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FIXED POINTS OF A CERTAIN CLASS OF ASYMPTOTICALLY REGULAR MAPPINGS

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ABSTRACT. In this paper, we study in Banach spaces the existence of fixed points of asymptotically regular mapping T satisfying: for each x, y in the domain and for $n = 1, 2, \cdots$,

$$\begin{aligned}\|T^n x - T^n y\| \le a_n\|x-y\| + b_n(\|x - T^n x\| + \|y - T^n y\|) \\ + c_n(\|x - T^n y\| + \|y - T^n x\|),\end{aligned}$$

where a_n, b_n, c_n are nonnegative constants satisfying certain conditions. We also establish some fixed point theorems for these mappings in a Hilbert space, in L^p spaces, in Hardy spaces H^p, and in Sobolev spaces $H^{k,p}$ for $1 < p < \infty$ and $k \ge 0$. We extend results from papers [10], [11], and others.

1. Introduction and preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let K be a nonempty subset of E. A mapping $T : E \to E$ is said to be asymptotically regular [2] if $\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$ for all $x \in E$. It is well known that if T is nonexpansive, then $T_t = t \cdot I + (1-t) \cdot T$ is asymptotically regular for all $0 < t < 1$ (cf. [9]).

Lin [14] constructed an asymptotically regular Lipschitzian mapping acting on a weakly compact subset of l^2 which has no fixed point. Górnicki gave the sufficient condition for the existence of fixed points

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of asymptotically regular mappings in L^p spaces. Recently, Górnicki [11] extended his own result [10] in p-uniformly convex Banach spaces.

In this paper, we extend all the above results for the class of mappings whose nth iterate T^n satisfy

$$\begin{aligned}\|T^n x - T^n y\| \le a_n\|x-y\| &+ b_n\Big(\|x-T^n x\| + \|y - T^n y\|\Big)\\ &+ c_n\Big(\|x - T^n y\| + \|y - T^n x\|\Big)\end{aligned} \tag{1}$$

for each $x,\ y \in K$ and $n = 1,\ 2,\ \cdots$, where $a_n,\ b_n,\ c_n$ are the nonnegative constants such that there exists an integer n_0 satisfying $b_n + c_n < 1$ for all $n \ge n_0$.

This class of mappings are more general than nonexpansive mappings. Also by taking $b_n = c_n = 0$, it will be seen that this class of mappings are more general than asymptotically nonexpansive mappings defined by Goebel and Kirk [8].

The normal structure coefficient $N(E)$ (cf. Bynum [3]) of E is the number:

$$N(E) = \inf\left\{\frac{\operatorname{diam}K}{r_K(K)} : K \text{ is a bounded convex subset of } E \text{ consisting of more than one point}\right\},$$

where $\operatorname{diam}K = \sup\{\|x-y\| : x,\ y \in K\}$ is the diameter of K and $r_K(K) = \inf_{x\in K}\left\{\sup_{y\in K}\|x-y\|\right\}$ is the Chebyshev radius of K relative to itself. E is said to have uniformly normal structure if $N(E) > 1$. It is known that a uniformly convex Banach space has uniformly normal structure (cf. Danes [5]) and for a Hilbert space H, $N(H) = \sqrt{2}$. Recently, Pichugov [16] (cf. Prus [17]) calculated that

$$N(L^p) = \min\{2^{\frac{1}{p}}, 2^{\frac{p-1}{p}}\},\ 1 < p < \infty.$$

Some estimates for normal structure coefficients in other Banach spaces may be found in [18].

Let $p > 1$ and denote by λ the number in $[0,1]$ and by $W_p(\lambda)$ the function $\lambda\cdot(1-\lambda)^p + \lambda^p\cdot(1-\lambda)$.

The functional $\|\cdot\|^p$ is said to be uniformly convex (cf. Zalinescu [23]) on the Banach space E if there exists a positive constant c_p such that for all $\lambda \in [0,1]$ and $x,\ y \in E$ the following inequality holds:

$$(2)\quad \|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)\cdot c_p \cdot \|x-y\|^p.$$

Xu [22] proved that the functional $\|\cdot\|^p$ is uniformly convex on the whole Banach space E if and only if E is p-uniformly convex, i.e., there exists a constant $c > 0$ such that the modulus of convexity (see [9]) $\delta_E(\varepsilon) \ge c \cdot \varepsilon^p$ for all $0 \le \varepsilon \le 2$.

2. Main results

Before presenting our main result, we need following lemmas.

LEMMA 1 ([22]). *Let $p > 1$ and let E be a p-uniformly convex* Banach *space, K a nonempty closed convex subset of E and $\{x_n\} \subset E$ a bounded sequence. Then there exists a unique point z in K such that*

$$(3)\qquad \limsup_{n\to\infty} \|x_n - z\|^p \le \limsup_{n\to\infty} \|x_n - x\|^p - c_p \cdot \|x - z\|^p$$

for every x in K, where c_p is the constant given in (2).

LEMMA 2 ([11]). *Let K be a nonempty closed convex subset of a* Banach *space E and $\{n_i\}$ an increasing sequence of natural numbers. Assume that $T : K \to K$ is an asymptotically regular mapping such that for some $m \in \mathbb{N}$, T^m is continuous. If*

$$\limsup_{i\to\infty} \|x - T^{n_i}u\| = 0$$

for some $u \in K$ and $x \in K$, then $Tx = x$.

Now we are in position to give our result.

THEOREM 1. *Let $p > 1$ and let E be a p-uniformly convex* Banach *space, K a nonempty closed convex subset of E, and $T : K \to K$*

an asymptotically regular mapping which holds the inequality (1) *such that*

$$\text{(C)} \qquad \left[\frac{(\alpha+\beta)^p(2^{p-1}\alpha^p - 1)}{(c_p - 2^{p-1}\beta^p)\cdot N^p} \right]^{\frac{1}{p}} < 1,$$

where

$$\alpha = \liminf_{n\to\infty} \frac{a_n + c_n}{1 - c_n}, \quad \beta = \liminf_{n\to\infty} \frac{b_n}{1 - c_n},$$

and N is the normal structure coefficient of E. Suppose that there is a z_0 in K for which $\{T^n z_0\}$ is bounded. Then T has a fixed point in K.

Proof. Let $\{n_i\}$ be a sequence of natural numbers such that

$$\alpha = \liminf_{n\to\infty} \frac{a_n + c_n}{1 - c_n} = \lim_{i\to\infty} \frac{a_{n_i} + c_{n_i}}{1 - c_{n_i}}$$

and

$$\beta = \liminf_{n\to\infty} \frac{b_n}{1 - c_n} = \lim_{i\to\infty} \frac{b_{n_i}}{1 - c_{n_i}}.$$

Since $\{T^n z_0\}$ is bounded (and hence $\{T^n z\}$ is bounded for any z in K), by Lemma 1, we can inductively construct a sequence $\{z_m\}$ such that z_m is the unique asymptotic center of the sequence $\{T^{n_i} z_{m-1}\}_{i\geq 1}$ with respect to the functional

$$\limsup_{i\to\infty} \|x - T^{n_i} z_{m-1}\|^p$$

over x in K.

Now for each $m \geq 1$, we set

$$D_m = \limsup_{i\to\infty} \|z_m - T^{n_i} z_m\|$$

and

$$r_m \doteq \limsup_{i\to\infty} \|z_{m+1} - T^{n_i} z_m\|.$$

Now, using (1), we have for each $x,\ y \in K$ and $k,\ l \geq 1$,

$$\begin{aligned}\|T^k x - T^l y\| &\leq \|T^k x - T^{l+k} y\| + \|T^{k+l} y - T^l y\| \\ &\leq a_k \|x - T^l y\| + b_k(\|x - T^k x\| + \|T^l y - T^{k+l} y\|) \\ &\quad + c_k(\|T^l y - T^k x\| + \|x - T^{k+l} y\|) + \|T^{k+l} y - T^l y\|\end{aligned}$$

which by simplification, gives

$$\begin{aligned}\|T^k x - T^l y\| \leq \frac{a_k + c_k}{1 - c_k} \cdot \|x - T^l y\| + \frac{b_k}{1 - c_k} \cdot \|x - T^k y\| \\ + \frac{1 + b_k + c_k}{1 - c_k} \cdot \|T^{k+l} y - T^l y\|.\end{aligned} \tag{4}$$

By inequality (4), the result of Casini and Maluta [4], and the asymptotic regularity of T, we have

$$\begin{aligned}r_m &\leq \frac{1}{N} \cdot \lim_{n\to\infty} (\sup \|T^{n_i} z_m - T^{n_j} z_m\| : n_i,\ n_j \geq n) \\ &\leq \frac{1}{N} \cdot \limsup_{i\to\infty} (\limsup_{j\to\infty} \|T^{n_i} z_m - T^{n_j} z_m\|) \\ &\leq \frac{1}{N} \cdot \limsup_{i\to\infty} \left[\limsup_{j\to\infty} \left(\frac{a_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_j} z_m\| \right.\right. \\ &\quad \left.\left. + \frac{b_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_i} z_m\| + \frac{1 + b_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \|T^{n_i + n_j} z_m - T^{n_j} z_m\| \right) \right] \\ &\leq \frac{1}{N} \cdot \limsup_{i\to\infty} \left[\limsup_{j\to\infty} \left(\frac{a_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_j} z_m\| \right.\right. \\ &\quad + \frac{b_{n_i}}{1 - c_{n_i}} \cdot \|z_m - T^{n_i} z_m\| \\ &\quad \left.\left. + \frac{1 + b_{n_i} + c_{n_i}}{1 - c_{n_i}} \cdot \sum_{l=0}^{n_i - 1} \|T^{n_j + l + 1} z_m - T^{n_j + l} z_m\| \right) \right]\end{aligned}$$

which implies

$$r_m \leq \frac{1}{N} \cdot (\alpha + \beta) \cdot D_m,\ m = 0,\ 1,\ 2,\ \cdots, \tag{5}$$

where N is the normal structure coefficient of E. For each $m \geq 1$ and all $n_i,\ n_j$, we have from (2) and (4)

$$\begin{aligned}
&\|\lambda z_{m+1} + (1-\lambda)T^{n_j} z_{m+1} - T^{n_i} z_m\|^p + c_p \cdot W_p(\lambda) \cdot \|z_{m+1} - T^{n_j} z_{m+1}\|^p \\
&\leq \lambda \cdot \|z_{m+1} - T^{n_i} z_m\|^p + (1-\lambda) \cdot \left[\frac{a_{n_j} + c_{n_j}}{1 - c_{n_j}} \cdot \|z_{m+1} - T^{n_i} z_m\| \right. \\
&\quad \left. + \frac{b_{n_j}}{1 - c_{n_j}} \cdot \|z_{m+1} - T^{n_j} z_{m+1}\| + \frac{1 + b_{n_j} + c_{n_j}}{1 - c_{n_j}} \cdot \|T^{n_i + n_j} z_m - T^{n_i} z_m\| \right]^p \\
&\leq \lambda \cdot \|z_{m+1} - T^{n_i} z_m\|^p + (1-\lambda) \cdot \left[\frac{a_{n_j} + c_{n_j}}{1 - c_{n_j}} \cdot \|z_{m+1} - T^{n_i} z_m\| \right. \\
&\quad + \frac{b_{n_j}}{1 - c_{n_j}} \cdot \|z_{m+1} - T^{n_j} z_{m+1}\| \\
&\quad \left. + \frac{1 + b_{n_j} + c_{n_j}}{1 - c_{n_j}} \cdot \sum_{l=0}^{n_j - 1} \|T^{n_i + l + 1} z_m - T^{n_i + l} z_m\| \right]^p .
\end{aligned}$$

Taking the limit superior as $i \to \infty$ on each side, by definition of z_m and the asymptotic regularity of T, we get

$$\begin{aligned}
&r_m^p + c_p \cdot W_p(\lambda) \cdot \|z_{m+1} - T^{n_j} z_{m+1}\|^p \\
&\leq \lambda r_m^p + (1-\lambda) \left[\frac{a_{n_j} + c_{n_j}}{1 - c_{n_j}} \cdot r_m + \frac{b_{n_j}}{1 - c_{n_j}} \cdot \|z_{m+1} - T^{n_j} z_{m+1}\| \right]^p \\
&\leq \lambda r_m^p + (1-\lambda) \left[2^{p-1} \left\{ \left(\frac{a_{n_j} + c_{n_j}}{1 - c_{n_j}} \right)^p \cdot r_m^p \right. \right. \\
&\qquad\qquad \left. \left. + \left(\frac{b_{n_j}}{1 - c_{n_j}} \right)^p \cdot \|z_{m+1} - T^{n_j} z_{m+1}\|^p \right\} \right].
\end{aligned}$$

It then follows from (5) that

$$r_m^p + c_p \cdot W_p(\lambda) \cdot D_{m+1}^p \leq \lambda r_m^p + (1-\lambda) \left[2^{p-1} \left\{ \alpha^p r_m^p + \beta^p \cdot D_{m+1}^p \right\} \right]$$

or

$$\begin{aligned}
D_{m+1}^p &\leq \frac{(1-\lambda) \cdot (2^{p-1} \cdot \alpha^p - 1)}{c_p \cdot W_p(\lambda) - (1-\lambda) \cdot 2^{p-1} \cdot \beta^p} \cdot r_m^p \\
&\leq \frac{(1-\lambda) \cdot (2^{p-1} \cdot \alpha^p - 1)}{\{c_p \cdot W_p(\lambda) - (1-\lambda) \cdot 2^{p-1} \cdot \beta^p\}} \cdot \frac{(\alpha + \beta)^p}{N^p} \cdot D_m^p .
\end{aligned}$$

Letting $\lambda \uparrow 1$, we conclude that

$$D_{m+1} \le \left[\frac{(\alpha+\beta)^p(2^{p-1}\cdot\alpha^p-1)}{(c_p-2^{p-1}\cdot\beta^p)\cdot N^p}\right]^{\frac{1}{p}} \cdot D_m$$
$$= A\cdot D_m, \qquad m=1,\ 2,\ \cdots,$$

where

$$A = \left[\frac{(\alpha+\beta)^p(2^{p-1}\cdot\alpha^p-1)}{(c_p-2^{p-1}\cdot\beta^p)\cdot N^p}\right]^{\frac{1}{p}} < 1$$

by the assumption of the theorem. Since

$$\|z_{m+1}-z_m\| \le r_m + D_m \le 2D_m \le \cdots \le 2\cdot A^m D_1 \to 0 \text{ as } m\to\infty,$$

it follows that $\{z_m\}$ is a Cauchy sequence. Let $z=\lim_{m\to\infty} z_m$. Then we have

$$\begin{aligned}
&\|z-T^{n_i}z\| \\
&\le \|z-z_m\| + \|z_m - T^{n_i}z_m\| + \|T^{n_i}z_m - T^{n_i}z\| \\
&\le \|z-z_m\| + \|z_m - T^{n_i}z_m\| + a_{n_i}\cdot\|z_m - z\| \\
&\quad + b_{n_i}\left(\|z_m-T^{n_i}z_m\| + \|z-T^{n_i}z\|\right) + c_{n_i}\left(\|z_m-T^{n_i}z\| + \|z-T^{n_i}z_m\|\right)
\end{aligned}$$

and so

$$\|z-T^{n_i}z\| \le \frac{1+a_{n_i}+2c_{n_i}}{1-b_{n_i}-c_{n_i}}\cdot\|z-z_m\| + \frac{1+b_{n_i}+c_{n_i}}{1-b_{n_i}-c_{n_i}}\cdot\|z_m-T^{n_i}z_m\|.$$

Taking the limit superior as $i\to\infty$ on each side, we get

$$\begin{aligned}
\limsup_{i\to\infty}\|z-T^{n_i}z\| &\le \limsup_{i\to\infty}\frac{1+a_{n_i}+2c_{n_i}}{1-b_{n_i}-c_{n_i}}\cdot\|z-z_m\| \\
&\quad + \limsup_{i\to\infty}\frac{1+b_{n_i}+c_{n_i}}{1-b_{n_i}-c_{n_i}}\cdot D_m \to 0 \ \text{ as } m\to\infty.
\end{aligned}$$

Therefore we obtain $Tz=z$ by Lemma 2. This completes the proof. □

Górnicki [11] proved the following theorem:

THEOREM ([Górnicki]). *Let $p > 1$ and let E be a p-uniformly convex Banach space, K a nonempty bounded closed convex subset of E, and $T : K \to K$ an asymptotically regular mapping. If*

$$\liminf_{n\to\infty} |||T^n||| = k < \left[\frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot c_p \cdot N^p}\right)\right]^{\frac{1}{p}},$$

$\Big($*where* $|||T^n|||$ *is the* Lipschitz *constant of* T^n, *i.e.,*

$$|||T^n||| = \sup\left\{\frac{\|T^n x - T^n y\|}{\|x - y\|} : x \neq y,\ x,\ \ y \in K\right\},$$

N is the normal structure coefficient of E, and c_p is the constant in (2)$\Big)$, *then T has a fixed point in K.*

If we put $b_n = c_n = 0$ in (1), then a_n is equal to $|||T^n|||$ and the condition of Górnicki [11] that

$$k < \left[\frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot c_p \cdot N^p}\right)\right]^{\frac{1}{p}} \text{ or equivalently } \left[\frac{k^p(k^p - 1)}{c_p \cdot N^p}\right]^{\frac{1}{p}} < 1$$

follows from condition (C) of Theorem 1, and hence the result of Górnicki [11] follows as special case of Theorem 1.

REMARK 1. In place of bounded subset K of [11], we have taken weaker assumption that there is an z_0 in K for which $\{T^n z_0\}$ is bounded.

3. Some Applications

In a Hilbert space H, the following equality holds:

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{6}$$

for all $x,\ y$ in H and $\lambda \in [0, 1]$.

By Theorem 1 and (6), we immediately obtain the following.

THEOREM 2. *Let K be a nonempty closed convex subset of a* Hilbert *space H and $T : K \to K$ an asymptotically regular mapping which holds the inequality* (1) *such that*

$$\left[\frac{(\alpha+\beta)^2(2\alpha^2-1)}{2(1-2\beta^2)}\right]^{\frac{1}{2}} < 1,$$

where α, β as in Theorem 1. *Suppose that there is a z_0 in K for which $\{T^n z_0\}$ is bounded. Then T has a fixed point in K.*

If we put $b_n = c_n = 0$ in (1), then from Theorem 2, we have the following result.

COROLLARY 1 ([11, Corollary 2]). *Let K be a nonempty bounded closed convex subset of a* Hilbert *space H. If $T : K \to K$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} |||T^n||| < \sqrt{2},$$

then T has a fixed point in K.

If $1 < p \le 2$, then we have for all x, y in L^p and $\lambda \in [0,1]$,

(7) $\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)(p-1)\|x-y\|^2.$

(The inequality (7) is contained in Lim, Xu and Xu [13] and Smarzewski [21].)

Assume that $2 < p < \infty$ and t_p is the unique zero of the function $g(x) = -x^{p-1} + (p-1)x + p - 2$ in the interval $(1,\infty)$. Let

$$c_p = (p-1)(1+t_p)^{2-p} = \frac{1+t_p^{p-1}}{(1+t_p)^{p-1}}.$$

Then we have the following inequality

(8) $\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)\cdot c_p \cdot \|x-y\|^p$

for all x, y in L^p and $\lambda \in [0,1]$. (The inequality (8) is essentially due to Lim [12].)

THEOREM 3. *Let K be a nonempty closed convex subset of L^p, $1 < p < \infty$, and $T : K \to K$ an asymptotically regular mapping which holds* (1) *such that*

$$\left[\frac{(\alpha+\beta)^2 \cdot (2\alpha^2 - 1)}{((p-1) - 2\beta^2) \cdot 2^{\frac{p-1}{p}}}\right]^{\frac{1}{2}} < 1 \qquad \text{for } 1 < p \le 2$$

and

$$\left[\frac{(\alpha+\beta)^p \cdot (2^{p-1}\alpha^p - 1)}{(c_p - 2^{p-1}\beta^p) \cdot 2}\right]^{\frac{1}{p}} < 1 \qquad \text{for } 2 < p < \infty,$$

where α, β as in Theorem 1. *Suppose that there is a z_0 in K for which $\{T^n z_0\}$ is bounded. Then T has a fixed point in K.*

If we put $b_n = c_n = 0$ in (1), then from Theorem 3, we have the following result.

COROLLARY 2 ([11, Corollary 3, 4]). *Let K be a nonempty bounded closed convex subset of L^p $(1 < p < \infty)$. If $T : K \to K$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} |||T^n||| = k < \left[\frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot (p-1) \cdot 2^{\frac{p-1}{p}}}\right)\right]^{\frac{1}{2}} \quad \text{for } 1 < p \le 2$$

and

$$\liminf_{n\to\infty} |||T^n||| = k < \left[\frac{1}{2}\left(1 + \sqrt{1 + 8 \cdot c_p}\right)\right]^{\frac{1}{p}} \quad \text{for } 2 < p < \infty,$$

then T has a fixed point in K.

Let H^p, $1 < p < \infty$, denote the Hardy space [7] of all functions x analytic in unit disc $|z| < 1$ of the complex plane and such that

$$\|x\| = \lim_{r\to 1^-} \left(\frac{1}{2\pi}\int_0^{2\pi} |x(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} < \infty.$$

Now, let Ω be an open subset of $\mathbb{R}^n$. Denote by $H^{k,p}(\Omega)$, $k \geq 0$, $1 < p < \infty$, the Sobolev space [1, p.149] of distributions x such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$ equipped with the norm

$$\|x\| = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha x(\omega)|^p d\omega \right)^{\frac{1}{p}}.$$

Let $(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, be a sequence of positive measure spaces, where index set Λ is finite or countable. Given a sequence of linear subspaces X_α in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < \infty$ and $q = \max\{2, p\}$ [15], the linear space of all sequences $x = \{x_\alpha \in X_\alpha : \alpha \in \Lambda\}$ equipped with the norm

$$\|x\| = \left(\sum_{\alpha \in \Lambda} \left(\|x_\alpha\|_{p,\alpha} \right)^q \right)^{\frac{1}{q}},$$

where $\| \cdot \|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$.

Finally, let $L_p = L^p(S_1, \sum_1, \mu_1)$ and $L_q = L^q(S_2, \sum_2, \mu_2)$, where $1 < p < \infty$, $q = \max\{2, p\}$, and $(S_i, \sum_i, \mu_i)$ are positive measure spaces. Denote by $L_q(L_p)$ the Banach spaces [6, III. 2.10] of all measurable L_p-value function x on S_2 such that

$$\|x\| = \left(\int_{S_2} \left(\|x(s)\|_p \right)^q \mu_2(ds) \right)^{\frac{1}{q}}.$$

These spaces are q-uniformly convex with $q = \max\{2, p\}$ [19, 20] and the norm in these spaces satisfies

$$\|\lambda x + (1 - \lambda) y\|^q \leq \lambda \|x\|^q + (1 - \lambda) \|y\|^q - d \cdot W_q(\lambda) \cdot \|x - y\|^q$$

with a constant

$$d = \begin{cases} \dfrac{p-1}{8} & \text{for } 1 < p \leq 2 \\ \dfrac{1}{p \cdot 2^p} & \text{for } 2 < p < \infty. \end{cases}$$

Now, from Theorem 1, we have the following result.

THEOREM 4. *Let K be a nonempty bounded closed convex subset of the space E, where $E = H^p$, or $E = H^{k,p}(\Omega)$, or $E = L_{q,p}$, or $E = L_q(L_p)$, and $1 < p < \infty$, $q = \max\{2, p\}$, $k \geq 0$. Let $T : K \to K$ be an asymptotically regular mapping which holds the inequality* (1) *such that*

$$\left[\frac{(\alpha+\beta)^q \cdot (2^{q-1} \cdot \alpha^q - 1)}{(d - 2^{q-1} \cdot \beta^q) \cdot N^q}\right]^{\frac{1}{q}} < 1,$$

where α, β as in Theorem 1. *Suppose that there is a z_0 in K for which $\{T^n z_0\}$ is bounded. Then T has a fixed point in K.*

If we put $b_n = c_n = 0$ in (1), then from Theorem 4, we have the following result:

COROLLARY 3 ([11, Corollary 5]). *Let K be a nonempty bounded closed convex subset of the space E, where E is as in* Theorem 4. *If $T : K \to K$ is an asymptotically regular mapping such that*

$$\liminf_{n\to\infty} |||T^n||| = k < \left[\frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot d \cdot N^q}\right)\right]^{\frac{1}{q}},$$

then T has a fixed point in K.

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