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CONJUGACY SEPARABILITY OF CERTAIN FREE PRODUCT AMALGAMATING RETRACTS

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Abstract. We find some conditions to derive the conjugacy separability of the free product of conjugacy separable split extensions amalgamated along cyclic retracts. These conditions hold for any double coset separable groups and free-by-cyclic groups with nontrivial center. It was known that free-by-finite, polycyclic-by-finite, and Fuchsian groups are double coset separable. Hence free products of those groups amalgamated along cyclic retracts are conjugacy separable.

1. Introduction

Two nonconjugate elements of a group G are called *conjugacy distinguished* (c.d.) if their images are not conjugate in some finite quotient of G. The whole group is termed *conjugacy separable* (c.s.) if each pair of its nonconjugate elements is c.d. Some known c.s. groups which are related to this paper are polycyclic-by-finite groups [7], free-by-finite groups [4], free-by-cyclic groups with nontrivial center [5], Fuchsian groups [6]. Dyer [5] showed that the free product of two free groups –or two finitely generated (f.g.) nilpotent groups– amalgamating cyclic subgroups is c.s. Also, in [11, 10, 18], the conjugacy separability of free products of c.s. groups amalgamating cyclic subgroups was considered.

The purpose of this paper is to investigate the conjugacy separability of free products of c.s. groups amalgamating along retracts. This is an

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extension of the Boler and Evans' result [3] that the free product of residually finite ($\mathcal{RF}$) groups amalgamated along retracts is $\mathcal{RF}$. Their proof was based on the fact that each split extension of a f.g. $\mathcal{RF}$ group by a $\mathcal{RF}$ group is $\mathcal{RF}$ [15, p.29]. However, C. F. Miller [15, p.28] constructed a split extension of a f.g. free group by a f.g. free group which is not c.s. Thus Boler and Evans' method can not be adapted to our study. In [1, 8], free products of π_c groups amalgamated along retracts are π_c. On the other hand free products of subgroup separable groups amalgamated along retracts may not be subgroup separable [1]. In this paper, we find some conditions for the free product of c.s. groups, amalgamated along cyclic retracts, to be c.s. as follows:

MAIN THEOREM. *Let $G_i = E_i \cdot H$ $(i \in I)$ be c.s., split extensions of E_i by a retract $H = \langle h \rangle$. Assume that, for each $i \in I$, G_i satisfies the following:*

- $D1$ *If there exist $u_i, v_i \in E_i$ such that $u_i \notin Hv_iH$ then there exists $P_i \lhd_f E_i$ such that $P_i \lhd G_i$ and $u_i \notin P_iHv_iH$;*
- $D2$ *If there exists $u_i \in E_i$ such that $[u_i, h^j] \neq 1$ for all $j \neq 0$ then, for any integer $\epsilon > 1$, there exists $P_i \lhd_f E_i$ such that $P_i \lhd G_i$ and $[u_i, h^j] \in P_i$ implies $\epsilon \mid j$.*

Then the free product G of the G_i $(i \in I)$ amalgamated along $H = \langle h \rangle$ is c.s.

$D1$ and $D2$ hold for any double coset separable group (Lemma 3.7). Note that free-by-finite [17], polycyclic-by-finite [12], and Fuchsian [16] groups are double coset separable. Hence those groups satisfy $D1$ and $D2$. We show that free-by-cyclic groups with nontrivial center also satisfy $D1$ and $D2$. Thus free products of those groups amalgamated along cyclic retracts are c.s.

Finally, we note that conditions $D1$ and $D2$ played an important role in [9] to study the conjugacy separability of certain one-relator groups.

We introduce some definitions and results that we shall use in this paper.

We write $x \sim_G y$ if there exists $g \in G$ such that $x = g^{-1}yg$ and we write $x \not\sim_G y$ otherwise. $\{x\}^G$ denotes the conjugacy class $\{y \in G : x \sim_G y\}$ of

x in G. We use $\langle X\rangle^G$ to denote the normal closure of X in G. We also use $[x,y]=x^{-1}y^{-1}xy$ and $C_H(K)=\{h\in H:[h,k]=1 \text{ for all } k\in K\}$.

We denote by $A*_H B$ the free product of A and B with their subgroup H amalgamated. If $G=A*_H B$ and $x\in G$ then $\|x\|$ denotes the amalgamated free product length of x in G. On the other hand we use $|x|$ to denote the order of x.

$N\lhd_f G$ denotes that N is a normal subgroup of finite index in G. If $\overline{G}$ is a homomorphic image of G then we use $\overline{x}$ to denote the image of $x\in G$ in $\overline{G}$.

Let H be a subgroup of G. Then we say that G is *H-separable* if to each $x\in G\backslash H$ there exists $N\lhd_f G$ such that $x\notin NH$. A group G is said to be *residually finite* ($\mathcal{RF}$) if G is $\langle 1\rangle$-separable. A group G is said to be *conjugacy separable* (c.s.) if G is $\{x\}^G$-separable for all $x\in G$. Clearly every c.s. group is $\mathcal{RF}$. We shall use the following theorems:

THEOREM 1.1.([3]) *The free product of $\mathcal{RF}$ groups amalgamated along retracts is $\mathcal{RF}$.*

THEOREM 1.2.([5]) *If A and B are c.s. and H is finite, then $A*_H B$ is c.s.*

As Dyer [5] mentioned, the main tool to prove the conjugacy separability of a free product with amalgamation is the following result, known as Solitar's theorem:

THEOREM 1.3.([14]) *Let $G=A*_H B$ and $x\in G$ be of minimal length in its conjugacy class. Suppose $y\in G$, y is cyclically reduced, and $x\sim_G y$.*

(1) *If $\|x\|=0$, then $\|y\|\le 1$ and if $y\in A$ say, there is a sequence $h_1,h_2,\dots,h_r$ of elements in H such that $y\sim_A h_1\sim_B h_2\sim_A\cdots\sim_B h_r=x$.*
(2) *If $\|x\|=1$, then $\|y\|=1$ and either $x,y\in A$ and $x\sim_A y$, or else $x,y\in B$ and $x\sim_B y$.*
(3) *If $\|x\|\ge 2$, then $\|x\|=\|y\|$ and $y\sim_H x^*$ where x^* is some cyclic permutation of x.*

2. Preliminary results

In this section, we find some basic results to study the conjugacy separability of free products of c.s. groups amalgamated along retracts. Throughout the paper $E = E_1 \cdot H$ and $F = F_1 \cdot H$ are split extensions of the normal subgroups E_1 and F_1 by a retract H and, by Theorem 1.2, we assume that the retract H is infinite. The following lemma was used implicitly to study split extensions [3, 15].

LEMMA 2.1. *If $N \lhd_f E = E_1 \cdot H$ then there exist $M_1 \lhd_f E_1$ and $M_2 \lhd_f H$ such that $M_1M_2 \lhd_f E$, $M_1M_2 \subset N$, and E/M_1M_2 is a split extension of the finite group E_1M_2/M_1M_2 by the finite group HM_1/M_1M_2.*

Using this, we can easily see the following.

LEMMA 2.2.([8]) *If $E = E_1 \cdot H$ is $\mathcal{R}F$, then E is H-separable.*

In view of [5, Lemma 5], we need the following lemma to derive the conjugacy separability of the free product of c.s. split extensions, amalgamated along retracts.

LEMMA 2.3. *Suppose that $E = E_1 \cdot H$ is c.s. and $x \in E$ such that $\{x\}^E \cap H = \emptyset$. Then there exists $N \lhd_f E$ such that $\{\overline{x}\}^{\overline{E}} \cap \overline{H} = \emptyset$, where $\overline{E} = E/N$.*

Proof. Let $x = x_1h$, where $x_1 \in E_1$ and $h \in H$. Now E is c.s. and $x \not\sim_E h$. It follows that there exists $M \lhd_f E$ such that $\tilde{x} \not\sim_{\tilde{E}} \tilde{h}$, where $\tilde{E} = E/M$. By Lemma 2.1, there is a homomorphism $\pi : E \to \overline{E}$ such that $\overline{E}$ is a split extension of a finite group $\overline{E_1}$ by a finite group $\overline{H}$ and Ker $\pi \subset M$. Thus $\overline{x} \not\sim_{\overline{E}} \overline{h}$. Then we can see that $\{\overline{x}\}^{\overline{E}} \cap \overline{H} = \emptyset$, where $\overline{E} = E/\text{Ker }\pi$. $\square$

In $E = E_1 \cdot H$, if $h \not\sim_H k$ for $h, k \in H$, then we have $h \not\sim_E k$. Thus we have the following.

LEMMA 2.4. *If $E = E_1 \cdot H$ is c.s. then H is c.s.*

Now we are ready to consider the conjugacy separability of the free product of c.s. split extensions amalgamated along retracts.

THEOREM 2.5. *Assume that $E = E_1 \cdot H$ and $F = F_1 \cdot H$ are c.s. Let x and y be nonconjugate elements of $G = E *_H F$, each of minimal length in its conjugacy class. Then x and y are c.d. unless $\|x\| = \|y\| \geq 2$.*

Proof. Since G is $\mathcal{RF}$ by Theorem 1.1, we may assume $x \neq 1 \neq y$.

Case 1. $\|x\| = 0$ and $\|y\| =1$ (or, similarly, $\|y\| = 0$ and $\|x\| = 1$).

Without loss of generality, we may assume $x \in H$ and $y \in E \backslash H$. Since E is H-separable (Lemma 2.2), there exists $N_1 \lhd_f E$ such that $y \notin N_1 H$. Now y has the minimal length 1 in its conjugacy class in G. This implies that $\{y\}^E \cap H = \emptyset$. Hence, by Lemma 2.3, there exists $N_2 \lhd_f E$ such that $\{\tilde{y}\}^{\tilde{E}} \cap \tilde{H} = \emptyset$, where $\tilde{E} = E/N_2$. Let $N = N_1 \cap N_2$ and $N_H = N \cap H$. Then $N \lhd_f E$ and $N_H \lhd_f H$. Since H is a retract of F, there exists $M \lhd_f F$ such that $M \cap H = N_H = N \cap H$. Let π be the natural homomorphism of $E *_H F$ onto $E/N *_{\overline{H}} F/M$, where $\overline{H} = NH/N = MH/M$. Clearly $y\pi \notin H\pi$ and $\{y\pi\}^{E\pi} \cap H\pi = \emptyset$. It follows from Theorem 1.3 that $y\pi$ has the minimal length 1 in its conjugacy class in $G\pi$. This implies that $y\pi \not\sim_{G\pi} x\pi$. Since $G\pi$ is c.s. by Theorem 1.2, x and y are c.d.

Case 2. $\|x\| \neq \|y\|$ and $\|x\| \geq 2$ (or, similarly, $\|x\| \neq \|y\|$ and $\|y\| \geq 2$).

Since x has the minimal length in its conjugacy class in G, x is cyclically reduced, say, $x = e_1 f_1 e_2 \cdots e_n f_n$ where $e_i \in E \backslash H$ and $f_i \in F \backslash H$. Let $y = a_1 b_1 \cdots$ where $a_j \in E \backslash H$ and $b_j \in F \backslash H$ (we note that y may have any length ≥ 0). Since E and F are H-separable (Lemma 2.2), there exist $N_1 \lhd_f E$ and $M_1 \lhd_f F$ such that $e_i, a_j \notin N_1 H$ and $f_i, b_j \notin M_1 H$, for all i, j. Now H is a retract of E and F, and $N_1 \cap M_1 \lhd_f H$. This follows that there exist $N_2 \lhd_f E$ and $M_2 \lhd_f F$ such that $N_2 \cap H = N_1 \cap M_1 = M_2 \cap H$. Let $N = N_1 \cap N_2$ and $M = M_1 \cap M_2$. Then clearly $N \cap H = M \cap H$. Thus we form a homomorphism $\pi : E *_H F \to E/N *_{\overline{H}} F/M$, where $\overline{H} = NH/N = MH/M$. Moreover, we have $\|x\pi\| = \|x\|$ and $\|y\pi\| = \|y\|$. Since $x\pi$ is cyclically reduced and of length ≥ 2, $x\pi$ has the minimal length ($\neq \|y\pi\|$) in its conjugacy class in $E/N *_{\overline{H}} F/M$. Note that $y\pi$ is cyclically reduced, since y has the minimal length in its conjugacy class in G. It follows from Theorem 1.3 that $x\pi \not\sim_{G\pi} y\pi$. Since $G\pi$ is c.s., x and y are c.d.

Case 3. $\|x\| = \|y\| = 0$.

In this case, we have $x, y \in H$ and $x \not\sim_H y$. Considering the homomorphism $\pi_0 : E *_H F \to H$ defined by $z\pi_0 = 1$ for all $z \in E_1 \cup F_1$, we have $x\pi_0 \not\sim_H y\pi_0$. Since H is c.s. by Lemma 2.4, x and y are c.d.

Case 4. $\|x\| = \|y\| = 1$.

Subcase 1. Both x and y are in $E \backslash H$ (or, similarly, both x and y are in $F \backslash H$). Since E is c.s. and H-separable, there exists $N_1 \lhd_f E$ such that $x, y \notin N_1 H$ and $N_1 x \not\sim_{E/N_1} N_1 y$. Now x has the minimal length 1 in its conjugacy class in G. This implies that $\{x\}^E \cap H = \emptyset$. By Lemma 2.3, there exists $N_2 \lhd_f E$ such that $\{\tilde{x}\}^{\tilde{E}} \cap \tilde{H} = \emptyset$, where $\tilde{E} = E/N_2$. Let $N = N_1 \cap N_2 \lhd_f E$. Since H is a retract of F and $N \cap H \lhd_f H$, there exists $M \lhd_f F$ such that $M \cap H = N \cap H$. Hence we have a homomorphism $\pi : E *_H F \to E/N *_{\overline{H}} F/M$ such that $\|x\pi\| = 1 = \|y\pi\|$ and $\{x\pi\}^{E\pi} \cap H\pi = \emptyset$, where $\overline{H} = HN/N = HM/M$. Thus, it follows from Theorem 1.3 that $x\pi$ has the minimal length 1 in its conjugacy class in $G\pi$. Since Ker $\pi \subset N_1$ and $N_1 x \not\sim_{E/N_1} N_1 y$, we have $x\pi \not\sim_{E\pi} y\pi$. Hence, by Theorem 1.3 again, we have $x\pi \not\sim_{G\pi} y\pi$. Since $G\pi$ is c.s., x and y are c.d.

Subcase 2. $x \in E \backslash H$ and $y \in F \backslash H$ (or, similarly, $x \in F \backslash H$ and $y \in E \backslash H$). As in Subcase 1, we can find $N \lhd_f E$ and $M \lhd_f F$ such that $x \notin NH$, $y \notin MH$, $N \cap H = M \cap H$, and $\{x\pi\}^{E\pi} \cap H\pi = \emptyset$. Then, as before, we have $x\pi \not\sim_{G\pi} y\pi$, hence x and y are c.d. This completes the proof. Then $N \cap H = N_3 \cap M_1 = M \cap H$. Thus, we have a homomorphism $\pi : E *_H F \to E/N *_{\overline{H}} F/M$ such that $\|x\pi\| = \|y\pi\| = 1$ and $\{x\pi\}^{E\pi} \cap H\pi = \emptyset$. It follows, as before, that x and y are c.d. This completes the proof. □

The following lemma gives a necessary condition to derive the conjugacy separability of free products with amalgamation.

LEMMA 2.6. *Suppose G contains two elements x, y and a subgroup H such that $x \notin HyH$. If there is no $N \lhd_f G$ such that $x \notin NHyH$, then $Q = G *_H G$ is not c.s.*

Proof. If G is not $\mathcal{RF}$, then clearly Q is not $\mathcal{RF}$, hence not c.s. Thus we may assume that G is $\mathcal{RF}$. If G is not H-separable, then Q is not $\mathcal{RF}$ [5, p.42]. Hence Q is not c.s. Thus we may assume that G is $\mathcal{RF}$

and H-separable. Then, it follows from our assumption that $x \notin H$ and $y \notin H$. Write $Q = G *_H G_1$, the free product of G and G_1 amalgamating subgroups H and $H\psi$, where $\psi : G \to G_1$ is an isomorphism under which $g_1 = g\psi$. Thus $x_1 = x\psi \notin H\psi$ and $y_1 = y\psi \notin H\psi$. It follows that $\|xx_1^{-1}\| = \|yy_1^{-1}\| = 2$. Since $x \notin HyH$, we have $xx_1^{-1} \not\sim_H yy_1^{-1}$. This implies by Theorem 1.3 that $xx_1^{-1} \not\sim_Q yy_1^{-1}$. We shall prove that the images of xx_1^{-1} and yy_1^{-1} are conjugate under any homomorphisms of Q with finite images. Let $\phi : Q \to F$ be a homomorphism, where F is finite. Let $N = \text{Ker } \phi \cap G \cap \psi^{-1}(\text{Ker } \phi \cap G_1)$. Then $N \lhd_f G$ and $(N \cap H)\psi = N\psi \cap H\psi$. It follows that there exists a homomorphism $\pi : G *_H G_1 \to G/N *_{\overline{H}} G_1/N_1$, where $\overline{H} = HN/N = H_1N_1/N_1$. Since $N \lhd_f G$, by assumption, we have $\overline{x} = \overline{hyk}$ for some $h, k \in H$ where $\overline{G} = G/N$. Thus $x^{-1}hyk \in N$. It follows that $(x^{-1}hyk)\psi \in N\psi$ and $\overline{x_1} = \overline{h_1y_1k_1}$ in $\overline{G_1} = G_1/N_1$. In $\overline{Q} = Q\pi$, we have $\overline{xx_1^{-1}} = \overline{hyk} \cdot (\overline{h_1y_1k_1})^{-1} = \overline{hyy_1^{-1}h^{-1}}$. It follows that $xx_1^{-1}(hyy_1^{-1}h^{-1})^{-1} \in \text{Ker } \pi$. Since $\text{Ker } \pi = \langle N, N_1 \rangle^Q \subset \text{Ker } \phi$, we have $(xx_1^{-1})\phi = h\phi \cdot (yy_1^{-1})\phi \cdot (h\phi)^{-1}$. This proves the lemma. $\square$

THEOREM 2.7. *Let $E = E_1 \cdot H$ and $F = F_1 \cdot H$, where E_1 and F_1 are finite and H is f.g. abelian. Then $E *_H F$ is c.s.*

Proof. Since E_1 is finite, it is not difficult to see that $C_H(E_1) \lhd_f E$. Similarly $C_H(F_1) \lhd_f F$. Hence E and F are finite extensions of f.g. abelian groups, whence E and F are c.s. By Theorem 2.5, we need only consider the case of $x \not\sim_G y$ where $\|x\| = \|y\| = 2n$. Let $x = h_1e_1f_1 \cdots e_nf_n$ and $y = h_2a_1b_1 \cdots a_nb_n$, where $h_1, h_2 \in H$, $e_i, a_i \in E_1$ and $f_i, b_i \in F_1$. If $h_1 \neq h_2$ then $\overline{x} = \overline{h_1} \not\sim_{\overline{G}} \overline{h_2} = \overline{y}$, where $\overline{G} = E/E_1 *_{\overline{H}} F/F_1 \cong H$. Then x and y are c.d., since $\overline{G} \cong H$ is c.s.

So we assume $h_1 = h_2$. Let $S = C_H(E_1) \cap C_H(F_1)$. Then $S \lhd_f E$ and $S \lhd_f F$. It follows by Theorem 1.2 that $\overline{G} = E/S *_{\overline{H}} F/S$ is c.s. Now we shall show that $\overline{x} \not\sim_{\overline{G}} \overline{y}$. Clearly $\|\overline{x}\| = \|\overline{y}\| = 2n$. Hence if $\overline{x} \sim_{\overline{G}} \overline{y}$, then $\overline{x} \sim_{\overline{H}} \overline{y}^*$ for some cyclic permutation $\overline{y}^*$ of $\overline{y}$. It follows that $(y^*)^{-1}h^{-1}xh \in S \subset H$ for some $h \in H$. Note $(y^*)^{-1}h^{-1}xh \in E_1 * F_1$. Hence $(y^*)^{-1}h^{-1}xh = 1$, since $(E_1 * F_1) \cap H = \langle 1 \rangle$. Thus $x \sim_G y$, contradicting our assumption. Therefore $\overline{x} \not\sim_{\overline{G}} \overline{y}$, whence x and y are c.d. $\square$

In general, the conjugacy separability of free products amalgamated along retracts is not easy. So in the next section we consider only the case that retracts are cyclic. However, if retracts are direct factors of c.s. groups then we can easily see that free products of c.s. groups amalgamated along direct factors are c.s. For, if the $G_i = E_i \times H$ $(i \in I)$ are c.s. then it is easy to see that the E_i are also c.s. Now the free product G of the G_i amalgamated along H is just a direct product of H and the free product of the E_i. Since free products and direct products of c.s. groups are c.s., G is c.s.

3. Amalgamating cyclic retracts

Let H be a retract of both $E = E_1 \cdot H$ and $F = F_1 \cdot H$. We define

$$\eta = \{(N, M) : N \lhd_f E_1, N \lhd E \text{ and } M \lhd_f F_1, M \lhd F\}.$$

Then, for each $(N, M) \in \eta$, we have a homomorphism

$$\pi_{N,M} : E *_H F \to E/N *_{\overline{H}} F/M, \tag{1}$$

where $\overline{H} = HN/N = HM/M$. Note that $\overline{H} \cong H$ is a retract of both $E/N = (E_1/N) \cdot \overline{H}$ and $F/M = (F_1/M) \cdot \overline{H}$. Then, by Theorem 2.7, $(E *_H F)\pi_{N,M}$ is c.s., if H is abelian. Hence we have the following lemma.

LEMMA 3.1. *Let $E = E_1 \cdot H$ and $F = F_1 \cdot H$ be c.s., where H is f.g. abelian. Then we have*
(a) *$(E *_H F)\pi_{N,M}$ is c.s. for each $(N, M) \in \eta$,*
(b) *$(\cap_{j=1}^n N_j, \cap_{j=1}^n M_j) \in \eta$ for each $(N_j, M_j) \in \eta$, and*
(c) *$\cap_{(N,M)\in\eta} N = \langle 1 \rangle$ and $\cap_{(N,M)\in\eta} M = \langle 1 \rangle$.*

For the split extension $E = E_1 \cdot H$, where $H = \langle h \rangle$, we shall consider the following conditions:

- $D1$ If there exist $u, v \in E_1$ such that $u \notin HvH$, then there exists $P \lhd_f E_1$ such that $P \lhd E$ and $u \notin PHvH$.
- $D2$ If there exists $u \in E_1$ such that $[u, h^j] \neq 1$ for all $j \neq 0$ then, for any integer $\epsilon > 1$, there exists $P \lhd_f E_1$ such that $P \lhd E$ and $[u, h^j] \in P$ implies $\epsilon \mid j$.

LEMMA 3.2. *Let $E = E_1 \cdot H$ and $F = F_1 \cdot H$ be $\mathcal{R}F$, where $H = \langle h \rangle$, and let E and F satisfy D2. If there exist $h^\alpha \in H$ and $u_1u_2\cdots u_k \in E_1 * F_1$ such that $h^\alpha \notin C_H(u_1\cdots u_{k-1})C_H(u_k)$, then there exists $(N, M) \in \eta$ such that $\overline{h}^\alpha \notin C_{\overline{H}}(\overline{u}_1\cdots\overline{u}_{k-1})C_{\overline{H}}(\overline{u}_k)$, where $\overline{G} = G\pi_{N,M}$.*

Proof. Let $h^\alpha \in H$ and $e_1f_1\cdots e_nf_n \in E_1 * F_1$ such that $h^\alpha \notin C_H(e_1f_1 \cdots f_{n-1}e_n)C_H(f_n)$, where $e_k \in E_1$ and $f_k \in F_1$ for $1 \le k \le n$ (the other cases are similar). Let $C_H(f_n) = \langle h^\beta \rangle$ and $C_H(e_1f_1\cdots f_{n-1}e_n) = \langle h^\gamma \rangle$. Then clearly $\gamma \ne 1 \ne \beta$ and $\alpha \ne 0$.

Case 1. $\beta \ne 0$ and $\gamma = 0$. Since $h^\alpha \notin C_H(e_1f_1\cdots f_{n-1}e_n)C_H(f_n) = \langle h^\beta \rangle$, β does not divide α. Now $C_H(e_1f_1\cdots f_{n-1}e_n) = \langle 1 \rangle$. It follows that $C_H(e_k) = \langle 1 \rangle$ for some $1 \le k \le n$, or $C_H(f_{k'}) = \langle 1 \rangle$ for some $1 \le k' \le n-1$. Here we assume that $C_H(e_k) = \langle 1 \rangle$ (the other cases are similar). By $D2$, there exists $P \lhd_f E_1$ such that $P \lhd E$ and $[e_k, h^j] \in P$ implies $\beta \mid j$. Since G is $\mathcal{R}F$ by Theorem 1.1 and $[h^l, f_n] \ne 1$ for $1 \le l < \beta$, there exists $L \lhd_f G$ such that $[h^l, f_n] \notin L$ and $e_s, f_s \notin L$ for all $1 \le l < \beta$ and $1 \le s \le n$. Let $N = L \cap P$ and $M = L \cap F_1$. Then $N \lhd_f E_1$, $N \lhd E$ and $M \lhd_f F_1$, $M \lhd F$. It follows that $(N, M) \in \eta$. Now we shall prove that $\overline{h}^\alpha \notin C_{\overline{H}}(\overline{e_1f_1\cdots f_{n-1}e_n})C_{\overline{H}}(\overline{f_n})$, where $\overline{G} = E/N *_{\overline{H}} F/M = G\pi_{N,M}$. First, we can easily see that $C_{\overline{H}}(\overline{f_n}) = \langle \overline{h}^\beta \rangle$ and $C_{\overline{H}}(\overline{e_1f_1\cdots f_{n-1}e_n}) = C_{\overline{H}}(\overline{e_1}) \cap C_{\overline{H}}(\overline{f_1}) \cap \cdots \cap C_{\overline{H}}(\overline{e_n}) \subset C_{\overline{H}}(\overline{e_k}) \subset \langle \overline{h}^\beta \rangle$. Since $|\overline{h}| = \infty$ and β does not divide α, we have $\overline{h}^\alpha \notin \langle \overline{h}^\beta \rangle = C_{\overline{H}}(\overline{e_1f_1\cdots f_{n-1}e_n})C_{\overline{H}}(\overline{f_n})$, as required.

Case 2. $\beta = 0$ and $\gamma = 0$. Since $C_H(e_1f_1\cdots f_{n-1}e_n) = \langle 1 \rangle$, there exists k or k' such that $C_H(e_k) = \langle 1 \rangle$ or $C_H(f_{k'}) = \langle 1 \rangle$, where $1 \le k \le n$, $1 \le k' \le n-1$. We consider the case when $C_H(e_k) = \langle 1 \rangle$ (the other case is similar). Choose an integer $s > |\alpha|$. Then $h^\alpha \notin \langle h^s \rangle$. By $D2$, there exists $P \lhd_f E_1$ such that $P \lhd E$ and $[e_k, h^j] \in P$ implies $s \mid j$. Similarly, there exists $Q \lhd_f F_1$ such that $Q \lhd F$ and $[f_n, h^j] \in Q$ implies $s \mid j$. Since G is $\mathcal{R}F$ (Theorem 1.1), there exists $L \lhd_f G$ such that $e_m, f_m \notin L$ for all $1 \le m \le n$. Let $N = L \cap P$ and $M = L \cap Q$. Then clearly $(N, M) \in \eta$. Moreover, we have $C_{\overline{H}}(\overline{f_n}) \subset \langle \overline{h}^s \rangle$ and $C_{\overline{H}}(\overline{e_1f_1\cdots f_{n-1}e_n}) \subset C_{\overline{H}}(\overline{e_k}) \subset \langle \overline{h}^s \rangle$, where $\overline{G} = E/N *_{\overline{H}} F/M = G\pi_{N,M}$. It follows that $C_{\overline{H}}(\overline{e_1f_1\cdots f_{n-1}e_n})C_{\overline{H}}(\overline{f_n}) \subset \langle \overline{h}^s \rangle$. Since $|\overline{h}| = \infty$ and $s > |\alpha|$, we have $\overline{h}^\alpha \notin C_{\overline{H}}(\overline{e_1f_1\cdots f_{n-1}e_n})C_{\overline{H}}(\overline{f_n})$.

Similar methods can be applied to the following cases:
Case 3. $\beta = 0$ *and* $\gamma \neq 0$.
Case 4. $\beta \neq 0$ *and* $\gamma \neq 0$.
This completes the proof. $\square$

Now we are ready to prove our main result.

THEOREM 3.3. *Let* $E = E_1 \cdot H$ *and* $F = F_1 \cdot H$ *be c.s. and satisfy* $D1$ *and* $D2$*, where* $H = \langle h \rangle$*. Then* $G = E *_H F$ *is c.s.*

Proof. Let x and y be nonconjugate elements in $G = E *_H F$, each of minimal length in its conjugacy class in G. By Theorem 2.5, we need only consider the case $\|x\| = \|y\| \geq 2$. Since $G = (E_1 * F_1) \cdot H$ is a split extension with a retract H, we may write $x = h^\alpha e_1 f_1 \cdots e_n f_n$ and $y = h^\beta a_1 b_1 \cdots a_n b_n$, where $e_j, a_j \in E_1$ and $f_j, b_j \in F_1$ for all $1 \leq j \leq n$. Now, if $\alpha \neq \beta$, then $x\pi_{E_1,F_1} \neq y\pi_{E_1,F_1}$, where $G\pi_{E_1,F_1} \cong H$. This implies that x and y are c.d. Therefore, we may assume that $x = h^\alpha e_1 f_1 \cdots e_n f_n$ and $y = h^\alpha a_1 b_1 \cdots a_n b_n$, where $a_j, e_j \in E_1$ and $b_j, f_j \in F_1$ for all j. From (b) and (c) in Lemma 3.1, we can find $(N', M') \in \eta$ such that $e_j, a_j \notin N'$ and $f_j, b_j \notin M'$, for all j.

Since $x \not\sim_G y$, by Theorem 1.3 we have $y \not\sim_H x^*$ for all cyclic permutations x^* of x. It follows that each of the equations

$$(2) \qquad (J:j)\;\; e_j f_j \cdots e_n f_n h^\alpha e_1 f_1 \cdots e_{j-1} f_{j-1} = h^{-i} h^\alpha a_1 b_1 \cdots a_n b_n h^i$$

has no solution $h^i \in H$ for each $1 \leq j \leq n$. Hence, we shall find $(N_j, M_j) \in \eta$ such that $N_j \subset N'$, $M_j \subset M'$ and $(J:j)\pi_{N_j,M_j}$ has no solution in $H\pi_{N_j,M_j}$ where π_{N_j,M_j} is as in (1). Then considering $N = \cap_{j=1}^n N_j$ and $M = \cap_{j=1}^n M_j$, $(J:j)\pi_{N,M}$ has no solution in $H\pi_{N,M}$ for all $1 \leq j \leq n$. Moreover, we have $\|x\pi_{N,M}\| = \|x\| = \|y\| = \|y\pi_{N,M}\| = 2n$. It follows from Theorem 1.3 that $\overline{x} \not\sim_{\overline{G}} \overline{y}$, where $\overline{G} = G\pi_{N,M}$. By (b) and (a) in Lemma 3.1, we have $(N, M) \in \eta$ and $G\pi_{N,M}$ is c.s. Therefore, x and y are c.d. This completes the proof once we find, for each $1 \leq j \leq n$, a suitable $(N_j, M_j) \in \eta$ such that $(J:j)\pi_{N_j,M_j}$ has no solution in $H\pi_{N_j,M_j}$. Here we only consider $j = 1$, since the other equation $(J:j)$ for $j > 1$ can be stated as $h^\alpha e'_j f'_j \cdots e'_n f'_n e_1 f_1 \cdots e_{j-1} f_{j-1} = h^{-i} h^\alpha a_1 b_1 \cdots a_n b_n h^i$, where $e'_k = h^{-\alpha} e_k h^\alpha \in E_1$ and $f'_k = h^{-\alpha} f_k h^\alpha \in F_1$ for $j \leq k \leq n$.

Since $(J:1)$ has no solution $h^i \in H$, which is equivalent to $e_1f_1\cdots e_nf_n \notin Ha_1b_1\cdots a_nb_nH$, one of the following will be true:

(1) $e_r \notin Ha_rH$ or $f_r \notin Hb_rH$ for some $1 \le r \le n$, or assuming that $e_r \in Ha_rH$ and $f_r \in Hb_rH$ for all $1 \le r \le n$, for the remaining cases,

(1′) $e_1f_1 \notin Ha_1b_1H$,

(2) $e_1f_1 \in Ha_1b_1H$ and $e_1f_1e_2 \notin Ha_1b_1a_2H$,

(2′) $e_1f_1e_2 \in Ha_1b_1a_2H$ and $e_1f_1e_2f_2 \notin Ha_1b_1a_2b_2H$,

$\vdots \quad \vdots$

(n') $e_1f_1\cdots e_n \in Ha_1b_1\cdots a_nH$ and $e_1f_1\cdots e_nf_n \notin Ha_1b_1\cdots a_nb_nH$.

If (1) is true, say, $e_r \notin Ha_rH$ for some r, then by $D1$ there exists $(P,Q) \in \eta$ such that $e_r \notin PHa_rH$. Let $N_1 = N' \cap P$ and $M_1 = M' \cap Q$. Then $(N_1, M_1) \in \eta$ and $(J:1)\pi_{N_1,M_1}$ has no solution in $H\pi_{N_1,M_1}$ as required.

If one of (1′), (2), ..., (n') is true, say $f_r \in Hb_rH$, $e_1f_1\cdots e_r \in Ha_1b_1\cdots a_rH$ and $e_1f_1\cdots e_rf_r \notin Ha_1b_1\cdots a_rb_rH$, then we have $e_1f_1\cdots e_r = h^{-s}a_1b_1\cdots a_rh^s$ and $f_r = h^{-t}b_rh^t$ for some s,t. Note that $e_1f_1\cdots e_rf_r \notin Ha_1b_1\cdots a_rb_rH$ if and only if $a_1b_1\cdots a_rh^sh^{-t}b_r \notin Ha_1b_1\cdots a_rb_rH$ if and only if $h^{s-t} \notin C_H(a_1b_1\cdots a_r)C_H(b_r)$. Hence, by Lemma 3.2, there exists $(P,Q) \in \eta$ such that $\tilde{h}^{s-t} \notin C_{\tilde{H}}(\tilde{a_1}\tilde{b_1}\cdots\tilde{a_r})C_{\tilde{H}}(\tilde{b_r})$ in $\tilde{G} = G\pi_{P,Q}$. Let $N_1 = N' \cap P$ and $M_1 = M' \cap Q$. Then $(N_1, M_1) \in \eta$ and $\overline{h}^{s-t} \notin C_{\overline{H}}(\overline{a_1b_1\cdots a_r})C_{\overline{H}}(\overline{b_r})$ in $\overline{G} = G\pi_{N_1,M_1}$. Hence, we have $\overline{e_1f_1\cdots e_rf_r} \notin \overline{Ha_1b_1\cdots a_rb_rH}$. Thus $\overline{e_1f_1\cdots e_nf_n} \notin \overline{Ha_1b_1\cdots a_nb_nH}$. This implies that $(J:1)\pi_{N_1,M_1}$ has no solution in $\overline{H} = H\pi_{N_1,M_1}$. This completes the proof. □

To generalize Theorem 3.3, we consider the following lemma.

LEMMA 3.4. *Let $E = E_1 \cdot H$ and $F = F_1 \cdot H$ be $\mathcal{RF}$, where $H = \langle t \rangle$, and satisfy $D1$ and $D2$. Then the split extension $G = E *_H F = (E_1 * F_1)\cdot H$ satisfies $D1$ and $D2$.*

Proof. For $D1$, let $y, w \in E_1 * F_1$ such that $y \notin HwH$.

Case 1. $\|y\| \neq \|w\|$ or $\|y\| = \|w\|$ and the first syllables of y and w are in the different factors of $E_1 * F_1$. Since E and F are H-separable (Lemma 2.2), there exist $N_1 \lhd_f E_1$ and $M_1 \lhd_f F_1$ such that $N_1 \lhd E$, $M_1 \lhd F$

and $\|y\| = \|\overline{y}\|, \|w\| = \|\overline{w}\|$, where $\overline{G} = E/N_1 *_{\overline{H}} F/M_1$ and $\overline{H} = HN_1/N_1 = HM_1/M_1$. This means that $\|\overline{y}\| \neq \|\overline{w}\|$ or $\|\overline{y}\| = \|\overline{w}\|$ and the first syllables of $\overline{y}$ and $\overline{w}$ are in the different factors of $\overline{E}_1 * \overline{F}_1$. It follows that $\overline{y} \notin \overline{HwH}$. Since $\overline{E_1} = E_1/N_1$ and $\overline{F_1} = F_1/M_1$ are finite, there exists a least integer $\gamma > 0$ such that $[\,\overline{t}^{\,\gamma}, \overline{z}] = 1$ for all $\overline{z} \in \overline{E_1} * \overline{F_1}$. Hence $\overline{y} \notin \overline{HwH}$ if and only if $\overline{y} \neq \overline{t^{-s}wt^s}$ for all $0 \leq s < \gamma$. Since $\overline{G}$ is $\mathcal{RF}$ by Theorem 1.1, there exists $\overline{L} \lhd_f \overline{G}$ such that $\overline{y(t^{-s}wt^s)^{-1}} \notin \overline{L}$ for all $0 \leq s < \gamma$. Let $\overline{L_1} = \overline{L} \cap (\overline{E_1} * \overline{F_1})$. Then $\overline{L_1} \lhd_f \overline{E_1} * \overline{F_1}$ and $\overline{L_1} \lhd \overline{G}$. Let R be the preimage of $\overline{L_1}$ in G. Then $R \lhd_f E_1 * F_1$, $R \lhd G$ and $y \notin RHwH$, as required.

Case 2. $\|y\| = \|w\|$ *and the first syllables of* y *and* w *are in the same factor of* $E_1 * F_1$. Let $y = e_1 f_1 \cdots e_n f_n$ and $w = a_1 b_1 \cdots a_n b_n$, where $e_k, a_k \in E_1$ and $f_k, b_k \in F_1$ (the other cases are similar). Since $y \notin HwH$, $y \neq h^{-1}wh$ for all $h \in H$. Thus, as in $(J:1)$ in Theorem 3.3, there exist $N \lhd_f E_1$ and $M \lhd_f F_1$ such that $N \lhd E$, $M \lhd F$ and $e_k, a_k \notin N$, $f_k, b_k \notin M$ for all $1 \leq k \leq n$ and $\overline{y} \notin \overline{HwH}$, where $\overline{G} = E/N *_{\overline{H}} F/M$. Now, as in the previous case, we can find the required R satisfying $D1$.

For $D2$, let $y \in E_1 * F_1$ such that $[y, t^j] \neq 1$ for all $j \neq 0$, and let ϵ be a given integer.

Case i. $\|y\| = 1$. Without loss of generality, we let $y \in E_1$. Then, by $D2$, there exists $P \lhd_f E_1$ such that $P \lhd E$ and $[y, t^j] \in P$ implies $\epsilon \mid j$. Let $R = \langle P * F_1 \rangle^G \cap (E_1 * F_1)$. Then $R \lhd_f E_1 * F_1$ and $R \lhd G$. Moreover, if $[y, t^j] \in R$, then $[y, t^j] \in R \cap E_1 = P$. It follows that, if $[y, t^j] \in R$, then $\epsilon \mid j$ as required.

Case ii. $\|y\| > 1$. Let $y = e_1 f_1 \cdots e_n f_n$ be a reduced word in $E_1 * F_1$, where $e_l \in E_1$ and $f_l \in F_1$ for $1 \leq l \leq n$ (other cases being similar). Since $[y, t^j] \neq 1$ for all $j \neq 0$, we have $\langle 1 \rangle = C_H(y) = C_H(e_1) \cap C_H(f_1) \cap \cdots \cap C_H(f_n)$. It follows that $C_H(e_k) = \langle 1 \rangle$ or $C_H(f_k) = \langle 1 \rangle$ for some k. We assume $C_H(e_k) = \langle 1 \rangle$ (other cases being similar). By $D2$, there exists $P \lhd_f E_1$ such that $P \lhd E$ and $[e_k, t^j] \in P$ implies $\epsilon \mid j$. Since E and F are $\mathcal{RF}$, we can find $P_1 \lhd_f E_1$ and $Q_1 \lhd_f F_1$ such that $P_1 \lhd E$, $Q_1 \lhd F$, $e_l \notin P_1$ and $f_l \notin Q_1$ for all $1 \leq l \leq n$. Let $N = P \cap P_1$ and $M = F \cap Q_1$ and consider the homomorphism $\pi : G \to E/N *_{\overline{H}} F/M$, where $\overline{H} = HN/N = HM/M$. Then $\|\overline{y}\| = \|y\|$ where $\overline{y} = y\pi$. Hence, if

$[\overline{y}, \overline{t}^j] = 1$, then $\overline{t}^j \in C_{\overline{H}}(\overline{y}) = C_{\overline{H}}(\overline{e_1}) \cap C_{\overline{H}}(\overline{f_1}) \cap \cdots \cap C_{\overline{H}}(\overline{f_n})$. It follows that $\overline{t}^j \in C_{\overline{H}}(\overline{e_k})$. Thus $[e_k, t^j] \in \text{Ker}\,\pi \cap E_1 = N \subset P$, which implies that $\epsilon \mid j$. Thus, if $[\overline{y}, \overline{t}^j] = 1$, then $\epsilon \mid j$. Since $\overline{E_1}$ and $\overline{F_1}$ are finite, there exists a least positive integer γ such that $[\overline{y}, \overline{t}^\gamma] = 1$. Now $[\overline{y}, \overline{t}^\ell] \neq 1$ for $1 \leq \ell < \gamma$ and $\overline{G}$ is $\mathcal{RF}$ by Theorem 1.1. It follows that there exists $\overline{L} \triangleleft_f \overline{G}$ such that $[\overline{y}, \overline{t}^\ell] \notin \overline{L}$ for all $1 \leq \ell < \gamma$. Let $\overline{L_1} = \overline{L} \cap (\overline{E_1} * \overline{F_1})$ and let R be the preimage of $\overline{L_1}$ in G. Then $R \triangleleft_f E_1 * F_1$, $R \triangleleft G$ and, if $[y, t^j] \in R$, then $\epsilon \mid j$ as required. $\square$

More generally we can state Theorem 3.3 as follows:

THEOREM 3.5. *Let $G_i = E_i \cdot H$ $(i \in I)$ be a c.s. and satisfy $D1$ and $D2$, where $H = \langle t \rangle$. Then the free product G of the G_i $(i \in I)$ amalgamated along H is c.s.*

Proof. First we can use Theorem 3.3 and Lemma 3.4 repeatedly to show that G is c.s. when I is a finite set. For an arbitrary set I, let $x \not\sim_G y$. Then, we can find a finite subset J of I such that x and y are contained in the free product G_J of the G_j $(j \in J)$ amalgamated along $\langle t \rangle$. Now, there exists a homomorphism $\theta : G \to G_J$ such that $e\theta = 1$ for all $e \in E_i$ $(i \in I \backslash J)$ and $w\theta = w$ for all $w \in G_j$ $(j \in J)$. Hence $x\theta \not\sim_{G\theta} y\theta$. Since $G\theta \cong G_J$ is c.s. by above, x and y are c.d. $\square$

In [9], conditions $D1$ and $D2$ are used to derive the conjugacy separability of certain 1-relator groups. It was also proved that finite extensions of free or f.g. nilpotent groups and certain 1-relator groups satisfy $D1$ and $D2$. Most of the above groups satisfying $D1$ and $D2$ are double coset separable.

DEFINITION 3.6. A group G is said to be *double coset separable* if for every pair H, K of f.g. subgroups of G, and any $g, x \in G$ such that $x \notin HgK$, there exists $N \triangleleft_f G$ such that $x \notin NHgK$.

For example, free-by-finite groups [17], polycyclic-by-finite groups [12] and f.g. Fuchsian groups [16] are double coset separable. Hence those groups satisfy $D1$ and $D2$ by the next observation.

LEMMA 3.7. *Let $E = E_1 \cdot \langle h \rangle$ be double coset separable. Then E satisfies $D1$ and $D2$.*

Proof. Clearly $D1$ holds. For $D2$, let $u \in E_1$ such that $[u, h^j] \neq 1$ for any $j \neq 0$. Let $\epsilon > 1$ be a given integer. Then $h^{-i}uh^i \notin \langle h^\epsilon \rangle u \langle h^\epsilon \rangle$ for $1 \leq i \leq \epsilon - 1$, since $\langle h \rangle$ is a retract. Then there exists $N_i \lhd_f E$ such that $h^{-i}uh^i \notin N_i \langle h^\epsilon \rangle u \langle h^\epsilon \rangle$ for each $1 \leq i \leq \epsilon - 1$. Let $P = \cap_{i=1}^{\epsilon-1} N_i \cap E_1$. Then $P \lhd_f E_1$ and $P \lhd E$. If $[u, h^j] \in P$ for $j = s\epsilon - k$, where $0 \leq k < \epsilon$, then $h^{-k}uh^k \in P\langle h^\epsilon \rangle u \langle h^\epsilon \rangle$. It follows that $k = 0$, whence $\epsilon \mid j$ as required. □

COROLLARY 3.8. *Let $G_i = E_i \cdot H$ $(i \in I)$ be c.s. and double coset separable, where $H = \langle t \rangle$. Then the free product G of the G_i $(i \in I)$ amalgamated along H is c.s.*

We shall prove that free-by-cyclic groups with nontrivial center satisfy $D1$ and $D2$.

LEMMA 3.9. *Let A be a free group and let $u, f, v \in A$ such that $u \neq f^{-j}vf^j$ for all integers j. Then there exists $N \lhd_f A$ such that $Nu \neq Nf^{-j}vf^j$ for all j.*

Proof. Let A be free on a set X of generators and $A_k = \langle x_1, \ldots, x_k \rangle$ be a subgroup of A generated by a subset $\{x_1, \ldots, x_k\}$ of X such that $u, f, v \in A_k$. Then there is a homomorphism $\xi : A \to A_k$ such that $x_i\xi = x_i$ for $1 \leq i \leq k$ and $y\xi = 1$ for all $y \in X \backslash \{x_1, \ldots, x_k\}$. First we are going to find the c−th term Γ_c of the lower central series of A_k such that $u \notin \Gamma_c f^{-j}vf^j$ for all j. Since free groups are residually nilpotent by [13], there exists an integer n_0 such that $f, u, v \notin \Gamma_{n_0}$. If $[v, f] = 1$ then we can choose $c \geq n_0$ such that $uv^{-1} \notin \Gamma_c$. It follows that $u \notin \Gamma_c v = \Gamma_c f^{-j}vf^j$ for all j. Hence, we may assume that $[v, f] \neq 1$. Choose (using [13]) $n_1 \geq n_0$ so that $[v, f] \notin \Gamma_{n_1}$. We shall prove that there exists $c \geq n_1$ such that $u \notin \Gamma_c f^{-j}vf^j$ for all j. If, for each $j \geq n_1$, there exists an integer s_j such that $u\gamma_j = f^{-s_j}vf^{s_j}$, for some $\gamma_j \in \Gamma_j$, then $\tilde{f}^{-s_{n_1}}\tilde{v}\tilde{f}^{s_{n_1}} = \tilde{f}^{-s_j}\tilde{v}\tilde{f}^{s_j}$ or $\tilde{v}^{-1}\tilde{f}^{s_j - s_{n_1}}\tilde{v} = \tilde{f}^{s_j - s_{n_1}}$, where $\tilde{A}_k = A_k/\Gamma_{n_1}$. Since $\tilde{A}_k$ is torsion-free nilpotent and $\tilde{f} \neq 1$, we have $s_j - s_{n_1} = 0$ or $\tilde{v}^{-1}\tilde{f}\tilde{v} = \tilde{f}$. It follows that $s_j - s_{n_1} = 0$, since $[v, f] \notin \Gamma_{n_1}$. This implies that $u^{-1}f^{-s_{n_1}}vf^{s_{n_1}} = u^{-1}f^{-s_j}vf^{s_j} = \gamma_j \in \cap_{j \geq n_1}\Gamma_j = \langle 1 \rangle$ by [13]. Hence, we have $u = f^{-s_{n_1}}vf^{s_{n_1}}$, contradicting our assumption. Therefore, there exists an integer c such that $u \notin \Gamma_c f^{-j}vf^j$ for all j. It follows that, in the f.g. nilpotent group $\overline{A_k} = A_k/\Gamma_c$, we have $\overline{u} \neq \overline{f^{-j}vf^j}$ for all j. Thus

$\overline{v}^{-1}\overline{u} \neq \overline{v^{-1}fv}^{-j}\overline{f}^j$ for all j. Now $\overline{A_k}$ is residually finite with respect to nests [19] and the set $\{(\overline{v^{-1}fv}^{-j}, \overline{f}^j) : j \in \mathbb{Z}\}$ is a nest in $\overline{A_k} \times \overline{A_k}$. Hence there exists $\overline{N} \triangleleft_f \overline{A_k}$ such that $\overline{N}\overline{v}^{-1}\overline{u} \neq \overline{N}(\overline{v^{-1}fv})^{-j}\overline{f}^j$ for all j. It follows that $\overline{N}\overline{u} \neq \overline{Nf^{-j}vf^j}$ for all j. Let N be the preimage of $\overline{N}$ in A. Then $N \triangleleft_f A$ and $Nu \neq Nf^{-j}vf^j$ for all j, as required. □

LEMMA 3.10. *Let A be a free group and $u, f \in A$ such that $[u, f^j] \neq 1$ for all $j \neq 0$. Then, for any integer $\epsilon > 1$, there exists $N \triangleleft_f A$ such that $[u, f^j] \in N$ implies $\epsilon \mid j$.*

Proof. For each $1 \leq i < \epsilon$ we have $f^{-i}uf^i \neq (f^\epsilon)^{-k}u(f^\epsilon)^k$ for all k, since $u \neq f^{-j}uf^j$ for all $j \neq 0$. Then, by Lemma 3.9, there exists $N_i \triangleleft_f A$ such that $N_i f^{-i}uf^i \neq N_i(f^\epsilon)^{-k}u(f^\epsilon)^k$ for all k. Let $N = \cap_{i=1}^{\epsilon-1} N_i$. Then $N \triangleleft_f A$ and $[u, f^j] \in N$ implies $\epsilon \mid j$. □

LEMMA 3.11. *Let $E = E_1 \cdot H$ be a cyclic extension of a free group E_1, with nontrivial center, where $H = \langle t \rangle$. Then E satisfies $D1$ and $D2$.*

Proof. For $D1$, let $u, v \in E_1$ such that $u \notin HvH$. Since E_1 is free, we may assume that $e^{-1}t^n \in Z(E)$, where $e \in E_1$ and n is a positive integer. Note that $u \notin \langle t \rangle v \langle t \rangle$ if and only if $u \neq t^{-m}t^{-nr}vt^{nr}t^m$ for all r and $0 \leq m < n$; equivalently, $u \neq t^{-m}e^{-r}ve^rt^m$ for all r and $0 \leq m < n$. Then, by Lemma 3.9, there exists $N_1 \triangleleft_f E_1$ such that $N_1 t^m ut^{-m} \neq N_1 e^{-r}ve^r$ for all r and $0 \leq m < n$. Let $N = \cap_{m=0}^{n-1} t^m N_1 t^{-m}$. Since $t^m N_1 t^{-m} \triangleleft_f E_1$, it is not difficult to see that $N \triangleleft_f E_1$, $N \triangleleft E$, and $u \notin NHvH$.

For $D2$, let $u \in E_1$ such that $[u, t^j] \neq 1$ for all $j \neq 0$ and let $\epsilon > 1$ be a given integer. As above we let $e^{-1}t^n \in Z(E)$, where $e \in E_1$ and n is a positive integer. Then we have $u \neq t^{-j}ut^j$ for all $j \neq 0$ if and only if $u \neq t^{-m}e^{-r}ue^rt^m$ for all r, $0 \leq m < n$ except $r = 0 = m$. By Lemma 3.9, for each $0 < m < n$, there exists $N_m \triangleleft_f E_1$ such that $t^m ut^{-m} \notin N_m e^{-r}ue^r$ for all r. Also, by Lemma 3.10, there exists $N_0 \triangleleft_f E_1$ such that $[u, e^r] \in N_0$ implies $\epsilon \mid r$. Let $N = \cap_{k=0}^{n-1} t^k (\cap_{m=0}^{n-1} N_m) t^{-k}$. Then $N \triangleleft_f E_1$, $N \triangleleft E$, and $[u, t^j] \in N$ implies $\epsilon \mid j$, as required. □

THEOREM 3.12. *Let $G_i = E_i \cdot \langle t \rangle$ $(i \in I)$ be an infinite cyclic extension of E_i, where each G_i satisfies one of the following:*

1. *G_i is free-by-finite.*

2. G_i *is polycyclic-by-finite.*
3. G_i *is a f.g.* Fuchsian *group.*
4. E_i *is free and* G_i *has nontrivial center.*
5. $G_i = \langle t, b \ : \ (t^{-1}b^{\alpha}tb^{\beta})^s \rangle$*, where* $s > 1$.
6. $G_i = \langle t, b \ : \ t^{-1}b^{\alpha}tb^{\beta} \rangle$*, where* $|\alpha| = |\beta|$ or $|\alpha| = 1$ *or* $|\beta| = 1$.

Then the free product G *of the* G_i $(i \in I)$ *amalgamated along the retract* $\langle t \rangle$ *is c.s.*

Proof. The conjugacy separability of the G_i in the theorem is known by [4, 7, 6, 5, 2, 9]. The G_i's in 1, 2, and 3 satisfy $D1$ and $D2$, since they are double coset separable (see p.823). The G_i in 4 satisfies $D1$ and $D2$ by Lemma 3.11. The 1-relator groups G_i in 5 and 6 also satisfy $D1$ and $D2$ by Lemma 5.5 and Theorem 6.4 in [9]. Hence the result follows from Theorem 3.5. $\square$

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