Comm. Korean Math. Soc. **15** (2000), No. 4, pp. 697–706

BAD PAIRS OF POLYNOMIAL ZEROS

SEON-HONG KIM

ABSTRACT. If an arithmetic progression F of length $2n$ and the number k with $2k \leq n$ are given, can we find two monic polynomials with the same degrees whose set of all zeros form F such that both the number of bad pairs and the number of nonreal zeros are $2k$? We will consider the case that both the number of bad pairs and the number of nonreal zeros are two. Moreover, we will see the fundamental relation between the number of bad pairs and the number of nonreal zeros, and we will show that the polynomial in x where the coefficient of x^k is the number of sequences having $2k$ bad pairs has all zeros real and negative.

1. Introduction

Throughout this paper, $n \geq 2$ is an integer. Let $P_A(x)$ and $P_B(x)$ be monic polynomials with degrees n having all zeros distinct and real. We begin with the definition of bad pairs.

DEFINITION 1.1. If U is a finite multiset of complex numbers, write

$$P_U(x) = \prod_{\alpha \in U} (x - \alpha).$$

If U and V are sets of real numbers, with no repeated elements, and moreover

$$|U| = |V| = n, \qquad U \cap V = \phi,$$

we may write

$$T := U \cup V = \{t_1, t_2, \cdots, t_{2n}\}$$

Received April 10, 2000. Revised June 27, 2000.
2000 Mathematics Subject Classification: Primary 30C15; Secondary 11B25.
Key words and phrases: bad pairs, good pairs, zeros, polynomials.

with $t_i < t_{i+1}$ for all i. Define

$$T_1 = \{\{t_1, t_2\}, \{t_3, t_4\}, \cdots, \{t_{2n-1}, t_{2n}\}\}.$$

We say that a U-bad pair for T or for (P_U, P_V) is a pair of T_1 such that both elements belong to U; let $N_U(U, V)$ denote the number of U-bad pairs. The number of bad pairs is defined by

$$N_U(U, V) + N_V(U, V).$$

Also a pair that is not bad is called a good pair.

How many real zeros does $P_A(x) + P_B(x)$ have? It is an easy consequence of Fell[1] that, if the all zeros of $P_A(x)$ and $P_B(x)$ form good pairs, $P_A(x)+P_B(x)$ has all its zeros real. But there has been no neat result for the case having bad pairs so far. We can check, by computational search, that an arithmetic progression $F = \{1, 2, \cdots, 2n\}$ of length ≤ 10 satisfies the following condition.

CONDITION 1. *Whenever $F = A \cup B$, $A \cap B = \emptyset$, $|A| = |B| = n$, the number of bad pairs of F is equal to the number of nonreal zeros of*

$$\prod_{a_i \in A} (x - a_i) + \prod_{b_j \in B} (x - b_j).$$

So one might ask: is there a set $F = \{r_1, r_2, \ldots, r_{2n}\}$ of real numbers in increasing order such that Condition 1 holds, or is an arithmetic progression such a set? But, an arithmetic progression F does not satisfy Condition 1. For example, the polynomial

$$(x-1)(x-2)(x-5)(x-6)(x-8)(x-9) + (x-3)(x-4)(x-7)(x-10)(x-11)(x-12)$$

has four bad pairs but two nonreal zeros, and, moreover, there are ten polynomials for the arithmetic progression of length 12 that do not satisfy Condition 1. In fact, most sets do not seem to satisfy Condition 1. So it would be natural to ask a weaker question: if an arithmetic progression of length $2n$ and the number of bad pairs are given, can we find two monic polynomials $P_A(x)$ and $P_B(x)$ with the same degrees whose set of all zeros form an arithmetic progression such that the number of bad pairs is equal to the number of nonreal zeros? On the other hand, of the $\binom{2n}{n}$ sequences with n zeros of $P_A(x)$ and n zeros of $P_B(x)$, there are

$$2^{n-2k}\binom{n}{k}\binom{n-k}{k}$$

sequences having $2k$ bad pairs. In Section 2, we will see the fundamental relation between the number of bad pairs and the number of nonreal zeros, and consider an arithmetic progression of length $4n+4$

$$\{-4n-3, -4n-1, \cdots, -1, 1, \cdots, 4n+1, 4n+3\}$$

and the polynomial

$$f_n(x) := (x+1)\prod_{k=0}^{n}(x+(4k+3))\prod_{k=1}^{n}(x-(4k+1)) + (x-1)\prod_{k=0}^{n}(x-(4k+3))\prod_{k=1}^{n}(x+(4k+1)).$$

In Section 3, we will show that the polynomial in x where the coefficient of x^k is the number of sequences having $2k$ bad pairs has all zeros real and negative.

2. The relation between the number of bad pairs and the number of nonreal zeros

We begin with a definition.

DEFINITION 2.1. Let A and B be different sets consisting of real numbers such that $A \cap B = \emptyset$ and $|A| = |B| = n$. Suppose that

$$P(x) = \prod_{a_i \in A}(x-a_i) + \prod_{b_j \in B}(x-b_j)$$

is the polynomial such that $A \cup B$ has $2k$ bad pairs and the number of real zeros of $P(x)$ is r. Then we say that $P(x)$ satisfies the triple (n, k, r).

Of the $\binom{2n}{n}$ sequences with n a's and n b's, where $a \in A$, $b \in B$, how many sequences have $2k$ bad pairs? Here is the answer.

PROPOSITION 2.2. *Of the* $\binom{2n}{n}$ *sequences with* n *a's and* n *b's, where* $a \in A$, $b \in B$, *the number of sequences having* $2k$ *bad pairs is*

$$2^{n-2k}\binom{n}{k}\binom{n-k}{k},$$

and

$$\binom{2n}{n} = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k}\binom{n}{k}\binom{n-k}{k}. \tag{2.1}$$

PROOF. For a sequence consisting of $2k$ bad pairs, $N_A(A,\ B) = N_B(A,\ B) = k$ and the sequence has $n - 2k$ good pairs. Hence, with n pairs, the number of different permutations is

$$\frac{n!}{(k!)^2(n-2k)!} = \binom{2k}{k}\binom{n}{2k} = \binom{n}{k}\binom{n-k}{k}.$$

Moreover, a good pair produces another good pair by switching the numbers. Hence the number of sequences having $2k$ bad pairs is

$$2^{n-2k}\binom{n}{k}\binom{n-k}{k}.$$

Since the number of bad pairs ranges from 0 to $\lfloor n/2 \rfloor$, we have (2.1). This completes the proof. □

PROPOSITION 2.3. *With the notations of* Definition 2.1, *suppose that* $P(x)$ *satisfies* (n, k, r). *Then we have*

$$n \leq 2k + r. \tag{2.2}$$

PROOF. If there is no good pair, then $n = 2k$ which proves the result. Without loss of generality, we assume that

$$\alpha < a_t < b_t < \beta,$$

where (a_t, b_t) is a good pair and there is no element of $A \cup B$ in open intervals (α, a_t) and (b_t, β). If $\gamma < c_t < d_t < \eta$, where (c_t, d_t) is a bad pair with $b_t < c_t$ and there is no element of $A \cup B$ in open intervals (γ, c_t) and (d_t, η), then $P_A(c)P_A(d) > 0$ and $P_B(c)P_B(d) > 0$ for any $c \in (\gamma, c_t)$, $d \in (d_t, \eta)$. So, in order to determine the sign of $P_A(b)$ and $P_B(b)$ for any $b \in (b_t, \beta)$, it is enough to consider only good pairs. Since P_A and P_B have positive leading coefficient and each good pair has one element of A and B, respectively, we can see that $P_A(a)P_B(a) > 0$, $P_A(b)P_B(b) > 0$ and $P_A(a)P_A(b) < 0$ for any $a \in (\alpha, a_t)$ and $b \in (b_t, \beta)$. Hence each good pair has at least one real zero. This proves the proposition. □

The following proposition is the best possible case of (2.2), i.e.

$$n - 2k = r,$$

i.e. the number of good pairs is equal to the number of real zeros. From the proof of Proposition 2.3, we have

PROPOSITION 2.4. *With the notations of* Definition 2.1, $P(x)$ *satisfies* $(n, 0, n)$.

Now we consider an arithmetic progression of length $4n + 4$

$$\{-4n - 3, -4n - 1, \cdots, -3, -1, 1, 3, \cdots, 4n + 1, 4n + 3\}. \tag{2.3}$$

One may ask whether there is a sum of two polynomials whose zeros are the elements of (2.3) satisfying $(2n + 2, 1, 2n)$. To obtain an affirmative answer, we use Descartes' rule of signs.

THEOREM 2.5. *Define, for an integer* $n \geq 1$,

$$\begin{aligned} f_n(x) = (x + 1) \prod_{k=0}^{n} (x + (4k + 3)) \prod_{k=1}^{n} (x - (4k + 1)) \\ + (x - 1) \prod_{k=0}^{n} (x - (4k + 3)) \prod_{k=1}^{n} (x + (4k + 1)) . \end{aligned}$$

Then $f_n(x)$ *has two bad pairs,* $2n$ *real zeros, and two nonreal zeros.*

PROOF. Let n be a positive integer. First we show that $f_n(x)$ has two bad pairs. Define

$$u_n(x) = (x+1)\prod_{k=0}^{n}(x+(4k+3))\prod_{k=1}^{n}(x-(4k+1)).$$

Then $u_n(x)$ and $u_n(-x)$ have degree $2n+2$ and $f_n(x) = u_n(x)+u_n(-x)$. Moreover, the union of zeros of $u_n(x)$ and $u_n(-x)$ is the arithmetic progression (2.3). This has cardinality $4n+4$. Among its elements there are exactly two bad pairs;

$$\{-1,-3\} \quad \text{and} \quad \{1,3\}.$$

Now we determine the sign change in $f_n(x)$ to show $f_n(x)$ has at least $2n$ real zeros. Note that both $u_n(x)$ and $u_n(-x)$ have the same sign in $(-\infty,-4n-3)\cup\bigcup_{k=1}^{n-1}(-4k-1,-4k+1)$ and $\bigcup_{k=1}^{n-1}(4k-1,4k+1)\cup(4n+3,\infty)$ with different signs alternatively. So we see that there is a real zero in each interval of

$$\bigcup_{k=1}^{n}(-4k-3,-4k-1)\cup\bigcup_{k=1}^{n}(4k+1,4k+3)$$

which means the existence of at least $2n$ real zeros. Since $f_n(x) = f_n(-x)$, $f_n(x)$ has at least n positive zeros and n negative zeros. Next we show that there are nonreal zeros of $f_n(x)$. Observe that

$$\begin{aligned} f_n(x) = (x^2+4x+3)&\prod_{k=1}^{n}\left(x^2+2x-(4k+1)(4k+3)\right) \\ &+ (x^2-4x+3)\prod_{k=1}^{n}\left(x^2-2x-(4k+1)(4k+3)\right) \end{aligned}$$

has only terms with even exponents of x. So regarding x^2 as a new variable gives a new polynomial, say F, of degree $n+1$. By above, we know that the F has at least n positive zeros. If F has all its zeros positive, then, by Descartes' rule of signs, the number of sign changes in its coefficients is $\geq n+1$, i.e. $n+1$. So it suffices to show that the signs of the constant term and the coefficients of x^2 of $f_n(x)$ are the same, since this implies that there are at most n sign changes in its coefficients. Now $f_n(x)$ has the constant term,

$$2 \cdot 1 \cdot 3 \cdot (-1)^n \prod_{k=1}^{n} (4k+1)(4k+3),$$

and the coefficient of x^2 is

$$2\left((-1)^n \prod_{k=1}^{n} (4k+1)(4k+3) + 8(-1)^{n-1} \sum_{i=1}^{n} \prod_{\substack{k=1 \\ k\neq i}}^{n} (4k+1)(4k+3) + \right.$$

$$\left. 3\left((-1)^{n-1} \sum_{i=1}^{n} \prod_{\substack{k=1 \\ k\neq i}}^{n} (4k+1)(4k+3) + 4(-1)^{n-2} \sum_{\substack{i,j=1 \\ i<j}}^{n} \prod_{\substack{k=1 \\ k\neq i,j}}^{n} (4k+1)(4k+3) \right) \right)$$

$$=2\left((-1)^n \prod_{k=1}^{n} (4k+1)(4k+3) + 11(-1)^{n-1} \sum_{i=1}^{n} \prod_{\substack{k=1 \\ k\neq i}}^{n} (4k+1)(4k+3) + \right.$$

$$\left. 12(-1)^{n-2} \sum_{\substack{i,j=1 \\ i<j}}^{n} \prod_{\substack{k=1 \\ k\neq i,j}}^{n} (4k+1)(4k+3) \right).$$

Then the constant term and the coefficient of x^2 have the same sign. In fact, in the coefficient of x^2,

$$\prod_{k=1}^{n} (4k+1)(4k+3) > 11 \sum_{i=1}^{n} \prod_{\substack{k=1 \\ k\neq i}}^{n} (4k+1)(4k+3),$$

since

$$11 \sum_{i=1}^{\infty} \frac{1}{(4i+1)(4i+3)} = \frac{11}{2}\left(\frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right) = \frac{11}{2}\left(\frac{\pi}{4} - 1 + \frac{1}{3}\right)$$
$$= 0.65302\cdots < 1,$$

and so $1 > 11 \sum_{i=1}^{n} \frac{1}{(4i+1)(4i+3)}$ for all n. This proves the result. □

3. Polynomial in x where the coefficient of x^k is the number of sequences having $2k$ bad pairs

In this section, we consider the polynomial in x where the coefficient of x^k is the number of sequences having $2k$ bad pairs. We begin with two known results on the Hadamard product.

DEFINITION 3.1. The Hadamard product of two series is

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \otimes \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} a_n b_n x^n.$$

In 1895, Maló[3] proved the following theorem.

THEOREM 3.2 **(Maló)**. *Let $p(x)$ and $q(x)$ be polynomials with real zeros only, and with all zeros of $p(x)$ having the same sign. Then $p(x) \otimes q(x)$ has only real zeros.*

For the proof of the following theorem, see [2].

THEOREM 3.3. *Let $p(x)$ and $q(x)$ be real polynomials. If every zero of $p(x)$ and $q(x)$ is in the closed left half of the complex plane, then so is every zero of $p(x) \otimes q(x)$.*

We now have the following result.

THEOREM 3.4. *If n is an integer ≥ 1, then all zeros of*

$$K_n(x) := \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k} \binom{n}{k} \binom{n-k}{k} x^k$$

are real and negative.

PROOF. Observe that

$$K_n(x) = 2^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k} \left(\frac{x}{2^2}\right)^k.$$

So it suffices to show this for

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k} x^k.$$

The above is simply the Hadamard product of

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^k \tag{3.1}$$

with

$$\sum_{k=0}^{n} \binom{n}{k} x^k. \tag{3.2}$$

Now (3.2) is simply $(x+1)^n$, i.e. (3.2) has only negative zeros. On the other hand (3.1) is equal to

$$(-x)^{\lfloor n/2 \rfloor} U_n \left(\frac{1}{2\sqrt{-x}} \right),$$

where U_n is the nth Chebyshev polynomial of the second kind, since

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}$$

and

$$\begin{aligned} U_n \left(\frac{1}{2\sqrt{-x}} \right) &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (-x)^{-1/2(n-2k)} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{2k-\lfloor n/2 \rfloor} \binom{n-k}{k} x^{k-\lfloor n/2 \rfloor} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{k-\lfloor n/2 \rfloor}. \end{aligned}$$

Since all zeros of U_n are real, (3.1) has only negative zeros. Hence, by Theorem 3.2 and Theorem 3.3, all zeros of $K_n(x)$ are real and negative.□

ACKNOWLEDGMENT. The author wishes to thank to Professor Kenneth B. Stolarsky for help and encouragement.

References

[1] H. J. Fell, *On the zeros of convex combinations of polynomials*, Pacific J. Math. **89** (1980), 43–50.

[2] J. Garloff, D. G. Wagner, *Hadamard products of stable polynomials are stable*, J. Math. Anal. Appl. **202** (1996), no. 3, 797–809.

[3] E. Maló, *Note sur les équations algébriques dont toutes les racines sont réeles*, J. Math. Spéc. **4** (1895), no. 4, 7–10.

Department of Mathematics
Seoul National University
Seoul, 151-742, Korea
E-mail: shkim@math.snu.ac.kr