Bull. Korean Math. Soc. **37** (2000), No. 4, pp. 765–770

AN APPLICATION OF CERTAIN LINEAR OPERATOR

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Abstract. The object of the present paper is to give an application of a linear operator $L_p(a,c)$ defined by means of a Hadamard product (or convolution) to a Miller and Mocanu's theorem.

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N = \{1, 2, \dots\}) \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. For the functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{p+n,j} z^{p+n} \quad (p \in N), \tag{1.2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = \sum_{n=0}^{\infty} a_{p+n,1} a_{p+n,2} z^{p+n}. \tag{1.3}$$

Let the function $\varphi_p\,(a, c; z)$ be defined by

Received July 26, 1999. Revised July 10, 2000.
2000 Mathematics Subject Classification: 30C45.
Key words and phrases: linear operators, Hadamard product, Ruscheweyh derivative.

$$\varphi_p(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{p+n} \ (z \in U) \tag{1.4}$$

for $c \neq 0, -1, -2, \ldots$, where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (\text{ if } n = 0 \ ), \\ x(x+1)\ldots(x+n-1) & (\text{ if } n \in N \ ). \end{cases} \tag{1.5}$$

Also, we define a linear operator $L_p(a,c)$ on $A(p)$ by

$$L_p(a,c)f(z) = \varphi_p(a,c;z) * f(z) = \left(\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{p+n} \right) * f(z) \tag{1.6}$$

for $f(z) \in A(p)$ and $c \neq 0, -1, -2, \ldots$.

REMARK 1. If $c > a > 0$, $L_p(a,c)$ has integral representation

$$L_p(a,c)f(z) = \int_0^1 u^{-p} f(uz)\, d\mu(u),$$

where μ satisfies $d\mu(u) = \frac{u^{a-1}(1-u)^{c-a-1}}{\beta(a,c-a)} du$ and $\int_0^1 d\mu(u) = 1$, where $\beta(a, c-a)$ is the familiar Beta function.

The operator $L_p(a,c)$ was introduced by Saitoh ([10], [11]). For p = 1, the operator $L_1(a,c)$ was introduced by Carlson and Shaffer [2] in their systematic investigation of certain interesting classes of starlike, convex, and prestarlike hypergeometric functions.

REMARK 2. For $f(z) \in A(p)$,

$$L_p(n+p,1)f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) = D^{n+p-1} f(z), \tag{1.7}$$

where $n > -p$, the symbol $D^{n+p-1}f(z)$ when $p = 1$ was introduced by Ruscheweyh [8], and the symbol $D^{n+p-1}f(z)$ was introduced by Goel and Sohi [3]. Therefore, one called the symbol $D^{n+p-1}f(z)$ the Ruscheweyh derivative of $(n+p-1)$ th order.

$$L_p(\upsilon+p, \upsilon+p+1)f(z) = \frac{\upsilon+p}{z^{\upsilon}} \int_0^z t^{\upsilon-1} f(t)dt = J_{\upsilon,p}(f(z)) \tag{1.8}$$

where $v+p>0$. The operator $J_{v,1}(v \in N)$ was introduced by Bernardi [1]. In particular, the operator $J_{1,1}$ was studied earlier by Libera [4] and Livingston [5]. Some results for the operator $J_{v,p}$ were showed by Saitoh [9] and Saitoh, Owa et al [12].

DEFINITION. Let H be the set of complex valued functions $h(r,s,t)$; $h(r,s,t){:}C^3 \to C$ (C is the complex plane) such tht
(i) $h(r,s,t)$ is continuous in a domain $D \subset C^3$,
(ii) $(0,0,0) \in D$ and $\left|h(0,0,0)\right| < 1$,
(iii) $\left|h\left(e^{i\theta}, \frac{a-p+k}{a}e^{i\theta}, \frac{(a+1-p)(a-p+2k)e^{i\theta}+M}{a(a+1)}\right)\right| > 1$ whenever $\left(e^{i\theta}, \frac{a-p+k}{a}e^{i\theta}, \frac{(a+1-p)(a-p+2k)e^{i\theta}+M}{a(a+1)}\right) \in D$ with $\mathrm{Re}(e^{i\theta}M) \geq k(k-1)$ for real θ, for real $k \geq p$ and $a \neq 0, -1$.

2. Main result

We begin with the statement of the following lemma due to Miller and Mocanu [6].

LEMMA. *Let a function $w(z)$ defined by*

$$w(z) = b_p z^p + b_{p+1} z^{p+1} + \dots \quad (p \in N) \tag{2.1}$$

be regular in the unit disc U with $w(z) \not\equiv 0$ $(z \in U)$. If $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ and

$$|w(z_0)| = \max_{|z| \leq |z_0|} |w(z)|,$$

then

$$z_0 w'(z_0) = k w(z_0) \tag{2.2}$$

and

$$\mathrm{Re}\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\} \geq k, \tag{2.3}$$

where k is real and $k \geq p \geq 1$.

Making use of the above lemma, we prove

THEOREM. *Let $h(r,s,t)$ be in H, and let $f(z)$ belonging to the class $A(p)$ satisfy*

(i) $(L_p(a,c)f(z), L_p(a+1,c)f(z), L_p(a+2,c)f(z)) \in D \subset C^3$
and
(ii) $|h\,(L_p(a,c)f(z), L_p(a+1,c)f(z), L_p(a+2,c)f(z))| < 1$
where $c \neq 0, -1, -2, ...,$ $a \neq 0, -1,$ *and* $z \in U$. *Then we have*

$$|L_p(a,c)f(z)| < 1 \quad (z \in U). \tag{2.4}$$

Proof. We define the function $w(z)$ by

$$L_p(a,c)f(z) = w(z) \ (c \neq 0, -1, -2, ...; z \in U) \tag{2.5}$$

for $f(z)$ belonging to the class $A(p)$. Then, it follows that $w(z) \in A(p)$ and $w(z) \not\equiv 0\,(z \in U)$. With the aid of the identity

(2.6)
$$z\left(L_p(a,c)f(z)\right)' = aL_p(a+1,c)f(z) - (a-p)L_p(a,c)f(z) \quad (\text{cf. [10], [11]}),$$

where $c \neq 0, -1, -2, ...,$ we have

$$L_p(a+1,c)f(z) = \frac{1}{a}\left\{(a-p)w(z) + zw'(z)\right\} \tag{2.7}$$

and

(2.8)
$$L_p(a+2,c)f(z) = \tfrac{1}{a(a+1)}\left\{(a-p)(a-p+1)w(z) + 2(a-p+1)zw'(z) + z^2w''(z)\right\}.$$

Suppose that $z_0 = r_0e^{i\theta}\,(0 < r_0 < 1)$ and

$$|w(z_0)| = \max_{|z| \leq |z_0|} |w(z)| = 1. \tag{2.9}$$

Letting $w(z_0) = e^{i\theta}$ and using (2.2) of the above lemma, we see that

$$L_p(a,c)f(z_0) = w(z_0) = e^{i\theta}, \tag{2.10}$$

$$L_p(a+1,c)f(z_0) = \frac{a-p+k}{a}w(z_0) = \frac{a-p+k}{a}e^{i\theta}, \tag{2.11}$$

and

$$\begin{aligned} &L_p(a+2,c)f(z_0) \\ &= \tfrac{1}{a(a+1)}\left\{(a-p+1)(a-p+2k)w(z_0) + z_0^2w''(z_0)\right\} \\ &= \tfrac{(a-p+1)(a-p+2k)e^{i\theta}+M}{a(a+1)}, \end{aligned} \tag{2.12}$$

where $M = z_0^2 w''(z_0)$ and $k \geq p \geq 1$.

Further, an application of (2.3) in the above lemma gives

$$\text{Re}\left\{\frac{z_0 w''(z_0)}{w'(z_0)}\right\} = \text{Re}\left\{\frac{z_0^2 w''(z_0)}{ke^{i\theta}}\right\} \geq (k-1), \tag{2.13}$$

or

$$\text{Re}\left\{e^{-i\theta} M\right\} \geq k(k-1). \tag{2.14}$$

Since $h(r,s,t) \in H$, we have

$$\begin{aligned} &\left|h\big(L_p(a,c)f(z), L_p(a+1,c)f(z_0), L_p(a+2,c)f(z)\big)\right| \\ &= \left|h\Big(e^{i\theta}, \tfrac{a-p+k}{a}e^{i\theta}, \tfrac{(a+1-p)(a-p+2k)e^{i\theta}+M}{a(a+1)}\Big)\right| > 1 \end{aligned} \tag{2.15}$$

which contradicts the condition (ii) of the theorem. Therefore, we conclude that

$$\left|w(z)\right| = \left|L_p(a,c)f(z)\right| < 1,$$

where $c \neq 0, -1, -2, \ldots$, and for all $z \in U$. □

REMARK 3. Putting $a = \alpha + p$ $(\alpha > -p)$ and $c = 1$ in the above theorem, we get the theorem obtained by Owa and Ren [7].

Letting $a = c = p$ in the theorem, we have:

COROLLARY 1. *Let $h(r,s,t)$ be in H, and let $f(z)$ belonging to the class $A(p)$ satisfy*

(i) $\left(f(z), \frac{zf'(z)}{p}, \frac{z^2 f''(z)+2zf'(z)}{p(p+1)}\right) \in D \subset C^3$

and

(ii) $\left|h\left(f(z), \frac{zf'(z)}{p}, \frac{z^2 f''(z)+2zf'(z)}{p(p+1)}\right)\right| < 1, \ z \in U.$

Then we have $|f(z)| < 1$ $(z \in U)$.

COROLLARY 2. *Let $h_0(r,s,t) = s$ and let $f(z)$ belonging to the class $A(p)$ satisfy the conditions in theorem. Then*

$$|L_p(a-i,c)f(z)| < 1,$$

$$a \neq 0, -1, \ \ c \neq 0, -1, -2, \ldots \text{ and } i \in N_0 = N \cup \{0\}.$$

Proof. *Note that* $h_0(r,s,t) = s$ *is in* H, *with the aid of the theorem, we have*

$$\begin{aligned} &|L_p(a+1,c)f(z)| < 1 \\ \Rightarrow &|L_p(a,c)f(z)| < 1,\ (a \neq 0, -1, c \neq 0, -1, -2, \dots) \\ \Rightarrow &|L_p(a-i,c)f(z)| < 1 \quad (\, i \in N_0). \end{aligned}$$ □

ACKNOWLEDGMENT. The authors thank the referee for his helpful suggestion.

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