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FIXED POINTS THEORY ON CLOSED 2-DIMENSIONAL MANIFOLDS

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ABSTRACT. Let $f : M \to M$ be a homotopically periodic self-map of a closed surface M. Except for $M = S^2$, the Nielsen number $N(f)$ and the Lefschetz number $L(f)$ of the self-map f are the same. This is a generalization of Kwasik and Lee's result to 2-dimensional case. On the 2-sphere S^2, $N(f) = 1$ and $L(f) = deg(f) + 1$ for any self-map $f : S^2 \to S^2$.

1. Introduction

Let M be a closed smooth manifold and let $f : M \to M$ be a continuous self-map. There are two well-known invariants in the fixed point theory; the $Lefschetz\ number L(f)$ and the $Nielsen\ number N(f)$.

The Nielsen number $N(f)$ of a self-map f gives a lower bound for the number of fixed points of f. Unfortunately the Nielsen number is difficult to calculate. On the other hand, the Lefschetz number $L(f)$ is readily computable. In [2], Brooks, Brown, Pak and Taylor show that for a self-map $f : M \to M$ on a torus, $N(f) = |L(f)|$. This result is extended to compact nilmanifolds ([1] and [5]) and solvmanifolds ([10]).

But Anosov shows that there exists a map $f : K \to K$ of a Klein bottle K such that $N(f) \neq |L(f)|$. Kwasik and Lee restrict the class of maps to homotopically periodic maps when extend the above result to infra-nilmanifolds [7]:

THEOREM 1.1. *Let M be an infra-nilmanifold and let $f : M \to M$ be a homotopically periodic self-map. Then $N(f) = L(f)$.*

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A continuous self-map $f : M \to M$ is called *homotopically periodic* if there exists an integer $k \geq 1$ such that f^k is homotopic to the identity.

The purpose of this paper is to study the fixed point set of self-map on closed 2-dimensional manifolds. We will show that the above result of Kwasik and Lee is true on a surface of nonpositive Euler characteristic.

2. Results

Each closed surface admits a metric of constant curvature. There are three cases depending on the Euler characteristic of the surface. If a 2-dimensional manifold X is simply connected then X is isometric to one of the Euclidean plane E^2, the unit sphere S^2 or the hyperbolic plane $\mathbb{H}^2$. If a group G acts properly discontinuously and freely on X as isometries of X, then the quotient space X/G is a Riemannian manifold and the natural map $X \to X/G$ is a covering map with covering group G. Conversely, a closed surface of constant curvature can be obtained as a quotient of X by a discrete subgroup π of isometries which act freely.

It is known that the isometry group of S^2 is $O(3)$, and $\mathbb{Z}_2$ is the only nontrivial group which can act freely on S^2. This implies that there are only two closed surfaces with positive Euler characteristic, namely the 2-sphere S^2 and the real projective plane $\mathbb{R}P^2$. If a closed surface has zero Euler characteristic, then it is the torus or Klein bottle. All the other closed surfaces have negative Euler characteristic and admit metrics of constant negative curvature, equal to -1.

The following is a basic notion in fixed point theory;

DEFINITION 2.1. Two fixed points x and y of M are said to be $f-$ *equivalent* if there is a path $\omega : [0,1] \to M$ joining x and y such that ω and $f \circ \omega$ are homotopic relative to the endpoints.

The fixed point set $\text{fix}(f) = \{x \in X | f(x) = x\}$ can be partitioned by the f-equivalence. Points in a path-component of $\text{fix}(f)$ are f-equivalent. Each fixed point class $\mathbf{F}$ is then an isolated fixed point set, with a fixed point index $\text{ind}(f, \mathbf{F})$ defined.

A fixed point class is *essential* if its fixed point index is non-zero and $N(f)$ is the number of essential fixed point classes. We begin with the following lemmas.

LEMMA 2.2. *Let $f : M \to M$ be an isometry on a closed surface M of nonpositive Euler characteristic. Suppose that f has only finite isolated fixed points. If x and y are distinct points of* $\mathrm{fix}(f)$, *then they are not f-equivalent.*

PROOF. Suppose that x and y are f-equivalent. Then there exists a path $\omega : [0,1] \to M$ from x to y which is homotopic to $f \circ \omega$ relative to endpoints. We may assume that ω is a geodesic. Consider the universal covering map $p : X \to M$. If $\tilde{x}$ lies over x, lift ω to a path $\tilde{\omega} : [0,1] \to X$ starting at $\tilde{x}$. Let $\tilde{y}$ be the end point of $\tilde{\omega}$. ω_1 will denote the reverse path of ω. Then $f \circ \omega_1$ is a path from y to x. There is the lift $\widetilde{f \circ \omega_1} : [0,1] \to X$ of $f \circ \omega_1$ which begins at $\tilde{y}$. If the end point of $\widetilde{f \circ \omega_1}$ is $\tilde{x}$, then $\tilde{\omega}$ and $\widetilde{f \circ \omega_1}$ must overlap, for there is only one geodesic joining $\tilde{x}$ and $\tilde{y}$ [4]. Therefore $\widetilde{f \circ \omega_1}(1) \neq \tilde{x}$. This implies the loop $\omega \cdot (f \circ \omega_1)$ is nontrivial, which is a contradiction. □

LEMMA 2.3. *Let M be a closed manifold of dimension ≥ 2 and nonpositive Euler characteristic and let f be an isometry on M. If f has only finite isolated fixed points then each isolated fixed point is essential.*

PROOF. Suppose x is an isolated fixed point which is inessential. If $\mathbf{A}$ is the Jacobian matrix of f at x, then $det(\mathbf{I} - \mathbf{A}) = 0$ ([6]), where $\mathbf{I}$ is the identity matrix. Then $\mathbf{A}$ has eigenvalue 1. Hence f must fix at least 1-dimensional subset. But every fixed point is isolated. □

We investigate whether the equality in the result of Kwasik and Lee does hold in the case of 2-dimensional manifold. First we will show that we can restrict our objects to isometries.

THEOREM 2.4. *Let $f : M \to M$ be a homotopically periodic self-map of a hyperbolic surface M. Then there exists a hyperbolic surface M' which is diffeomorphic to M (via $g : M \to M'$), and an isometry $f' : M' \to M'$ such that $g \circ f = f' \circ g$.*

PROOF. We see that $M = \mathbb{H}^2/\pi$ where π is isomorphic to a subgroup of $PSL_2\mathbb{R}$. Let $\varepsilon(M)$ be the H-space of all self-homotopy equivalences of M. The homotopical periodicity of f implies that $f \in \varepsilon(M)$. The image of f under the natural map

$$\varepsilon(M) \to \pi_0(\varepsilon(M)),$$

where $\pi_0(\varepsilon(M))$ is the group of homotopy classes of self-homotopy equivalences, generates a subgroup $\mathbb{Z}_k = \langle [f] \rangle$ of $\pi_0(\varepsilon(M))$. Since M is a $K(\pi, 1)$, $\mathbb{Z}_k$ induces an abstract kernel

$$\phi : \mathbb{Z}_k \to Out(\pi) \cong \pi_0(\varepsilon(M)).$$

In order to realize $\mathbb{Z}_k$ as a group action on M, it is necessary for the abstract kernel to have an extension

$$1 \to \pi \to E \to \mathbb{Z}_k \to 0$$

([8]). Since π is centerless, the abstract kernel $\phi : \mathbb{Z}_k \to Out(\pi)$ has such an extension. It is well-known that such a group E can be embedded into $PSL_2\mathbb{R}$ as a discrete subgroup, say $\varphi : E \to PSL_2\mathbb{R}$. Moreover, there exists a diffeomorphism $\widetilde{g} : \mathbb{H}^2 \to \mathbb{H}^2$ which conjugates π into $\varphi(\pi)$.

Then clearly, $M' = \mathbb{H}^2/\varphi(\pi)$ is a hyperbolic surface, and $\widetilde{g}$ induces a diffeomorphism $g : M \to M'$. Let $\widetilde{f} \in E$ be a preimage of the generator $[f]$ of $\mathbb{Z}_k$ and let $f' : M' \to M'$ be the map on M' induced by $\varphi(\widetilde{f})$. Then f' is an isometry of M', and $g \circ f = f' \circ g$ holds. This completes the proof. □

We are ready to plunge into our main theorem.

THEOREM 2.5. *If a self-map $f : M \to M$ on a closed 2-dimensional manifold M of nonpositive Euler characteristic is homotopically periodic, then $N(f) = L(f)$.*

PROOF. Every 2-dimensional flat manifold is an infra-nilmanifold. So the equality comes from Theorem 1.1. We shall work with a self-map f on a closed hyperbolic surface M. By Theorem 2.4, we may think of f as an isometry. Suppose that f is an orientation preserving isometry. The fixed point set has codimension 2 and hence it consists of isolated points. In the proof of the preceding theorem, we saw $\mathbb{Z}_k = \langle [f] \rangle$ acts on M. Let $M^{\mathbb{Z}_k}$ be the fixed point set of that action. Using the following fact [3];

$$L(f) = \chi(M^{\mathbb{Z}_k}),$$

where χ is the Euler characteristic, we can figure out the Lefschetz number easily. By Lemma 2.2, any pair of the fixed point set are not f-equivalent. Hence

$$L(f) = \text{the number of isolated fixed points of } f.$$

Since each isolated fixed point is essential from Lemma 2.3,

$$N(f) = \text{the number of isolated fixed points of } f.$$

Therefore, it follows that the equality $N(f) = L(f)$ holds.

Now consider the case of an orientation reversing isometry f. The fixed point set of f is the union of isolated fixed points and mutually disjoint 1-dimensional submanifolds. In order to remove 1-dimensional submanifolds which is fixed by f, we make use of homotoping f to a map g such that $\mathrm{Fix}(g)$ is exactly the isolated fixed points of f. The result then follows if we proceed as the case of orientation preserving map.

Let $C_1, \ldots, C_m$ be the components of $\mathrm{Fix}(f)$ with dim $C_i = 1$ and N_1 be the total space of a small normal segment bundle around C_1, i.e., a 1-dimensional tubular neighborhood of C_1. Then $f|_{N_1} : N_1 \to N_1$ and $f|_{\partial N_1}$ is fixed point free. We will alter the map f only inside N_1. Since N_1 is strong deformation retracts onto C_1, there is a map f_1, which needs not be an isometry, with the following properties:

1. $f = f_1$ on $M - N_1$;
2. $f \simeq f_1$ on N_1 relative to ∂N_1; and
3. $f|_{N_1}$ has no fixed points.

This is possible because C_1 is a circle which has Euler characteristic 0. Repeating the procedure finitely many times, the isometry f can be homotoped to a self-map g such that $\mathrm{Fix}(g)$ consists of isolated fixed points of f. □

The following shows that the restriction "nonpositive Euler characteristic" in the above theorem is essential.

EXAMPLE 2.6. Consider an isometry $f : S^2 \to S^2$ which is a rotation through π about the z-axis. There are 2 fixed points, i.e., the north and south poles. They are f-equivalent and the north pole is essential. So $N(f) = 1$. Since f_* is the identity on both $H_0(S^2;\mathbb{Q})$ and $H_2(S^2;\mathbb{Q})$, $L(f) = 1 - 0 + 1 = 2$. Hence $N(f) \neq L(f)$.

We investigate the relation between the Nielsen number $N(f)$ and the Lefschetz number $L(f)$ of a self-map $f : S^2 \to S^2$.

LEMMA 2.7. *For a self-map $f : S^2 \to S^2$ which has degree d, $N(f) = 1$ and $L(f) = d + 1$. Therefore if $d \neq 0$, then $N(f) \neq L(f)$.*

PROOF. Let $g_d : S^1 \to S^1$ is the complex map $z \to z^d$ by which the complex unit circle covers itself d times. Then f is homotopic to the suspension $S(g_d) : S^2 \to S^2$. We view $S^2 \subset \mathbb{C} \times \mathbb{R}$ so that the points of S^2 are of the form (z, t), where $z \in \mathbb{C}, t \in \mathbb{R}$, and $\|z\|^2 + |t|^2 = 1$. $S(g_d)$ is defined by

$$S(g_d)(z,t) = \begin{cases} (z,t) & \text{if } z = 0 \\ (\|z\| \cdot g_d(z/\|z\|), t) & \text{if } z \neq 0. \end{cases}$$

Since the induced homomorphism f_* is the identity on $H_0(S^2; \mathbb{Q})$ and (d) on $H_2(S^2; \mathbb{Q})$, $L(f) = 1 - 0 + d = d + 1$. By the essential of the fixed point class, $N(f) = 1$. We obtain the conclusion. □

The following shows that the equality $N(f) = L(f)$ for a homotopically periodic self-map $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ on the real projective plane $\mathbb{R}P^2$.

LEMMA 2.8. *For a homotopically periodic self-map $f : \mathbb{R}P^2 \to \mathbb{R}P^2$, $N(f) = L(f) = 1$.*

PROOF. Suppose $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ is homotopically periodic. So a lift $\tilde{f} : S^2 \to S^2$ of f is also homotopically periodic, and the degree of $\tilde{f}$ is ± 1. Since the other lift of f is $\tilde{f} \circ A$, where A is the antipodal map on S^2, we may assume that the degree of $\tilde{f}$ is 1 without loss of generality. Then $\tilde{f}$ is homotopic to the rotation R through $\pi/2$ about the z-axis. Obviously $\mathrm{Fix}(R)$ consists of the north and the south poles. On the other hand, the lift $\tilde{f} \circ A$ of f is homotopic to $R \circ A$ and $\mathrm{Fix}(R \circ A) = \emptyset$. The self map on $\mathbb{R}P^2$ of which lifts are R and $R \circ A$ is homotopic to f and has a single fixed point which is essential. Hence $N(f) = L(f) = 1$. □

We close this paper by saying the conclusion of this paper.

COROLLARY 2.9. *Let M be a closed surface and $f : M \to M$ a self-map. Exclude the case $id : S^2 \to S^2$ (i.e., $M = S^2$ and $f = id$). If f is homotopically periodic, then $N(f) = L(f)$.*

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