Bull. Korean Math. Soc. **37** (2000), No. 4, pp. 711–719

A GENERALIZATION OF GIESEKER'S LEMMA

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Abstract. We generalize Gieseker's lemma and use it to compute Picard number of a complete intersection surface.

1. Introduction

We work over the complex numbers $\mathbb{C}$. In [2], J. Harris gave a proof of the following Gieseker's lemma using monodromy:

Gieseker's Lemma. *Let $W \subseteq H^0(O_{\mathbb{P}^1}(d-1))$ be a linear system and $V \subseteq H^0(O_{\mathbb{P}^1}(d))$ be a linear system containing the image of W under the multiplication map μ*

$$\mu : W \otimes H^0(O_{\mathbb{P}^1}(1)) \to H^0(O_{\mathbb{P}^1}(d)).$$

Then either $\dim V \geq \dim W + 2$ *or $|V|$ equals the complete series $|O_{\mathbb{P}^1}(l-1)|$ plus $d-l+1$ fixed points, where $l = \dim V$.*

Though this looks simple, it has been used explicitly and implicitly in the proofs of important results. (See, for example, [6]). We generalize the lemma as follows:

Theorem 1. *Let $2 \leq d_1 \leq \cdots \leq d_{n-2}$, $n \geq 3$ and $E = \bigoplus_{j=1}^{n-2} O_{\mathbb{P}^1}(d_j)$. Let $W \subset H^0(\mathbb{P}^1, E)$ denote a subspace such that the evaluation map*

$$f : W \otimes O_{\mathbb{P}^1,x} \to E_x$$

is surjective for all $x \in \mathbb{P}^1$ and $\operatorname{codim} W \geq 1$. *Let μ denote the multiplication map*

Received January 29, 2000. Revised March 29, 2000.
2000 Mathematics Subject Classification: 14J10, 14C30.
Key words and phrases: Gieseker's lemma, Picard number.

$$\mu : W \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1)) \to H^0(\mathbb{P}^1, E \otimes O_{\mathbb{P}^1}(1)).$$

Then $\dim(\mathrm{im}\mu(W \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1))) \geq m + n - 1$, *where* $m = \dim W \geq 1$.

We will give an elementary proof of this theorem. As an application of this theorem, we will show that the Picard number of a general complete intersection surface in $\mathbb{P}^n$ containing a line is 2.

2. Proof of the Theorem 1

GENERAL STRATEGY. *We will show that a basis of* W *can be divided into at least* $n-1$ *disjoint subsets with the property that the map* μ *operates on each disjoint subset creating* 1 *extra dimension, resp. Here, we note that for the above map* f *to be surjective as in the hypothesis,* $\dim W \geq n - 1$.

NOTATION. *Let* z_0, z_1 *denote homogeneous coordinates for* $\mathbb{P}^1$. *Let* $B = \{v_1, \dots, v_m\}$ *be a basis of* W *in the* Theorem 1. *Each* v_i *can be written as*

$$v_i = z_0^{i_0} z_1^{i_1} (p_i^1, p_i^2, \dots, p_i^{n-2}),$$

where neither z_0 *nor* z_1 *is a common factor of* $p_i^1, p_i^2, \dots, p_i^{n-2}$. *We denote by*

$$p_i^* = (p_i^1, p_i^2, \dots, p_i^{n-2}).$$

Here, p_i^k *is a homogeneous polynomial of degree* $d_k - (i_0 + i_1)$ *for* $i = 1, \dots, m$, *and* $k = 1, \dots, n-2$.

DEFINITION 1. Define for $v_i, v_j \in B, i \neq j$,

$$\mathrm{Edge}(v_i, v_j) = 1 \quad \text{if } i_0 - j_0 = j_1 - i_1 \text{ and } p_i^* = p_j^*$$

$$\mathrm{Edge}(v_i, v_j) = 0 \quad \text{otherwise.}$$

Note that $\mathrm{Edge}(v_i, v_j) = \mathrm{Edge}(v_j, v_i)$ and that for each $v_j \in B$, there can be at most two v_i's in B such that $\mathrm{Edge}(v_i, v_j) = 1$.

DEFINITION 2. For a $v_l \in B$, we define a subset $[v_l]$ of B recursively as follows:

(1) $v_l \in [v_l]$.

(2) $v_i \in [v_l]$ if $\mathrm{Edge}(v_l, v_i) = 1$, and $v_i \in B - [v_l]$.

(3) $v_j \in [v_l]$ if $\mathrm{Edge}(v_k, v_j) = 1$ for some $v_k \in [v_l]$, $v_j \in B - [v_l]$.

(4) Repeat (3) until there remains no such v_j's.

FACTS. One can easily observe the following facts:
1. Each element v_k of $[v_l]$ can be written as $v_k = z_0^{k_0} z_1^{k_1} p_l^*$. That is, k_o and k_1 depend on v_k, but $p_k^* = p_l^*$ for any $v_k \in [v_l]$.
2. If there is a $v_k \in B - [v_l]$, one can construct another subset $[v_k]$. By construction, $[v_l]$ is disjoint with $[v_k]$. Also, for any element v_i of $[v_l]$ and any element v_j of $[v_k]$, $z_h v_i \neq z_\nu v_j$, where $h, \nu \in \{0, 1\}$. Thus W can be divided into disjoint subset $[v_i]$'s.
3. The map μ operates on each disjoint $[v_i]$ creating 1 extra dimension respectively. That is, let $V_i \subset H^0(\mathbb{P}^1, E)$ be the subspace generated by the elements of $[v_i]$. Then $\dim V_i = |[v_i]|$ = the number of elements of v_i and

$$\dim\Big(\mathrm{im}\mu\big(V_i \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1)\,)\big)\Big) = \dim V_i + 1.$$

Moreover, if B is the union of disjoint subsets, say $[v_1], \ldots, [v_k]$, then

$$\begin{aligned}
&\dim\Big(\mathrm{im}\mu\big(W \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1)\,)\big)\Big)\\
&= \sum_{i=1}^{k} \dim\Big(\mathrm{im}\mu\big(V_i \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1))\big)\Big)\\
&= \sum_{i=1}^{k} (|[v_i]| + 1) = \dim W + k.
\end{aligned}$$

From the above facts, we can see that to prove the theorem, all we need to show is the following:

LEMMA 1. *There are at least $n-1$ disjoint subset $[v_i]$'s in B.*

Proof. We will show this for $n = 3$ and then for any $n \geq 4$. Though the proof need not be separated for these 2 cases, we provide the proof for $n = 3$ as an illustration for the idea of the proof.

A. For $n = 3$, we claim that B is the union of at least 2 disjoint subsets, say, $[v_1]$ and $[v_k]$. If not, then $B = [v_1]$ and $v_i = z_0^{i_0} z_1^{i_1} p_1^1$, $1 \leq i \leq m$, and p_1^1 is a homogeneous polynomial which is not divisible by either z_0 or z_1. Moreover, v_i's can be rearranged so that

$$\begin{aligned} v_1 &= z_0^{\alpha+m} z_1^{\beta} p_1^1 \\ v_2 &= z_0^{\alpha+m-1} z_1^{\beta+1} p_1^1 \\ &\vdots \\ v_m &= z_0^{\alpha+1} z_1^{\beta+m-1} p_1^1 \end{aligned}$$

for an integer $\alpha \geq -1$ and for some nonnegative integer β.

If $\deg p_1^1 \geq 1$, then a zero of p_1^1 is a base point of W, at which the map f is not surjective.

If $\deg p_1 = 0$, then $m + \alpha + \beta = d_1$. For the evaluation map f to be surjective at $P = (1, 0)$ and at $Q = (0, 1)$, we should have $\beta = 0$ and $\alpha = -1$. Hence we get $m - 1 = d_1$, i.e., $\operatorname{codim} W = 0$, which is a contradiction. □

B. If $n \geq 4$, we will show that B is a union of at least $n - 1$ disjoint $[v_i]$'s.

If B is a union of k disjoint $[v_i]$'s, then without loss of generality, we may assume that $B = \cup_{i=1}^k [v_i]$ and

$$\begin{aligned} [v_1] &= \{ z_0^{\gamma_1} z_1^{\delta_1} p_1^*, z_0^{\gamma_1 - 1} z_1^{\delta_1 + 1} p_1^*, \dots, z_0^{\gamma_1 - \alpha_1} z_1^{\gamma_1 + \alpha_1} p_1^* \} \\ &\vdots \\ [v_k] &= \{ z_0^{\gamma_k} z_1^{\delta_k} p_k^*, z_0^{\gamma_k - 1} z_1^{\delta_k + 1} p_k^*, \dots, z_0^{\gamma_k - \alpha_k} z_1^{\delta_k + \alpha_k} p_k^* \} \end{aligned}$$

for nonnegative integers γ_i, δ_i, and α_i, $1 \leq i \leq k$ satisfying the following conditions:

(a) $\sum_{i=1}^k (\alpha_i + 1) = m$

(b) $\gamma_i - \alpha_i \geq 0$

(c) Fix a j with $1 \leq j \leq n-2$. Then,

$\gamma_i + \delta_i + \deg p_i^j = d_j$ for any i with $p_i^j \neq 0$.

We will show $k \geq n-1$.

(1) If $k \leq n-3$, then, at $P = (z_0, z_1)$ with $z_0 \neq 0$ and $z_1 \neq 0$, the rank of the evaluation map f is at most $k \leq n-3$, which contradicts the hypothesis.

(2) If $k = n-2$, then we will find a point where the map f is not surjective. We consider the following matrix:

$$\begin{pmatrix} p_1^1 & p_1^2 & \cdots & p_1^{n-2} \\ p_2^1 & p_2^2 & \cdots & p_2^{n-2} \\ \vdots & \vdots & \ddots & \\ p_{n-2}^1 & p_{n-2}^2 & \cdots & p_{n-2}^{n-2} \end{pmatrix}$$

(i) If $\deg p_i^j > 0$ for some i and j, then at the zeros of the determinant of the above matrix, the evaluation map f is not surjective.

(ii) If $\deg p_i^j = 0$ for every i and j in $\{1, \ldots, n-2\}$, then either $p_i^j = 0$ or $p_i^j = 1$. If the rank of the above matrix is $< n-2$, then the map f is not surjective at any point $P = (z_0, z_1)$ where $z_0 \neq 0$ and $z_1 \neq 0$. So it contradicts the hypothesis of the theorem and the proof is done. But, for the above matrix to be of rank $n-2$, the determinant of the matrix should not be equal to 0. This can happen when

$$p_1^{j_1} p_2^{j_2} \ldots p_{n-2}^{j_{n-2}} \neq 0$$

for at least one permutation $(j_1 \ldots j_{n-2})$ of $\{1, ..., n-2\}$. In this case, $\deg p_i^j = 0$ implies $d_{j_i} = \gamma_i + \delta_i$ by the above condition (c). For the map f to be surjective at $(1,0)$ and at $(0,1)$, $\delta_i = 0$ for all i and $\gamma_i = \alpha_i$. So

$$\begin{aligned} m = \sum_{i=1}^{n-2} (\alpha_i + 1) &= \sum_{i=1}^{n-2} (\gamma_i + 1) \\ &= \sum_{i=1}^{n-2} (d_{j_i} + 1) = \sum_{i=1}^{n-2} (d_i + 1). \end{aligned}$$

This implies $\operatorname{codim} W = 0$, which is a contradiction. □

3. An Application

Using, the above theorem, we will show that the Picard number of a general complete intersection surface S containing a line in $\mathbb{P}^n$ is 2, that is, Pic(S) is generated by the hyperplane section curve and the line.

Let $2 \leq d_1 \leq d_2 \leq \cdots \leq d_{n-2}$ and $Y_{n,d_1,d_2,\cdots,d_{n-2}} = \Big\{$ smooth complete intersection surfaces of type $(d_1, \ldots, d_{n-2})$ in $\mathbb{P}^n \Big\}$. The *Noether-Lefschetz locus* is $\Sigma = \{S \in Y_{n,d_1,d_2,\cdots,d_{n-2}} \mid \mathrm{Pic}(S) \not\cong \mathbb{Z}\}$.

THEOREM 2. *Let $\sum_{i=1}^{n-2} d_i \geq n+2$ and $n \geq 3$. Let Z_1 denote an irreducible component of Σ whose generic member contains a line. Then, for a general S in Z_1, the* Picard *number is* 2.

The word "general" is used in the sense that a property is said to hold at a general point of a projective variety V if the property holds at all the points of V but the points in a countable union of subvarieties of V.

It is known that the codimension of Z_1 is $\geq \sum_{i=1}^{n-2} d_i - n$(cf. [3]).

Using deformation theoretic technique, Lopez [5] figured the generators of the Picard group of a general complete intersection surface containing a fixed curve. In [5], he showed that for a general projectively Cohen-Macaulay surface X in $\mathbb{P}^4$ defined by the maximal minors of a matrix with no zeros, $Pic(X) \cong \mathbb{Z}^2$ generated by $O_X(1)$ and K_X unless X is the Castelnuovo or Bordiga surface. He [6] also gave a new proof of the above Theorem 2 for a general surface in $\mathbb{P}^3$ containing a plane curve, which is infinitesimal Hodge theoretic and completely different from the one in [5].

Following Lopez's idea for the case $n = 3$ in [6], we can reduce Theorem 2 to Theorem 1.

Proof of Theorem 2. Let $z_0, \ldots, z_n$ denote homogeneous coordinates for $\mathbb{P}^n$. C be the line with equations $z_0 = z_1 = \cdots = z_{n-2} = 0$ in $\mathbb{P}^n$. For a generic $S \in Z_1$, let $S = \bigcap_{i=1}^{n-2}\{F_i = 0\}$, where $F_i = \sum_{j=0}^{n-2} z_j G_j^i = 0$ and F_i is an irreducible homogeneous polynomial of degree d_i, $i = 1, \ldots, n-2$. Without loss of generality, we may assume that $\bigcap_{j=1}^{k}\{F_j = 0\}$ are smooth for $k = 1, \dots, n-2$, and that, for $i = 1, \dots, n-2$, $\bigcap_{j=0}^{n-2}\{G_j^i = 0\} \cap C = \phi$.

Let $H^{1,1}_{prim}(S) \subset H^1(S, \Omega^1_S)$ denote the primitive $(1,1)$-cohomology of S. Let $L = O_S(C)$. $\gamma = c_1(L) \in H^{1,1}_{prim}(S)$ defines an extension M of the tangent sheaf Θ_S of S by the structure sheaf O_S, i.e. M is defined by the exact sequence

$$0 \to O_S \to M_S \to \Theta_S \to 0$$

with the extension class γ. The induced map $H^1(S, \Theta_S) \to H^2(S, O_S)$ is given by the cup product with γ. By dualizing the map,

$$H^1(S, \Theta_S) \otimes H^{1,1}_{prim}(S) \to H^2(S, O_S),$$

we get

$$H^1(S, \Theta_S) \otimes H^{2,0}(S) \to H^{1,1}_{prim}(S)^*.$$

Let $E = \oplus_{i=1}^{n-2} O_{\mathbb{P}^n}(d_i)$, $E(k) = E \otimes O_{\mathbb{P}^n}(k)$, and ν denote the number $\nu = \sum_{i=1}^{n-2} d_i - n - 1$.

By algebraic identifications (cf. [1] or [5] for $n = 3$, [3] for $n \geq 4$), the above map is the multiplication map

$$\frac{H^0(\mathbb{P}^n, E)}{J} \otimes \frac{H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu))}{I} \to \frac{H^0(\mathbb{P}^n, E(\nu))}{J'}$$

where J, I, J' denote the appropriate subspaces; For $n = 3$, J is the Jacobian ideal of S in degree d_1 and $I = 0$. For $n \geq 4$, $I = im H^0(\mathbb{P}^n, \oplus_{i=1}^{n-2} O_{\mathbb{P}^n}(\nu - d_i)) = \{\sum_{i=1}^{n-2} a_i F_i | a_i \in H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu - d_i))\}$, $a_i = 0$ for $\nu - d_i \leq 0$. J is generated by $\{\frac{\partial F_i}{\partial z_j} e_i | 1 \leq i \leq n-2, 0 \leq j \leq n\}$. Here $e_i = (e_i^1, \dots, e_i^{n-2})$, $e_i^k = 1$ if $i = k$, $e_i^k = 0$ otherwise (For precise definitions, see [3]).

Let $W'_\gamma \subset H^1(S, \Theta_S)$ be the Zariski tangent space to Z_1 keeping γ of type $(1,1)$. Then the image of $W'_\gamma \otimes H^{2,0}(S)$ is contained in $(\gamma)^\perp$. Let W_γ be the preimage of W'_γ under the projection $H^0(\mathbb{P}^n, E) \to \frac{H^0(\mathbb{P}^n, E)}{J}$, and $R_{J'} = \frac{H^0(\mathbb{P}^n, E(\nu))}{J'}$. Then we have a map

$$\lambda : W_\gamma \otimes H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu)) \to H^0(\mathbb{P}^n, E(\nu)) \to R_{J'}.$$

It is known that the evaluation map $W_\gamma \otimes O_{\mathbb{P}^n, x} \to E_x$ is surjective for every $x \in \mathbb{P}^n$(cf. [3]).

LEMMA 2. $\operatorname{codim}_{R'_J} \operatorname{im}\lambda(W_\gamma \otimes H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu))) \leq 1$.

Proof. Let $W = \operatorname{im}(W_\gamma \otimes H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu)) \to H^0(\mathbb{P}^n, E(\nu)))$ and $R = H^0\Big(\mathbb{P}^n,\ E(\nu)\Big)$. By definition, $J' \subset W$ and hence it is enough to show $\operatorname{codim}_R W \leq 1$.

Let $R|_C$ be the restriction of R to C, and $W|_C, W_\gamma|_C$, the restriction of W, W_γ to C, resp. Recall that $C = \mathbb{P}^1$ with homogeneous coordinates z_{n-1}, z_n.

Note $R|_C = H^0(\mathbb{P}^1,\ E(\nu) \otimes O_{\mathbb{P}^1})$, and $W|_C = \operatorname{im}(W_\gamma|_C \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\nu)) \to R|_C)$. Let I_C be the ideal sheaf of C, and $I(C) = \operatorname{im}(H^0(\mathbb{P}^n, I_C \otimes E) \to H^0(E))$. By construction, $I(C) \subset W_\gamma$ and this implies $\operatorname{codim}_R W = \operatorname{codim} R|_C W|_C$. So it suffices to show $\operatorname{codim} R|_C W|_C \leq 1$.

On the other hand, $\{z_k(G_j^1, \dots, G_j^{n-2}) | 0 \leq j \leq n-2, k = n-1, n\} \subset W_\gamma|_C$ and so $W_\gamma|_C = W' \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(1))$ for some $W' \subset H^0(\mathbb{P}^1, E \otimes O_{\mathbb{P}^1}(-1))$ containing $\{(G_j^1, \dots, G_j^{n-2}) \mid 0 \leq j \leq n-2\}$. So the evaluation map of W' is surjective and $\dim W' \geq n-1$. By applying Theorem 1 $(\nu+1)$ times,

$$\dim W|_C = \dim\Big(\operatorname{im}\big(W' \otimes H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\nu+1))\big)\Big) \geq n-1+(n-1)(\nu+1).$$

Hence $\operatorname{codim}_{R|_C} W|_C \leq 1$. □

The rest of the proof of the theorem uses the idea of Lopez's proof for $n = 3$ which we restate: By the semicontinuity theorem, it is enough to prove that for each $\gamma' \in H^{1,1}_{prim}(S) - \mathbb{C}\gamma$, there exists a deformation $\eta \in W_\gamma$ such that, when we deform S in the direction of η to a surface S', the class γ' is not of type $(1,1)$. That is, it is enough to show $W_\gamma \not\subseteq W_{\gamma'}$. By Lemma 2, $\gamma' \in \operatorname{im}(W_\gamma \otimes H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu)))$. Therefore, if $\gamma' \neq 0$ and $W_\gamma \subset W_{\gamma'}$, then

$$\gamma' \in \operatorname{im}(W_\gamma \otimes H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu))) \subset \operatorname{im}(W_{\gamma'} \otimes H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(\nu))) \subset (\gamma')^\perp,$$

which is a contradiction. □

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