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MULTIPLICITY OF SOLUTIONS AND SOURCE TERMS IN A NONLINEAR PARABOLIC EQUATION UNDER DIRICHLET BOUNDARY CONDITION

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ABSTRACT. We investigate the existence of solutions of the nonlinear heat equation under Dirichlet boundary condition on Ω and periodic condition on the variable t, $Lu - D_t u + g(u) = f(x,t)$. We also investigate a relation between multiplicity of solutions and the source terms of the equation.

0. Introduction

In this paper, we investigate multiplicity of solutions $u(x,t)$ for a nonlinear perturbation $g(u)$ of the parabolic operator $(L-D_t)$ under Dirichlet boundary condition on Ω and periodic condition on the variable t,

$$
\begin{array}{rcll}
Lu - D_t u + g(u) & = & f(x,t) & \text{in } \Omega \times R, \\
u & = & 0 & \text{on } \partial\Omega, \\
u(x,t) & = & u(x,t+T), &
\end{array}
\tag{0.1}
$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$ and the nonlinear perturbation $g(u)$ is piecewise linear one $bu^+ - au^-$ with $a < \lambda_{01} < b < \lambda_{02}$. Here L is a second order elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact linear inverse, with

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eigenvalues $-\lambda_i$, each repeated as often as multiplicity

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \longrightarrow +\infty.$$

Let H be the Hilbert space defined by

$$H = \{u \in L^2(\Omega \times [0,T])| \quad \text{u is } T-\text{periodic in } t\}.$$

Then equation (0.1) is represented by

$$Lu - D_t u + bu^+ - au^- = f(x,t) \text{ in } H. \tag{0.2}$$

In [6], the author showed by degree theory that equation (0.2), with the forcing term f is supposed to be a multiple of the first positive eigenfunction, has at least two solutions if n is even, and at least three solutions if n is odd.

We suppose that $a < \lambda_{01} < b < \lambda_{02}$ and the source term f is generated by φ_{01} and φ_{02}. Our goal is to investigate a relation between multiplicity of solution and source terms in equation (0.2) when f belongs to the two-dimensional subspace of H that spanned by φ_{01} and φ_{02}.

Let V be the two dimensional subspace of H spanned by φ_{01} and φ_{02}. Let P be the orthogonal projection H onto V. Let $\Phi : V \to V$ be a map (cf. (1.7)) defined by

$$\Phi(v) \doteq Lv - D_t v + P(b(v+\theta(v))^+ - a(v+\theta(v))^-), \quad v \in V.$$

In section 1, we suppose that the nonlinearity $-(bu^+ - au^-)$ crosses the eigenvalue λ_{01}. And we use the variational reduction method to reduce the problem from an infinite dimensional one to a finite dimensional one. In section 2, we investigate the properties of the map Φ and we reveal a relation between multiplicity of solutions and source terms in equation (0.2) when $f(x,t)$ belongs to the two-dimensional space V.

1. A variational reduction

We consider the parabolic equation under Dirichlet boundary condition and periodic condition on the variable t,

$$(1.1)\qquad \begin{aligned} Lu - D_t u + g(u) &= f(x,t) && \text{in } \Omega \times R,\\ u &= 0 && \text{on } \partial\Omega,\\ u(x,t) &= u(x,t+T). \end{aligned}$$

Here the nonlinear term $g(u)$ is piecewise linear $bu^+ - au^-$ with $a < \lambda_{01} < b < \lambda_{02}$. We consider the boundary problem

$$(1.2)\qquad \begin{aligned} Lu - D_t u + bu^+ - au^- &= f(x,t) && \text{in } \Omega \times R,\\ u &= 0 && \text{on } \partial\Omega,\\ u(x,t) &= u(x,t+T). \end{aligned}$$

We denote φ_n to be the eigenfunctions corresponding to eigenvalues λ_n and $\varphi_1(x) > 0$ in Ω. Let H be the Hilbert space defined by

$$H = \{u \in L^2(\Omega) \times [0,T] \mid u \text{ is } T\text{-periodic in } t\}.$$

Then the set $\{\varphi_{mn} = \frac{1}{\sqrt{2\pi}}\varphi_n(x)e^{imt} \mid n \geq 1, m = 0, \pm 1, \pm 2, \cdots\}$ is orthogonal in H and $\varphi_{01} > 0$.

We are concerned with the multiplicity of solutions of (1.2) only when f is generated by the eigenfunctions φ_{01} and φ_{02}. That is, we study the equation

$$(1.3)\qquad Lu - D_t u + bu^+ - au^- = f \text{ in } H,$$

where $f = s_1\varphi_{01} + s_2\varphi_{02} (s_1, s_2 \in R)$.

THEOREM 1.1. *If $s_1 < 0$, then (1.3) has no solution.*

Proof. We rewrite (1.3) as

$$(L - D_t + \lambda_{01})u + (b - \lambda_{01})u^+ - (a - \lambda_{01})u^- = s_1\varphi_{01} + s_2\varphi_{02}.$$

Multiply across by φ_{01} and integrate over H. Since $(L - D_t + \lambda_{01})\varphi_{01} = 0$ and $((L - D_t + \lambda_{01})u, \varphi_{01}) = 0$, we have

$$\int_\Omega \{(b - \lambda_{01})u^+ - (a - \lambda_{01})u^-\}\varphi_{01} = (s_1\varphi_{01} + s_2\varphi_{02}, \varphi_{01}) = s_1 \int_\Omega \varphi_{01}^2 = s_1.$$

However, we know that $(b - \lambda_{01})u^+ - (a - \lambda_{01})u^- \geq 0$ for all real valued function u. Also $\varphi_{01} > 0$ in H. Therefore

$$\int_\Omega \{(b - \lambda_{01})u^+ - (a - \lambda_{01})u^-\}\varphi_{01} \geq 0.$$

Hence, there is no solution of (1.3) if $s_1 < 0$. □

To study equation (1.3), we use the contraction mapping theorem to reduce the problem from an infinite-dimensional one to a finite-dimensional one.

Let V be two-dimensional subspace of H spanned by $\{\varphi_{01}, \varphi_{02}\}$ and W be the orthogonal complement of V in H. Let P be the orthogonal projection of H onto V. Then every $u \in H$ can be written as $u = v + w$, where $v = Pu$ and $w = (I - P)u$. Hence equation (1.3) is equivalent to a system

$$Lw - D_t w + (I - P)(b(v+w)^+ - a(v+w)^-) = 0, \tag{1.4}$$

$$Lv - D_t v + P(b(v+w)^+ - a(v+w)^-) = s_1\varphi_{01} + s_2\varphi_{02}. \tag{1.5}$$

LEMMA 1.2. *For a fixed $v \in V$, equation (1.4) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is* Lipschitz *continuous (with respect to the L^2-norm) in v.*

The proof of the lemma is similar to that of [5].

By Lemma 1.2, the study of multiplicity of solutions of (1.3) is reduced to one of an equivalent problem

$$Lv - D_t v + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) = s_1\varphi_{01} + s_2\varphi_{02} \tag{1.6}$$

defined on the two dimensional subspace V spanned by $\{\varphi_{01}, \varphi_{02}\}$.

While one feels intuitively that (1.6) ought to be easier to solve than (1.3), there is the disadvantage of an implicitly defined term $\theta(v)$ in the equation. However, in our case, it turns out that we know $\theta(v)$ for some special c's.

COROLLARY. *If $v \geq 0$ or $v \leq 0$, then $\theta(v) \equiv 0$.*

Proof. Now, take $v \geq 0$ and $\theta(v) = 0$ since $v \in V, (I - P)v = 0$. Then equation (1.4) is reduced to

$$(L - D_t) \cdot 0 + (I - P)(bv^+ - av^-) = 0$$

because $v^+ = v, v^- = 0$ and $(I - P)v = 0$. By Lemma 1.2, $\theta(v) \equiv 0$. □

Since $V = span\{\varphi_{01}, \varphi_{02}\}$ and φ_{01} is a positive eigenfunction, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\varphi_{01} + c_2\varphi_{02} | \quad c_1 \geq 0, |c_2| \leq \varepsilon_0 c_1\}$$

for some $\varepsilon_0 > 0$, so that $v \geq 0$ for all $v \in C_1$, and a cone C_3 defined by

$$C_3 = \{v = c_1\varphi_{01} + c_2\varphi_{02} | \quad c_1 \leq 0, |c_2| \leq \varepsilon_0 |c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$. Thus, we do not know $\theta(v)$ for all $v \in PH$, but we know $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$. And C_2 and C_4 are defined as follows

$$\begin{aligned} C_2 &= \{v = c_1\varphi_{01} + c_2\varphi_{02} | c_2 \geq 0, c_2 \geq \varepsilon_0 |c_1|\}, \\ C_4 &= \{v = c_1\varphi_{01} + c_2\varphi_{02} | c_2 \leq 0, |c_2| \geq \varepsilon_0 |c_1|\}. \end{aligned}$$

Then the union of C_1, C_3 and C_2, C_4 is the space V. Now we define a map $\Phi : V \longrightarrow V$ given by

$$\Phi(v) = Lv - D_t v + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), v \in V. \tag{1.7}$$

Then Φ is continuous on V, since θ is continuous on V and we have the following lemma.

LEMMA 1.3. *For $v \in V$ and $c \geq 0$, $\Phi(cv) = c\Phi(v)$.*

Proof. Let $c \geq 0$. If v satisfies

$$L\theta(v) - D_t\theta(v) + (I - P)(b(v + \theta(v))^+ - a(v + \theta(v))^-) = 0,$$

then

$$L(c\theta(v) - D_t(c\theta(v)) + (I - P)(b(cv + c\theta(v))^+ - a(cv + c\theta(v))^-) = 0$$

and hence $\theta(cv) = c\theta(v)$. Therefore we have

$$\begin{aligned} \Phi(cv) &= L(cv) - D_t(WV) + P(b(cv + \theta(cv))^+ - a(cv + \theta(cv))^-) \\ &= L(cv) - D_t(cv) + P(b(cv + c\theta(v))^+ - a(cv + c\theta(v))^-) \\ &= cL(v) - cD_t v + cP(b(v + \theta(v))^+ - a(v + \theta(v))^-) \\ &= c\Phi(v). \end{aligned}$$

□

2. Multiplicity of solutions and source terms

Now we investigate the image of the cone C_1, C_3 under Φ. First we consider the image of C_1 under Φ. If $v = c_1\varphi_{01} + c_2\varphi_{02}$, then we have

$$\begin{aligned}\Phi(v) &= Lv - D_t v + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) \\ &= -c_1\lambda_{01}\varphi_{01} - c_2\lambda_{02}\varphi_{02} + b(c_1\varphi_{01} + c_2\varphi_{02}) \\ &= c_1(b-\lambda_{01})\varphi_{01} + c_2(b-\lambda_{02})\varphi_{02}.\end{aligned}$$

Thus the image of the rays $c_1\varphi_{01} \pm \varepsilon_0 c_2\varphi_{02} (c_1 \geq 0)$ can be explicitly calculated and they are

$$c_1(b-\lambda_{01})\varphi_{01} \pm \varepsilon_0 c_1(b-\lambda_{02})\varphi_{02} \quad (c_1 \geq 0).$$

Therefore if $a < \lambda_{01} < b < \lambda_{02}$, then Φ maps C_1 onto the cone

$$R_1 = \left\{ d_1\varphi_{01} + d_2\varphi_{02} \,|\, \quad d_1 \geq 0, |d_2| \leq \varepsilon_0 \left(\frac{\lambda_{02} - b}{b - \lambda_{01}}\right) d_1 \right\}.$$

Second, we consider the image of C_3. If $v = -c_1\varphi_{01} + c_2\varphi_{02} \leq 0 \quad (c_1 \geq 0, |c_2| \leq \varepsilon_0 c_1)$, then we have

$$\begin{aligned}\Phi(v) &= Lv - D_t v + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) \\ &= Lv - D_t v + P(av) \\ &= c_1\lambda_{01}\varphi_{01} - c_2\lambda_{02}\varphi_{02} - ac_1\varphi_{01} + ac_2\varphi_{02} \\ &= c_1(\lambda_{01} - a)\varphi_{01} + c_2(a - \lambda_{02})\varphi_{02}.\end{aligned}$$

Thus the image of the rays $-c_1\varphi_{01} \pm \varepsilon_0 c_1\varphi_{02}$ can be explicitly calculated and they are

$$c_1(\lambda_{01} - a)\varphi_{01} \pm \varepsilon_0 c_1(a - \lambda_{02})\varphi_{02} \;\; (c_1 \geq 0).$$

Therefore Φ maps C_3 onto the cone

$$R_3 = \left\{ d_1\varphi_{01} + d_2\varphi_{02} \,|\, \quad d_1 \geq 0, |d_2| \leq \varepsilon_0 \left(\frac{\lambda_{02} - a}{\lambda_{01} - a}\right) d_1 \right\}.$$

Here we have three cases, which are $R_1 \subset R_3$, $R_3 \subset R_1$, and $R_1 = R_3$. The first relation $R_1 \subset R_3$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b > \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$. The second relation $R_3 \subset R_1$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b < \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$.

The last case $R_1 = R_3$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b = \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$.

LEMMA 2.1. *For every* $v = c_1\varphi_{01} + c_2\varphi_{02} \in V$, *there exists a constant* $d > 0$ *such that* $(\Phi(v), \varphi_{01}) \geq d|c_2|$.

Lemma 2.1 tells us that the image of Φ is contained in the right half-plane of V. That is, $\Phi(C_2)$ and $\Phi(C_4)$ are the cone in the right half-plane of V.

We consider the restriction $\Phi|_{C_i}(1 \leq i \leq 4)$ of Φ to the cone C_i. Let $\Phi_i = \Phi|_{C_i}(0 \leq i \leq 4)$, i.e.,

$$\Phi_i : C_i \longrightarrow V.$$

First, we consider Φ_1. It maps C_1 onto R_1. Let l_1 be the segment defined by

$$l_1 = \left\{ \varphi_{01} + d_2\varphi_{02} \mid \ |d_2| \leq \varepsilon_0 \left(\frac{\lambda_{02} - b}{b - \lambda_{01}} \right) \right\}.$$

Then the inverse image $\Phi^{-1}(l_1)$ is the segment

$$L_1 = \Phi_1^{-1}(l_1) = \left\{ \frac{1}{b - \lambda_{01}}(\varphi_{01} + c_2\varphi_{02}) \mid \ |c_2| \leq \varepsilon_0 \right\}.$$

By Lemma 1.3, $\Phi_1 : C_1 \longrightarrow R_1$ is bijective.

Next we consider Φ_3. It maps C_3 onto R_3. Let l_3 be the segment defined by

$$l_3 = \left\{ \varphi_{01} + d_2\varphi_{02} \mid \ |d_2| \leq \varepsilon_0 \left(\frac{\lambda_{02} - a}{a - \lambda_{01}} \right) \right\}.$$

Then the inverse image $\Phi_3^{-1}(l_3)$ is the segment

$$L_3 = \Phi_3^{-1}(l_3) = \left\{ \frac{1}{a - \lambda_{01}}(\varphi_{01} + c_2\varphi_{02}) \mid \ |c_2| \leq \varepsilon_0 \right\}.$$

By Lemma 1.3, $\Phi_3 : C_3 \longrightarrow R_3$ is bijective.

2.1. The nonlinearity $-(bu^+ - au^-)$ satisfies $b > \frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$

The relation $R_1 \subset R_3$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b > \frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$. Now we find the images of the cones C_2 and C_4 under Φ, where

$$\begin{aligned} C_2 &= \{v = c_1\varphi_{01} + c_2\varphi_{02} \mid c_2 \geq 0, \varepsilon_0|c_1| \leq c_2\}, \\ C_4 &= \{v = c_1\varphi_{01} + c_2\varphi_{02} \mid c_2 \leq 0, \varepsilon_0|c_1| \leq |c_2|\}. \end{aligned}$$

By Theorem 1.1 and Lemma 1.2, the image of C_2 under Φ is a cone containing

$$R_2 = \left\{ d_1\varphi_{01} + d_2\varphi_{02} \mid d_1 \geq 0, \varepsilon_0\left(\frac{\lambda_{02}-b}{b-\lambda_{01}}\right)d_1 \leq d_2 \leq \varepsilon_0\left(\frac{\lambda_{02}-a}{\lambda_{01}-a}\right)d_1 \right\}$$

and the image of C_4 under Φ is a cone containing

$$R_4 = \left\{ d_1\varphi_{01} + d_2\varphi_{02} \mid d_1 \geq 0, -\varepsilon_0\left(\frac{\lambda_{02}-a}{\lambda_{01}-a}\right)d_1 \leq d_2 \leq -\varepsilon_0\left(\frac{\lambda_{02}-b}{b-\lambda_{01}}\right)d_1 \right\}.$$

We consider the restrictions Φ_2 and Φ_4, and define the segments l_2, l_4 as follows:

$$\begin{aligned} l_2 &= \left\{ \varphi_{01} + d_2\varphi_{02} \mid \varepsilon_0\left(\frac{\lambda_{02}-b}{b-\lambda_{01}}\right) \leq d_2 \leq \varepsilon_0\left(\frac{\lambda_{02}-a}{\lambda_{01}-a}\right) \right\}, \\ l_4 &= \left\{ \varphi_{01} + d_2\varphi_{02} \mid \varepsilon_0\left(\frac{a-\lambda_{02}}{\lambda_{01}-a}\right) \leq d_2 \leq \varepsilon_0\left(\frac{b-\lambda_{02}}{b-\lambda_{01}}\right) \right\}. \end{aligned}$$

We investigate the inverse image $\Phi_2^{-1}(l_2)$ and $\Phi_4^{-1}(l_4)$. Hence, we want to prove that Φ_2 and Φ_4 are surjective.

LEMMA 2.2. *Let $\gamma_i (i = 2, 4)$ be any simple path in R_i with end points on ∂R_i, where each ray (starting from the origin) in R_i intersects only one point of γ_i. Then the inverse image $\Phi_i^{-1}(\gamma_i)$ of γ_i is also a simple path in C_i with end points on ∂C_i, where any ray (starting from the origin) in C_i intersects only one point of this path.*

Proof. We note that $\Phi_i^{-1}(\gamma_i)$ is closed since Φ is continuous and γ_i is closed in V. Suppose that there is a ray (starting from the origin) in C_i which intersects two points of $\Phi_i^{-1}(\gamma_i)$, say p and $\alpha p (\alpha > 1)$. Then, by lemma 3.1.3 $\Phi_i(\alpha p) = \alpha\Phi_i(p)$ which implies that $\Phi_i(p) \in \gamma_i$ and

$\Phi_i(\alpha p) \in \gamma_i$. This contradicts the assumption that each ray (starting from the origin) in C_i intersects only one point of γ_i.

We regard a point p as a radius vector in the plane V. Then for a point p in V, we define the argument arg p of p by the angle from the positive φ_{01}-axis to p.

We claim that $\Phi_i^{-1}(\gamma_i)$ meets all the rays (starting from the origin) in C_i. If not, $\Phi_i^{-1}(\gamma_i)$ is disconnected in C_i. Since $\Phi_i^{-1}(\gamma_i)$ is closed and meet at most one point of any ray in C_i, there are two points p_1 and p_2 in C_2 such that $\Phi_i^{-1}(\gamma_i)$ does not contain any point $p \in C_i$ with

$$\arg p_1 < \arg p < \arg p_2.$$

On the other hand, if we set l be the segment with end points p_1 and p_2. then $\Phi_i(l)$ is a path in R_i, where $\Phi_i(p_1)$ and $\Phi_i(p_2)$ belong to γ_i. Choose a point q in $\Phi_i(l)$ such that arg q is between arg $\Phi_i(p_1)$ and arg $\Phi_i(p_2)$. Then there exist a point q' of γ_i such that $q' = \beta q$ for some $\beta > 0$. Hence $\Phi_i^{-1}(q)$ and $\Phi_i^{-1}(q')$ are on the same ray (starting from the origin) in C_i and

$$\arg p_1 < \arg \Phi_i^{-1}(q') < \arg p_2,$$

which is a contraction. This completes the proof. □

Lemma 2.2 implies that $\Phi_i (i = 2, 4)$ is surjective. Hence we have the following theorem.

THEOREM 2.3. *For $1 \le i \le 4$, the restriction Φ_i maps C_i onto R_i. Therefore, Φ maps V onto R_3. In particular, Φ_1 and Φ_3 are bijective.*

The above theorem also implies the following result.

THEOREM 2.4. *Suppose $a < \lambda_{01} < b < \lambda_{02}$ and $b > \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$. Let $f = s_1\varphi_{01} + s_2\varphi_{02}$. Then we have:*

(1) If $f \in \bar{R}_1$, then (1.3) has exactly two solutions, one of which is positive and the other is negative.

(2) If f belongs to interior of R_2 or interior of R_4, then (1.3) has a negative solution and at least one sign changing solution.

(3) If f belongs to boundary of R_3, then (1.3) has a negative solution.

(4) If f does not belong to R_3, then (1.3) has no solution.

2.2. The nonlinearity $-(bu^+ - au^-)$ satisfies $b < \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$

The relation $R_3 \subset R_1$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b < \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$. We investigate the image of the cones C_2 and C_4 under Φ, where

$$\begin{aligned} C_2 &= \{v = c_1\varphi_{01} + c_2\varphi_{02} \mid \ c_2 \geq 0, \varepsilon_0|c_1| \leq c_2\}, \\ C_4 &= \{v = c_1\varphi_{01} + c_2\varphi_{02} \mid \ c_2 \leq 0, \varepsilon_0|c_1| \leq |c_2|\}. \end{aligned}$$

By Theorem 1.1 and Lemma 1.2, the image of C_2 under Φ is a cone containing

$$R_2' = \left\{ d_1\varphi_{01} + d_2\varphi_{02} \mid \ d_1 \geq 0, \varepsilon_0 \left(\frac{\lambda_{02} - a}{\lambda_{01} - a} \right) d_1 \leq d_2 \leq \varepsilon_0 \left(\frac{\lambda_{02} - b}{b - \lambda_{01}} \right) d_1 \right\}$$

and the image of C_4 under Φ is a cone containing

$$R_4' = \left\{ d_1\varphi_{01} + d_2\varphi_{02} \mid \ d_1 \geq 0, \varepsilon_0 \left(\frac{b - \lambda_{02}}{b - \lambda_{01}} \right) d_1 \leq d_2 \leq \varepsilon_0 \left(\frac{a - \lambda_{02}}{\lambda_{01} - a} \right) d_1 \right\}.$$

We consider the restrictions Φ_2 and Φ_4, and define the segments l_2' and l_4' as follows:

$$\begin{aligned} l_2' &= \left\{ \varphi_{01} + d_2\varphi_{02} \mid \ \varepsilon_0 \left(\frac{\lambda_{02} - a}{\lambda_{01} - a} \right) \leq d_2 \leq \varepsilon_0 \left(\frac{\lambda_{02} - b}{b - \lambda_{01}} \right) \right\}, \\ l_4' &= .\left\{ \varphi_{01} + d_2\varphi_{02} \mid \ \varepsilon_0 \left(\frac{b - \lambda_{02}}{b - \lambda_{01}} \right) \leq d_2 \leq \varepsilon_0 \left(\frac{a - \lambda_{02}}{\lambda_{01} - a} \right) \right\}. \end{aligned}$$

We investigate the inverse images $\Phi_2^{-1}(l_2')$ and $\Phi_4^{-1}(l_4')$. We note that $\Phi_2(C_2)$ and $\Phi_4(C_4)$ contains R_2' and R_4'.

LEMMA 2.5. *For $i = 2, 4$, let γ' be a simple path in R_i' with end points on $\partial R_i'$, where each ray in R_i' (starting from the origin) intersects only one point of γ'. Then the inverse image $\Phi_i^{-1}(\gamma')$ of γ' is also simple path in C_i with end point on ∂C_i, where any ray in C_i (starting from the origin) intersects only one point of this path.*

Proof. The proof is similar to that of Lemma 2.2. □

Lemma 2.5 implies that Φ_2 and Φ_4 are surjective. Hence we have the following theorem.

THEOREM 2.6. *For $i = 2, 4$, the restriction Φ_i maps C_i onto R_i'. And Φ_1 and Φ_3 are bijective. Therefore, Φ maps V onto R_1.*

With the above theorem, we have the following results.

THEOREM 2.7. *Suppose $a < \lambda_{01} < b < \lambda_{02}$ and $b < \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$. Let $f = s_1\varphi_{01} + s_2\varphi_{02} \in V$. Then we have*

(1) If $f \in \bar{R}_3$, then (1.3) has exactly two solutions one of which is positive and the other is negative.

(2) If f belongs to interior of R_2' or interior R_4', then (1.3) has a negative solution and at least one sign changing solution.

(3) If f belongs to boundary of R_1, then (1.3) has a negative solution.

(4) If f does not belong to R_1, then (1.3) has no solution.

2.3. The nonlinearity $-(bu^+ - au^-)$ satisfies $b = \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$

The relation $R_1 = R_3$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b = \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$. Consider the map $\Phi : V \longrightarrow V$ defined by

$$\Phi(v) = Lv - D_t v + P\big(b(v + \theta(v))^+ - a(v + \theta(v))^-\big), \quad v \in V,$$

where $a < \lambda_{01} < b < \lambda_{02}$ and $b = \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$. Now we want to investigate the images of the cone C_2 and C_4 under Φ. For fixed v, we define a map

$$\Phi_v : (\lambda_{01}, \lambda_{02}) \longrightarrow V$$

as follows

$$\Phi_v(b) = Lv - D_t v + P\big(b(v + w)^+ - a(v + w)^-\big), \quad b \in (\lambda_{01}, \lambda_{02}),$$

where $v \in V$ and a is fixed.

LEMMA 2.8. *If a is fixed and $\lambda_{01} < b < \lambda_{02}$, then Φ_v is continuous at $b_0 = \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$.*

Proof. Let $\delta = \frac{a+b_0}{2}$ and $\lambda_{01} < b < \lambda_{02}$. Rewrite (1.4) as

$$(2.1) \quad (-L + D_t - \delta)w = (I - P)(b(v + w)^+ - a(v + w)^- - \delta(v + w)),$$

or equivalently

$$(2.2) \qquad w = (-L + D_t - \delta)^{-1}(I - P)g(b, w),$$

where

$$g(b, w) = b(v+w)^+ - a(v+w)^- - \delta(v+w).$$

By Lemma 1.2, (2.2) has a unique solution $w = \theta_b(v)$, for a fixed b. Let $w_0 = \theta_{b_0}(v)$. Then we have

$$\begin{aligned} w - w_0 &= S\big[g(b,w) - g(b_0, w_0)\big] \\ &= S\big[g(b,w) - g(b,w_0) + g(b,w_0) - g(b_0,w_0)\big] \\ &= S\big[g(b,w) - g(b,w_0)\big] \\ &\quad + S\big[g(b,w_0) - g(b_0,w_0)\big], \end{aligned}$$

where $S = (-L + D_t - \delta)^{-1}(I - P)$. Since

$$\|g(b,w) - g(b,w_0)\| \le \max\{|b-\delta|, |\delta - a|\}\|w - w_0\|$$

and

$$\gamma = \frac{1}{|\lambda_{02} - a|}\max\{|b-\delta|, \delta - a|\} < 1,$$

we have

$$\|w - w_0\| \le \gamma\|w - w_0\| + \frac{1}{|\lambda_{02} - a|}\|w - w_0\| \cdot |b - b_0|.$$

Hence

$$\|w - w_0\| \le \frac{1}{|\lambda_{02} - a||1-\gamma|}\|v + w_0\| \cdot |b - b_0|,$$

which shows that $\theta_b(v)$ is continuous at $b_0 = \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$. Thus $\Phi_v(b)$ is continuous at b_0. □

First, we investigate the image of the cone C_2 under Φ. Let $q_1 = \varphi_{01} + \varepsilon_0\frac{\lambda_{02}-b}{b-\lambda_{01}}\varphi_{02}$ and $q_2 = \varphi_{01} + \varepsilon_0\frac{\lambda_{02}-a}{\lambda_{01}-a}\varphi_{02}$. We fix a and define

$$\theta = \begin{cases} \arg q_1 - \arg q_2, & \text{if } b > \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a} \\ \arg q_2 - \arg q_1, & \text{if } b < \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}. \end{cases}$$

Then $0 \le \theta \le \frac{\pi}{2}$ and

$$\tan\theta = \left|\frac{\varepsilon_0(\lambda_{02}-b)(\lambda_{01}-a) - \varepsilon_0(\lambda_{02}-a)(b-\lambda_{01})}{(b-\lambda_{01})(\lambda_{01}-a) + \varepsilon_0^2(\lambda_{02}-b)(\lambda_{02}-a)}\right|.$$

When b converges to $\frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$, $\tan\theta$ converges to 0. Hence θ converges to 0 since $0 \le \theta \le \frac{\pi}{2}$. We note that Φ_2 maps C_2 onto R_2 when $b >$

$\frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$ and that Φ_2 maps C_2 onto R_2' when $b < \frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$. So if b converges to $\frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$, the angle of two lines consisting ∂R_2 and $\partial R_2'$ converges to 0. Since Φ_2 is continuous at $b = \frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$, Φ_2 maps C_2 onto the ray

$$S_2 = \left\{ d_1\varphi_{01} + d_2\varphi_{02} | \quad d_1 \geq 0, d_2 = \varepsilon_0 \left(\frac{\lambda_{02} - b}{b - \lambda_{01}} \right) d_1 \right\}.$$

Second we investigate the image of the cone C_4 under Φ. Let $\gamma_1 = \varphi_{01} - \varepsilon_0 \frac{\lambda_{02}-b}{b-\lambda_{01}} \varphi_{02}$ and $\gamma_2 = \varphi_{01} - \varepsilon_0 \frac{\lambda_{02}-a}{\lambda_{01}-a} \varphi_{02}$. We fix a. Define

$$\theta' = \begin{cases} \arg \gamma_1 - \arg \gamma_2, & \text{if } b > \frac{\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}; \\ \arg \gamma_2 - \arg \gamma_1, & \text{if } b < \frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}. \end{cases}$$

Then $0 \leq \theta' \leq \frac{\pi}{2}$ and

$$\tan \theta' = \left| \frac{\varepsilon_0(\lambda_{02} - b)(\lambda_{01} - a) - \varepsilon_0(\lambda_{02} - a)(b - \lambda_{01})}{(b - \lambda_{01})(\lambda_{01} - a) + \varepsilon_0^2(\lambda_{02} - b)(\lambda_{02} - a)} \right|.$$

When b converges to $\frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$, $\tan\theta'$ converges to 0. Hence θ' converges to 0 since $0 \leq \theta' \leq \frac{\pi}{2}$. We note that Φ_4 maps C_4 onto R_4 when $b > \frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$ and that Φ_4 maps C_4 onto R_4' when $b < \frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$. So if b converges to $\frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$, the angle of two lines consisting ∂R_4 and $\partial R_4'$ converges to 0. Since Φ_4 is continuous at $b = \frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$, Φ_4 maps C_4 onto the ray

$$S_4 = \left\{ d_1\varphi_{01} + d_2\varphi_{02} | \quad d_1 \geq 0, d_2 = \varepsilon_0 \left(\frac{\lambda_{02} - a}{\lambda_{01} - a} \right) d_1 \right\}.$$

Hence we have the following results.

THEOREM 2.9. *For $i = 2, 4$, the restriction Φ_i maps C_i onto S_i. And Φ_1 and Φ_3 are bijective. Therefore, Φ maps V onto R, where $R = R_1 = R_3$.*

THEOREM 2.10. *Suppose $a < \lambda_{01} < b < \lambda_{02}$ and $b = \frac{2\lambda_{01}\lambda_{02}-a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$. Let $f = s_1\varphi_{01} + s_2\varphi_{02} \in V$. Then we have*

(1) *If f belongs to interior of R, then (1.3) has exactly two solutions, one of which is positive and the other is negative.*

(2) *If f belongs to boundary of R, then* (1.3) *has a positive solution and a negative solution, and infinitely may sign changing solutions.*

(3) *If f does not belong to R, then* (1.3) *has no solution.*

References

[1] A. Ambrosetti and G. Prodi, *A primer of nonlinear analysis*, Cambridge, University Press, Cambridge Studies in Advanced Math, **34** (1993).

[2] Q. H. Choi, S. Chun, and T. Jung, *The multiplicity of solutions and geometry of a nonlinear ellliptic equation*, Studia Math. **120** (1996), 259-270.

[3] Q. H. Choi and T. S. Jung, *On periodic solutions of the nonlinear suspension bridge equation*, Differential and Integral Equations, **4** (1991), 383-396.

[4] ———, *An application of a variational reduction method to a nonlinear wave equation*, J. Differential Equations, **117** (1995), 390-410.

[5] ———, *A nonlinear beam equation with nonlinearity crossing an eigenvalue*, Dynam. Contin. Discrete Impulsive Systems, **4** (1998), 331-350.

[6] P. J. McKenna, *Topological Methods for Asymmetric Boundary Value Problems,* Lecture Notes Ser. 11, Res, Inst, Math, Global Analysis Res. Center, Seoul National University, 1993.

[7] P. J. McKenna, R. Redlinger, and W. Walter, *Multiplicity results for asymptotically homogeneous semilinear boundary value problems*, Ann. Mat. Pura Appl. **143** (1988), no. 4, 347-257.

[8] J. Schröder, *Operator Inequalities*, Academic Press, New York, 1980.

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