi, \mu) = S_n r(\xi, \mu) \quad \text{if } \ \sigma_A^n \xi = \xi. \tag{3.17}$$

This relation shows that, in order to consider the zeta function $\zeta(\mu)$, $r(\xi, \mu)$ may be replaced by $\tilde{r}(\xi, \mu)$, which is a function of Σ_A^+, and we may consider $\zeta(\mu)$ on the space Σ_A^+. Also, to make analysis in Σ_A^+, it is known that a linear operator in $\mathcal{F}_\theta(\Sigma_A^+)$ called Ruelle operator plays a very important role.

The Ruelle operator $\mathcal{L}_\mu$ is defined as follows:

$$(3.18) \qquad (\mathcal{L}_\mu h)(\xi) = \sum_{\sigma_A \eta = \xi} e^{\tilde{r}(\eta,\mu)} h(\eta) \quad \text{for } \ h \in \mathcal{F}_\theta(\Sigma_A^+).$$

Related to $\mathcal{L}_\mu$, we introduce another operator $|\mathcal{L}_\mu|$ defined by

$$(3.19) \qquad (|\mathcal{L}_\mu|\, h)(\xi) = \sum_{\sigma_A \eta = \xi} |e^{\tilde{r}(\eta,\mu)}|\, h(\eta) \quad \text{for } \ h \in \mathcal{F}_\theta(\Sigma_A^+).$$

The Perron-Frobenius Theorem implies that there exists a real number α_0 such that

> $|\mathcal{L}_{\alpha_0}|$ has 1 as a simple eigenvalue and all the other spectrum are contained in $\{s \in \boldsymbol{C};\ |s| < 1 - \varepsilon\}$ for some $\varepsilon > 0$.

Concerning the relationship between $\zeta(\mu)$ and Ruelle operator, it is known that the abscissa of absolute convergence μ_0 of $\zeta(\mu)$ coincides with this α_0, and that the following fact holds(Haydn[2], Pollicott[7]):

THEOREM 3.1. *There exists $\varepsilon_0 > 0$ such that, in a slab domain $\{\mu \in \boldsymbol{C};\ \mu_0 - \varepsilon_0 \le \Re\mu \le \mu_0\}$, the necessary and sufficient condition for μ to be a pole of $\zeta(\mu)$ is that* Ruelle *operator $\mathcal{L}_\mu$ has 1 as an eigenvalue.*

3.3. Representation of $u(x,z)$ by Ruelle operator

By the argument in [5, Section 4] the asymptotic solutin $u(x,z)$ given by (3.12) for $\Im z < -\mu_0$ can be represented as

$$(3.20) \qquad u(x,z) = \sum_{j=1}^{J} \left(\mathcal{R}_\mu^{(x)} \sum_{n=0}^{\infty} \mathcal{L}_\mu^n h_0 \right) (\xi^{(j)}) + w(x,z),$$

where we take $\mu = iz$, and $\xi^{(j)}$ is a fixed element in $\mathcal{F}_\theta(\Sigma_A^+)$ such that $\xi_0^{(j)} = j$. In (3.20), we have the following:

(i) $w(\cdot\,, z)$ is $L^2(\Omega)$-valued holomorphic function in $\Im z \le -\mu_0 + \varepsilon_0$.

(ii) $\mathcal{R}_\mu^{(x)}$ is a bounded operator defined in $\Re\mu \ge \mu_0 - \varepsilon_0$ depending smoothly on $x \in \Omega$.

(iii) $h_0 \in \mathcal{F}_\theta(\Sigma_A^+)$ is determined by φ and g in (3.3).

Similarly, where the error term in (3.14) also has the following expression

$$\sum_{j\in I} e^{-iz\varphi_j(x)}\triangle v_j(x) = \sum_{j=1}^{J}\left(\tilde{\mathcal{R}}_\mu^{(x)}\sum_{n=0}^{\infty}\mathcal{L}_\mu^n h_0\right)(\xi^{(j)}) + \tilde{w}(x,z), \tag{3.21}$$

where $\tilde{\mathcal{R}}_\mu^{(x)}$ and $\tilde{w}(x,z)$ have the same properties as $\mathcal{R}_\mu^{(x)}$ and $w(x,z)$ respectively.

4. Proof of the theorem

Suppose that $\mu = s_0$ satisfying

$$\mu_0 - \varepsilon_0 \le \Re s_0 \le \mu_0$$

is a pole of $\zeta(\mu)$. Then from Theorem 3.1, $\mathcal{L}_{s_0}$ has 1 as an eigenvalue. Here, for the sake of simplicity, we assume that 1 is a simple eigenvalue and $\mathcal{L}_{s_0}$ has the following decomposition:

$$\mathcal{L}_{s_0} = 1\mathcal{P}_{s_0} + \mathcal{Q}_{s_0}$$

where $\mathcal{P}_{s_0}$ and $\mathcal{Q}_{s_0}$ satisfies

$$\mathcal{P}_{s_0}\mathcal{Q}_{s_0} = \mathcal{Q}_{s_0}\mathcal{P}_{s_0} = 0,$$
$$\dim \operatorname{range} \mathcal{P}_{s_0} = 1,$$
$$\text{spectral radius of } \mathcal{Q}_{s_0} < 1.$$

Then by the perturbation theory of bounded operators, we have for all μ very close to s_0

$$\mathcal{L}_\mu = \lambda_\mu\mathcal{P}_\mu + \mathcal{Q}_\mu \tag{4.1}$$

such that

$$\mathcal{P}_\mu\mathcal{Q}_\mu = \mathcal{Q}_\mu\mathcal{P}_\mu = 0,$$
$$\dim \operatorname{range} \mathcal{P}_\mu = 1,$$
$$\text{spectral radius of } \mathcal{Q}_\mu < 1,$$
$$\lambda_\mu \ne 1 = \lambda_{s_0} \quad \text{if } \mu \ne s_0,$$
$$\lambda_\mu \to \lambda_{s_0} \quad \text{when} \quad \mu \to s_0.$$

If the decompositon (4.1) can be continued analytically from $\Re\mu > \mu_0$ to s_0, the right hand side of

$$\sum_{n=0}^{\infty} \mathcal{L}_\mu^n = \frac{\mathcal{P}_\mu}{1-\lambda_\mu} + \sum_{n=1}^{\infty} \mathcal{Q}_\mu^n$$

also can be continued analytically near to s_0. Since the spectral radius of $\mathcal{Q}_\mu$ is less than 1, $\sum_{n=1}^{\infty} \mathcal{Q}_\mu^n$ is bounded for all μ near s_0.

Let us choose the function $g(x)$ in (3.1) in such a way that the corresponding h_0 satisfies

$$\mathcal{P}_{s_0} h_0 \neq 0.$$

Then, even though the proof is very complicated and long, we get

$$|| \left(\mathcal{R}_{s_0}^{(x)} \mathcal{P}_{s_0} h_0\right) (\xi^{(j)})||_{L^2(\Omega)} \geq c_1 > 0. \tag{4.2}$$

Concerning the detailed proof of this estimate, we are preparing a paper([6]). On the other hand we have

$$|| \left(\tilde{\mathcal{R}}_{s_0}^{(x)} \mathcal{P}_{s_0} h_0\right) (\xi^{(j)})||_{L^2(\Omega)} \leq c_2. \tag{4.3}$$

Moreover, between the constants c_1 and c_2, we have an estimate

$$c_2 \leq C_0 c_1, \tag{4.4}$$

where C_0 can be chosen as a uniform constant in $\mu_0 - \varepsilon_0 \leq \Re\mu < \mu_0$.

Then, for μ close to s_0 we have

$$||u(\,\cdot\,, z)||_{L^2(\Omega)} \geq \frac{c_1}{|1-\lambda_\mu|} - C_1. \tag{4.5}$$

On the other hand, we have

$$||f(\,\cdot\,, z) + \sum_{j\in I} e^{\varphi_j(\,\cdot\,)} \Delta v_j(\,\cdot\,)||_{L^2(\Omega)} \leq \frac{c_2}{|1-\lambda_\mu|} + C_2. \tag{4.6}$$

Recall that the problem (3.1) has an estimate (3.2). Then, the estimates (4.5) and (4.6) give an inequality

$$\frac{c_1}{|1-\lambda_\mu|} - C_1 \leq \frac{1}{2(\mu_0 - \varepsilon_0)|\Im s_0|} \left(\frac{c_2}{|1-\lambda_\mu|} + C_2\right).$$

μ tending to s_0, we have from the above estimate

$$c_1 \leq \frac{1}{2(\mu_0 - \varepsilon_0)|\Im s_0|} c_2.$$

The above inequality and the relation (4.4) give us

$$|\Im s_0| \leq \frac{C_0}{2(\mu_0 - \varepsilon_0)}.$$

This estimates shows that, in a slab domain $\mu_0 - \varepsilon_0 < \mu \leq \mu_0$, the imaginary part of a pole of $\zeta(\mu)$ is bounded. This is what we want to show.

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