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STABILITY ON SOLUTION OF POPULATION EVOLUTION EQUATIONS WITH APPLICATIONS

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Abstract. We prove that the non-homogeneous boundary value problem for population evolution equations is well-posed in Sobolev space $H^{\frac{3}{2},\frac{3}{2}}(\Omega)$. It provides a strictly mathematical basis for further research of population control problems.

0. Introduction

In this paper, we study the non-stationary population evolution equation

$$(0.1)\quad \begin{aligned} &\tfrac{\partial P}{\partial r} + \tfrac{\partial P}{\partial t} + B(r,t)P = A(r,t)P + G(r,t) && \text{in} \quad \Omega \times (0,T), \\ &P(r,0) = P_0(r) && \text{in} \quad \Omega, \\ &P^T(0,T) = V(t) = (v_1(t), \cdots, v_n(t)) && \text{in} \quad (0,T), \\ &v_i(t) = \beta_i(t) \textstyle\int_{r_{i_1}}^{r_{i_2}} h_i(r,t)k_i(r,t)p_i(r,t)\,dr && \text{in} \quad (0,T). \end{aligned}$$

Olsder and Strijbos [6] have proved that if $B(r,t) \equiv B(r)$ and $G(r,t) \equiv 0$, then (0.1) has a unique solution. Also, when $B(r,t) \equiv B(r)$ and $G(r,t) \equiv 0$, the explicit analytic expression of solution for (0.1) was given by Gao and Chen [2].

In this paper, we shall use the theories of generalized differential equation and the methods of the functional analysis to give the explicit analytic

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expression of the solution for (0.1) and to show the stability of the solution of control system (0.1).

1. Problem for population evolution equations

The mathematical continuous models of population control systems may be obtained from [2], which has raised the large-scale systems of non-stationary population control. The mathematical continuous model is the following non-homogeneous boundary value problem for the system of integro-differential equations,

$$\begin{aligned}
&\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} + B(r,t)P = A(r,t)P + G(r,t) && \text{in} \quad \Omega \times (0,T), \\
&P(r,0) = P_0(r) && \text{in} \quad \Omega, \\
(1.1) \qquad &P^T(0,t) = V(t) = (v_1(t), \cdots, v_n(t)) && \text{in} \quad (0,T),
\end{aligned}$$

$$v_i(t) = \beta_i(t) = \int_{r_{i_1}}^{r_{i_2}} h_i(r,t)k_i(r,t)p_i(r,t)\,dr \quad \text{in} \quad (0,T),$$

where

$$P(r,t) = \begin{pmatrix} p_1(r,t) \\ \vdots \\ p_n(r,t) \end{pmatrix}, G(r,t) = \begin{pmatrix} g_1(r,t) \\ \vdots \\ g_n(r,t) \end{pmatrix}, P_0(r) = \begin{pmatrix} p_{1_0}(r,t) \\ \vdots \\ p_{n_0}(r,t) \end{pmatrix},$$

$$B(r,t) = \begin{pmatrix} \mu_1(r,t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n(r,t) \end{pmatrix}, \Omega = \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_n \end{pmatrix},$$

$$A(r,t) = \begin{pmatrix} a_{11}(r,t) & \cdots & a_{1n}(r,t) \\ \ddots & \vdots & \ddots \\ a_{n1}(r,t) & \cdots & a_{nn}(r,t) \end{pmatrix}.$$

Here $\Omega_i = (0, r_{i_m})$ and r_{i_m} is the highest age ever attained by individuals of the $i-$th population group, $p_i(r,t)$, $g_i(r,t)$, $v_i(t)$, $p_{i_0}(r)$, $\beta_i(t)$, $k_i(r,t)$, $h_i(r,t)$ denote respectively the age density functions, the disturbance functions, the absolute infant fertility rate, the initial age pattern, the specific fertility rate of females, the female sex ratio over the total population, the fertility pattern of females function at age r and time t in the $i-$th population group and the functions $h_i(r,t)$ subjected to the condition of normalization

$$\int_{r_{i_1}}^{r_{i_2}} h_i(r,t)dr = 1,$$

$[r_{i_1}, r_{i_2}]$ denotes the fecundity period of females of the i−th population group, $a_{ij}(r,t)$ denotes the migration rate functions for group j from group j to group i. And $\mu_i(r,t)$ denotes the mortality functions of the i−th population group at age r and time t, satisfies conditions

$$\int_0^r \mu_i(r,t)dr < +\infty, \quad \text{if} \ \ r < r_{i_m},$$

$$\int_0^r \mu_i(r,t)dr \to +\infty, \quad \text{if} \ \ r < r_{i_m}, \tag{1.2}$$

and we assume that

$$\mu_i(r,t) \in H^{\frac{3}{2},\frac{3}{2}}(Q_{i_0}), \tag{1.3}$$

where $Q_{i_0} = \Omega_{i_0} \times (0,T), \Omega_{i_0} = [0, r'_{i_m}]$ for all $r'_{i_m} < r_{i_m}$.

According to the view point of modern control theory, the system governed by the system of equation (1.1) is a closed distributed parameter system with positive feedback on the boundary.

In this paper, we suppose that $n = 2$ and $G(r,t) = 0$. Under this assumption, the model (1.1) just describes the non-stationary urban and rural population control system. For $i = 1, 2, p_i(r,t), \mu_i(r,t), h_i(r,t), k_i(r,t)$ denote the urban and rural population functions respectively. To be specific, $a_{11}(r,t)$ denotes the migration rate function for urban population group migrated out urban (to rural areas), $a_{12}(r,t)$ denotes the migration rate which is migrated from rural into urban areas for rural population, $a_{21}(r,t)$ denotes the migration rate which is migrated from urban into rural areas for urban population, $a_{22}(r,t)$ denotes migration rate that is migrated out rural (to urban areas) for rural population. Because urban and rural population system is a closed system, we have that

$$-a_{11}(r,t) = a_{21}(r,t) \ \ \text{and} \ \ a_{12} = -a_{22}(r,t). \tag{1.4}$$

But, for a society which has larger difference between urban and rural areas, the migration rate from rural to urban will be far larger than the migration rate from urban to rural, that is $a_{12}a_{21}$. Therefore, we may assume that $a_{21}(r,t) = 0$. Then, the system of (1.1) $(n = 2)$ which describes the non-stationary urban and rural population control system is equivalent to the system;

$$
(1.5)\qquad
\begin{aligned}
&\frac{\partial p_1}{\partial r}+\frac{\partial p_1}{\partial t}+\mu_1(r,t)p_1=a_{12}(r,t)p_2 && \text{in} && Q_1,\\
&p_1(r,0)=p_{10}(r) && \text{in} && \Omega_1,\\
&p_1(0,t)=v_1(t) && \text{in} && (0,T),\\
&v_1(t)=\beta_1(t)\int_{r_{11}}^{r_{12}} h_1(r,t)k_1(r,t)p_1(r,t)\,dr && \text{in} && (0,T),
\end{aligned}
$$

and

$$
(1.5)'\qquad
\begin{aligned}
&\frac{\partial p_2}{\partial r}+\frac{\partial p_2}{\partial t}+\mu_2(r,t)p_2=-a_{12}(r,t)p_2 && \text{in} && Q_2,\\
&p_2(r,0)=p_{20}(r) && \text{in} && \Omega_2,\\
&p_2(0,t)=v_2(t) && \text{in} && (0,T),\\
&v_2(t)=\beta_2(t)\int_{r_{21}}^{r_{22}} h_2(r,t)k_2(r,t)p_2(r,t)dr && \text{in} && (0,T).
\end{aligned}
$$

In general case, $r_{1m}\neq r_{2m}$. The survey data shows that $r_{1m}\le r_{2m}$, but from the system of (1.5), we can see that since $a_{12}(r,t)\ge 0$, $r_{1m}\doteq r_{2m}$ if $t>0$. Then we may assume $r_{1m}=r_{2m}$ with no loss of generality. Therefore, we set

$$\Omega=\Omega_1=\Omega_2=(0,r_{1m})=(0,r_{2m})$$

and

$$Q=Q_1=Q_2=\Omega\times(0,T).$$

The problems of this kind were first studied by Song and Yu [8]. They have considered problems (1.5) and (1.5)$'$ under the hypotheses that $u_i(r,t)=u_i(r), k_i(r,t)=k_i(r), h_i(r,t)=h_i(r)(i=1,2)$ are independent of time t, which is the stationary case. But, population functions, for example $\mu_i(r,t)$, are dependent on time t. Thus, non-stationary urban and rural population systems (1.5) and (1.5)$'$ can reflect the reality of urban and rural population control evolution process finely, thus it has an important practical significance to research for the non-stationary urban and rural population system.

2. Solutions of systems

We consider the regular generalized solutions of integro-partial differential equations (1.5) and (1.5)′ in Sobolev space $H^{\frac{3}{2},\frac{3}{2}}(Q)$. First, we give the following definition.

DEFINITION. we define

$$\beta_{cr}^{-}(T)=\Big[\max_{0\le t\le T}\int_{r_1}^{r_2}h(r,t)k(r,t)e^{-\int_0^r\mu(s,s+t-r)ds}dr\Big]^{-1}, \tag{2.1}$$

where $\beta_{cr}^{-}(T)$is called the lower critical value of specific fertility rate of females.

LEMMA 2.1. *Let $\mu(r,t)$ satisfy conditions (1.2) $\sim$ (1.3) with $v(t)\in H^1(0,T)$, $p_0(r)\in H^1(\Omega), h(r,t)\in H^{\frac{3}{2},\frac{3}{2}}(Q), k(r,t)\in H^{\frac{3}{2},\frac{3}{2}}(Q), f(r,t)\in H^{1,1}(Q)$. And let $\{p_0(r),v(t),f(r,t)\}$ satisfy compatibility relations. Then, if $\beta(t)=\beta<\beta_{cr}^{-}(T)$, there exists a unique solution $p(r,t)$ in* Sobolev *space $H^{\frac{3}{2},\frac{3}{2}}(Q)$ satisfying the system of integro-partial differential equations*

$$\begin{aligned}
&\frac{\partial p}{\partial r}+\frac{\partial p}{\partial t}+\mu(r,t)p=f(r,t) && in\quad Q,\\
&p(r,0)=p_0(r) && in\quad \Omega,\\
&p(0,t)=v(t) && in\quad (0,T),\\
&v(t)=\beta(t)\int_{r_1}^{r_2}h(r,t)k(r,t)p(r,t)dr && in\quad (0,T).
\end{aligned}\tag{2.2}$$

Furthermore $p(r,t)$ can be expressed explicitly by the expansion in series

$$\begin{aligned}
p(r,t)=e^{-\int_0^r u(s,s+t-r)ds}\cdot\Big[&p_0(r-t)+\sum_{n=0}^{\infty}y_n(t-r)\beta^{n+1}\\
&+\int_0^r f(x,x+t-r)e^{\int_0^x\mu(s,s+t-r)\,ds}\,dx\Big],
\end{aligned}\tag{2.3}$$

where

$$y_0(t) = \int_{r_1}^{r_2} h(r,t)k(r,t)e^{-\int_0^r \mu(s,s+t-r)\,ds} \cdot \Big[p_0(r-t) + \int_0^r f(x, x+t-r)e^{\int_0^x \mu(s,s+t-r)ds}dx\Big]\, dr, \tag{2.4}$$

$$y_n(t) = \int_{t-r_2}^{t-r_1} K(t,z)y_{n-1}(z)dz \quad (n = 1, 2, \cdots), \tag{2.5}$$

$$K(t,z) = h(t-z,t)k(t-z,t)e^{-\int_0^{t-z} \mu(s,s+z)ds}. \tag{2.6}$$

Proof. To slove the problem $(2,2)$, we will first solve the problem

$$\begin{aligned} &\frac{\partial p}{\partial r} + \frac{\partial p}{\partial t} + u(r,t)p = f(r,t) && \text{in} \quad Q, \\ &p(r,0) = 0 && \text{in} \quad \Omega, \\ &p(0,t) = v(t) && \text{in} \quad (0,T). \end{aligned} \tag{2.7}$$

By Chen and Gao [2], (2.7) has a unique solution in Sobolev space $H^{\frac{3}{2},\frac{3}{2}}(Q)$ and it can be expressed as
(2.8)

$$p_1(r,t) = e^{-\int_0^r \mu(s,s+t-r)ds}\Big[v(t-r) + \int_0^r f(\xi, \xi+t-r)e^{\int_0^x u(s,s+t-r)ds}d\xi\Big],$$

where $v(t)$ is given by (2.2).

By similar calculation, the unique solution of the problem

$$\begin{aligned} &\frac{\partial p}{\partial r} + \frac{\partial p}{\partial t} + u(r,t)p = 0 && \text{in} \quad Q, \\ &p(r,0) = p_0(r) && \text{in} \quad \Omega, \\ &p(0,t) = v(t) && \text{in} \quad (0,T), \end{aligned} \tag{2.9}$$

is found to be

$$\begin{aligned} p_2(r,t) &= p_0(r-t)e^{-\int_0^t \mu(s+t-r,s)ds} \\ &= p_0(r-t)e^{-\int_0^r \mu(s.s+r-t)ds}. \end{aligned} \tag{2.10}$$

The unique solution of problem (2.2) is clearly the sum of solutions of problems (2.7) and (2.9), that is,

$$p(r,t) = p_1(r,t) + p_2(r,t).$$

Furthermore, $p(r,t)$ must satisfy the following relation

$$(2.11)\quad p(r,t) = e^{-\int_0^r \mu(s,s+t-r)ds}\Big[\beta(t)\int_{r_1}^{r_2} h(\xi,t-r)k(\xi,t-r)p(\xi,t-r)d\xi + p_0(r-t) + \int_0^r f(\xi,\xi+t-r)e^{\int_0^\xi \mu(s,s+t-r)ds}d\xi\Big].$$

Multiply across by $h(r,t)\cdot k(r,t)$ to the both sides and integrate from r_1 to r_2, and make the change of variable in the first term on the right-hand side $t-r=\eta$. Then we have

$$(2.12)\quad \begin{aligned}\int_{r_1}^{r_2} h(r,t)k(r,t)p(r,t)dr = \beta &\int_{t-r_2}^{t-r_1}\Big[\int_{r_1}^{r_2} h(\xi,\eta)k(\xi,\eta)p(\xi,\eta)\,d\xi\Big] \\ &\cdot h(t-\eta,t)k(t-\eta,t)e^{-\int_0^{t-\eta}\mu(s,s+\eta)\,ds}\,d\eta \\ &+\int_{r_1}^{r_2} e^{-\int_0^r \mu(s,s+t-r)ds}\Big[p_0(r-t) \\ &+\int_0^r f(\xi,\xi+t-r) \\ &\cdot e^{\int_0^\xi \mu(s,s+t-r)\,ds}\,d\xi\Big]h(r,t)k(r,t)\,dr.\end{aligned}$$

Note that integrand of (2.12) is nonnegative, and by property of Lebesgue integral, we have that equation (2.12) is equivalent to equation (2.11).

If we let

$$(2.13)\qquad Y(t) \equiv \int_{r-1}^{r_2} h(r,t)k(r,t)p(r,t)dr,$$

$$(2.14)\qquad K(t,\eta) \equiv h(t-\eta,t)k(t-\eta,t)e^{-\int_0^{t-\eta}\mu(s,s+\eta)ds},$$

$$(2.15)\quad \begin{aligned}F(t) \equiv \int_{r_1}^{r_2} & e^{-\int_0^r \mu(s,s+t-r)\,ds}\Big[p_0(r-t) \\ &+\int_0^r f(\xi,\xi+t-r)e^{\int_0^\xi \mu(s,s+t-r)ds}\,d\xi\Big]\cdot h(r,t)k(r,t)\,dr,\end{aligned}$$

then (2.12) is equivalent to

$$(2.16)\qquad Y(t) = \beta\int_{t-r_2}^{t-r_1} K(t,\eta)Y(\eta)d\eta + F(\eta).$$

This is the Volterra integral equation.

We apply the Picard's iteration method to solve the series solution of integral equation (2.16)

$$Y(t) = \sum_{n=0}^{\infty} Y_n(t)\beta^n. \tag{2.17}$$

Substituting the equation (2.17) for $Y(t)$ in the integral equation (2.16) and integrating term by term yields

$$\begin{aligned} Y_0(t) &= F(t), \\ Y_n(t) &= \int_{t-r_2}^{t-r_1} K(t-\eta)Y_{n-1}(\eta)d\eta,\ n = 1, 2, \cdots . \end{aligned} \tag{2.18}$$

If series (2.17) uniformly converges to $Y(t)$, then it follows that $Y(t)$ is a solution of the integral equation (2.16) and it is unique.

Recall that

$$k(r,t) \leq 1,$$

$$e^{-\int_0^r \mu(s,s+t-r)ds} \leq 1,$$

$$e^{-\int_\xi^r \mu(s,s+t-r)ds} \leq 1.$$

Then by (2.15) and Hölder inequality, we obtain the estimation for $t \in (0,T)$,

$$\begin{aligned} Y_0(t) &= F(t) \\ &\leq \int_{r_1}^{r_2} h(r,t)p_0(r-t)dr + \int_{r_1}^{r_2}\int_0^r f(\xi,\xi+t-r)h(r,t)d\xi dr \\ &\leq \|h(r,t)\|_{L^2(\Omega)}\big(\|p_0(r)\|_{L^2(\Omega)} + \|f(r,t)\|_{L^2(\Omega)}\big) \\ &\leq c_1(\|p_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}), \end{aligned}$$

where constant c_1 is independent of t. Thus

$$Y_1(t) \leq c_1(\|p_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}) \int_{r_1}^{r_2} h(r,t)k(r,t)e^{-\int_0^r \mu(s,s+t-r)ds}dr \tag{2.19}$$

$$\leq c_1(\|p_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)})(\beta_{cr}^-)^{-1}, \quad \forall t \in (0,T). \tag{2.20}$$

Repeating this process we obtain a sequence of approximations $Y_0(t), Y_1(t), Y_2(t), \cdots, Y_n(t), \cdots$ such that

$$Y_n(t) \leq c_1(\|p_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)})(\beta_{cr}^-)^{-n}, \forall t \in (0,T).$$

Therefore,

$$\sum_{n=1}^{\infty} Y_n(t)\beta^n \leq c_1(\|p_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}) \cdot \sum_{n=1}^{\infty}(\frac{\beta}{\beta_{cr}^-})^{-n}, \quad \forall \in (0,T).$$

Clearly, series (2.17) uniformly converges to $Y(t)$ with respect to t when $\beta < \beta_{cr}^-$, hence $Y(t)$ is an unique solution of the integral equation (2.18).

Substituting (2.17) into (2.11), we obtain the solution of the system of equation (2.2) and it can be expressed by

$$\begin{aligned} p(r,t) = \; & e^{-\int_0^r \mu(s,s+t-r)ds}\Big[p_0(r-t) + \sum_{n=0}^{\infty} Y_n(t-r)\beta^{n+1} \\ & + \int_0^r f(\xi,\xi+t-r)e^{\int_0^\xi \mu(s,s+t-r)ds}d\xi\Big] \end{aligned} \tag{2.21}$$

where $Y_0(t-r)$ and $Y_n(t-r)$ are given by (2.18). This completes the proof. □

DEFINITION. We define

$$\begin{aligned} \beta_{2c}^-(T) &= \Big[\max_{0\leq t\leq T}\int_{r_{21}}^{r_{22}} h_2(r,t)k_2(r,t)e^{-\int_0^r[\mu_2+a_{12}][s,s+t-r]ds}dr\Big]^{-1}, \\ \beta_{1c}^-(T) &= \Big[\max_{0\leq t\leq T}\int_{r_{11}}^{r_{12}} h_1(r,t)k_1(r,t)e^{-\int_0^r \mu[s,s+t-r]ds}dr\Big]^{-1}, \end{aligned}$$

where β_{2c}^- and β_{1c}^- is called the lower critical value of specific fertility rate of rural females and the lower critical value of specific fertility rate of urban females respectively.

The above lemma implies the following theorem.

THEOREM 2.2. *Let $p_{20}(r), v_2(t), h_2(r,t), k_2(r,t), a_{12}(r,t)$ and $\mu_2(r,t)$ be given, and $[u_1+a_{12}](r,t)$ satisfy conditions (1.2) and (1.3). And let $p_{20}(r) \in H^1(\Omega), v_2(t) \in H^1(0,t), h_2(r,t) \in H^{\frac{3}{2},\frac{3}{2}}(Q), k_2(r,t) \in H^{\frac{3}{2},\frac{3}{2}}(Q)$. Then, if $\beta_2(t) = \beta_2 < \beta_{2c}^-(t)$, there exists a unique function $p_2(r,t)$ in* Sobolev *space $H^{\frac{3}{2},\frac{3}{2}}(Q)$ satisfying the system of equation $(1.5)'$. Furthermore, $p_2(r,t)$ can be expressed explicitly by the expansion in series*

$$p_2(r,t) = e^{-\int_0^r[u_2+a_{12}](s,s+t-r)ds}\Big[p_{20}(r-t) + \sum_{n=0}^{\infty} Y_{2,n}(t-r)\beta_2^{n+1}\Big], \tag{2.22}$$

where

$$(2.23)\quad \begin{aligned} Y_{2,0}(t) &= \int_{r_{21}}^{r_{22}} h_2(r,t)k_2(r,t)p_{2,0}(r-t)e^{-\int_0^r[\mu_2+a_{12}](s,s+t-r)ds}dr \\ Y_{2,n}(t) &= \int_{t-r_{22}}^{t-r_{21}} k_2(t,z)Y_{2,n-1}(z)dz, \qquad (n=1,2,\cdots), \\ K_2(t,z) &= h_2(t-z,t)k_2(t-z,t)e^{-\int_0^{t-z}[\mu_2+a_{12}](s,s+z)ds}. \end{aligned}$$

The above lemma also implies the following result.

THEOREM 2.3. *Let* $\mu_1(r,t), h_1(r,t), k_1(r,t), p_{10}(r), v_1(t)$ *and* $a_{12}(r,t)$ *be given,* $p_2(r,t)$ *be obtained from* Theorem 2.2, *and* $\mu_1(r,t)$ *satisfy conditions* (1.2) *and* (1.3) *with* $h_1(r,t) \in H^{\frac{3}{2},\frac{3}{2}}(Q)$, $k_1(r,t) \in H^{\frac{3}{2},\frac{3}{2}}(Q), p_{10}(r) \in H^1(\Omega), v_1(t) \in H^1(0,t), a_{12}(r,t) \cdot p_2(r,t) \in H^{1,1}(Q)$. *Then, if* $\beta_1(t) = \beta_1 < \beta_{1c}^-(T)$ *and* $\beta_2(t) = \beta_2 < \beta_{2c}^-(T)$, *there exists a unique function* $p_1(r,t)$ *in Sobolev space* $H^{\frac{3}{2},\frac{3}{2}}(Q)$ *satisfying the system of equation* (1.5), *and* $p_1(r,t)$ *can be expressed as*

$$(2.24)\quad \begin{aligned} p_1(r,t) = \quad & e^{-\int_0^r \mu_1(s,s+t-r)ds}[p_{10}(r-t) + \sum_{n=0}^{\infty} Y_{1,n}(t-r)\beta_1^{n+1}] \\ & + \int_0^r e^{-\int_x^r \mu_1(s,s+t-r)ds} a_{12}(x,x+t-r)p_2(x,x+t-r)dx, \end{aligned}$$

where

$$\begin{aligned} Y_{1,0}(t) = \quad & \int_{r_{11}}^{r_{12}} h_1(r,t)k_1(r,t)e^{-\int_0^r \mu_1(s,s+t-r)ds}[p_{10}(r-t) \\ & + \int_0^r a_{12}(x,x+t-r)p_2(x,x+t-r)e^{\int_0^r \mu_1(s,s+t-r)ds}dx]dr, \\ Y_{1,n}(t) = \quad & \int_{t-r_{12}}^{t-r_{11}} k(t,z)Y_{1,n-1}(z)dz, (n=1,2,\cdots), \\ K(t,z) = \quad & h_1(t-z,t)k_1(t-z,t)e^{-\int_0^{t-z}\mu_1(s,s+z)ds}. \end{aligned}$$

3. An example of application

In section 2, we have obtained the expansion in series for the solutions $p_1(r,t)$ and $p_2(r,t)$ of system of equation (1.1). These expansion in series has an important application to research of non-stationary urban and rural population control system. To make use of it, we not only make approximate calculation on population evolution state, but also discuss the stability of population control system and so on.

According to the definition of stability in sense of Lyapunov, if $T \to \infty$, consequently $T \to \infty$, $\|p\|_{L^2(\Omega)}$ is bounded, then, the corresponding population control system is stable.

Now, we shall explain that, if the specific fertility rate of rural females $\beta_2 < \beta_{2c}^-$, then rural population control system is stable for the arbitrary initial condition $p_{20}(r)$, where $\beta_{2c}^- = \lim_{T\to\infty} \beta_{2c}^-(T)$.

As a matter of fact, since T is arbitrary and $\lim_{T\to\infty} \beta_{2c}^-(T) = \beta_{2c}^-$, by (2.23) and Hölder inequality, we can obtain that

$$Y_{2,0}(t) \le c\|p_{20}(r)\|_{L^2(\Omega)}, \forall\ t \in (0,\infty). \tag{3.1}$$

In general, we have that

$$Y_{2,n}(t) \le c_1\|p_{20}(r)\|_{L^2(\Omega)}(\beta_{2c}^-)^{-n}, \forall\ t \in (0,\infty). \tag{3.2}$$

Consequently, by (2.22) we have

$$\begin{aligned} p_2(r,t) &\le p_{20}(r-t) + \beta_2 \sum_{n=0}^{\infty} \beta_2^n Y_{2,n}(t-r) \\ &\le p_{20}(r-t) + c_1\|p_{20}(r)\|_{L^2(\Omega)} \cdot \beta_2 \sum_{n=0}^{\infty} (\tfrac{\beta_2}{\beta_{2c}^-})^n. \end{aligned}$$

If $\beta_2 < \beta_{2c}^-$, the series $\sum_{n=0}^{\infty}(\frac{\beta_2}{\beta_{2c}^-})^n$ is convergent, thus

$$\|p_2(r,t)\|_{L^2(\Omega)} \le c_2\|p_{20}(r)\|_{L^2(\Omega)}, \forall t \in (0,\infty).$$

This proves that rural population control system is stable if $\beta_2 < \beta_{2c}^-$. Similarly, we may discuss the stability of urban population system.

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