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ON THE JUMP NUMBER OF SPLITS OF ORDERED SETS

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ABSTRACT. In this paper, we consider the jump number of the split $P[S]$ of a subset S of an ordered set P. For $x \in P$, we show that $s(P) \leq s(P[x]) \leq s(P) + 2$ and give a necessary and sufficient condition for which $s(P[x]) = s(P)$.

Let P be a finite ordered set and let $|P|$ be the number of elements in P. An ordered set Q is called an *induced subset* of P provided that Q is a nonempty subset of P and that $x < y$ in Q if and only if $x < y$ in P for any elements x and y in Q. A *chain* C in P is an induced subset of P whose order is linear. If a and b are in P, then b *covers* a, written $a \prec b$, provided that $a < b$ and $a < c \leq b$ implies that $c = b$. A *linear extension* of an ordered set P is a linear order L on the elements of P such that $x < y$ in P implies $x < y$ in L. Szpilrajn [5] showed that any ordered set has a linear extension. In this paper, every ordered set is assumed to be finite.

Let P and Q be two disjoint ordered sets. The *disjoint sum* $P+Q$ of P and Q is the ordered set on $P \cup Q$ such that $x < y$ if and only if $x, y \in P$ and $x < y$ in P or $x, y \in Q$ and $x < y$ in Q. The *linear sum* $P \oplus Q$ of P and Q is obtained from $P + Q$ by adding the new relations $x < y$ for all $x \in P$ and $y \in Q$. An arbitrary linear extension of P is usually denoted by $L = C_1 \oplus \cdots \oplus C_m$ with chains $C_1, \cdots, C_m$ in P. A (P, L)-*chain* is a maximal sequence of elements $z_1, z_2, \ldots, z_k$ such that $z_1 \prec z_2 \prec \cdots \prec z_k$ in both L and P. Let $c(L)$ be the number of (P, L)-chains in L. A consecutive pair (x, y) of elements in L is called a *jump* (*setup*) of P in

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L if x is not comparable to y in P. The jumps induce a decomposition $L = C_1 \oplus \cdots \oplus C_m$ of L into (P, L)-chains $C_1, \ldots, C_m$, where $m = c(L)$ and $(\sup C_i, \inf C_{i+1})$ is a jump of P in L for $i = 1, \ldots, m-1$. Let $s(L, P)$ be the number of jumps of P in L and let $s(P)$ be the minimum of $s(L, P)$ over all linear extensions L of P. The number $s(P)$ is called the *jump (setup) number* of P. If $s(L, P) = s(P)$ then L is called an *optimal linear extension* of P. The jump number is a kind of measure between a given ordered set and its nearest linear extensions [1]. A practical motivation for studying the jump number of an ordered set comes from scheduling problems subject to precedence constraints. Namely, no task can be scheduled until all of its predecessors are scheduled. If a certain task is scheduled not immediately after one of its predecessors, then a jump occurs.

Now it is quite interesting to determine the jump number of variously constructed ordered sets from given ordered sets such as products, lexicographic sums, etc. For a natural number n, we denote the n-element chain by $\mathbf{n}$. An ordered set is called an *upward rooted tree* if it contains a least element and no induced subset isomorphic to $(\mathbf{1}+\mathbf{1}) \oplus \mathbf{1}$. Jung [2] estimated the jump number of the product of an upward rooted tree and a chain. Recently, Jung and Lee [3] estimated the jump number of the lexicographic sum of ordered sets.

Let P be an ordered set and let $S = \{x_1, x_2, ..., x_k\}$ be a nonempty subset of P. Kimble [4] defined the *split* of S in P, denoted by $P[S]$, as an ordered set on $(P - S) \cup \{x_1', x_1'', x_2', x_2'', ..., x_k', x_k''\}$ with the following order:

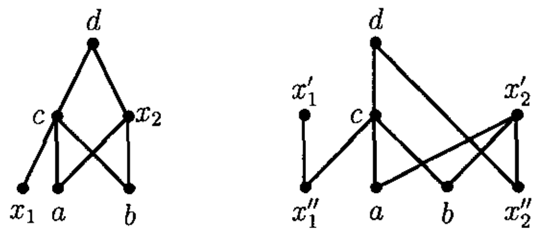
(1) $u \leq v$ in $P[S]$ if and only if $u \leq v$ in P, for all $u, v \in P - S$;

(2) $x_i'' < x_j'$ in $P[S]$ if and only if $x_i \leq x_j$ in P, for $i, j = 1, 2, ..., k$;

(3) $x_i'' < v$ in $P[S]$ if and only if $x_i < v$ in P, for $i = 1, 2, ..., k$ and for all $v \in P - S$;

(4) $u < x_j'$ in $P[S]$ if and only if $u < x_j$ in P, for $j = 1, 2, ..., k$ and for all $u \in P - S$;

EXAMPLE. Let P be the ordered set shown below on the left and let $S = \{x_1, x_2\}$. Then the $P[S]$ is shown on the right.



In this paper, we estimate the jump number of splits of ordered sets. We begin with a simple but very useful observation.

LEMMA 1. *Let P be an ordered set and $x \in P$. Then*

$$s(P) - 1 \leq s(P - \{x\}) \leq s(P).$$

Proof. Clearly, $s(P - \{x\}) \leq s(P)$. On the other hand, suppose that L be an optimal linear extension of $P - \{x\}$ with n (P, L)-chains. Then we can obtain a linear extension L' by putting x just above the highest y which is below x in L so that L' has at most $n + 1$ (P, L')-chains. $\square$

OBSERVATION. Let P be an ordered set and let S_1 and S_2 be disjoint subsets of P. Then $P[S_1 \cup S_2] = (P[S_1])[S_2]$.

By the above observation, we only consider the case when $S = \{x\}, x \in P$. In this case, we write $P[x]$ for $P[\{x\}]$. Now we prove the following theorem.

THEOREM 1. *Let P be an ordered set and $x \in P$. Then*

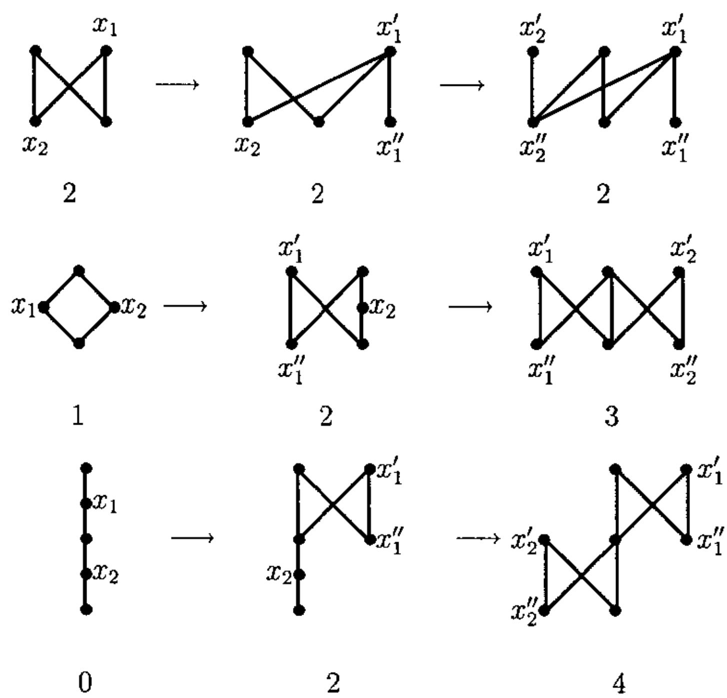
$$s(P) \leq s(P[x]) \leq s(P) + 2.$$

Proof. By Lemma 1, $s(P[x]) \leq s(P - \{x', x''\}) + 2 \leq s(P) + 2$. On the other hand, suppose that $m = s(P[x]) < s(P)$. Let $L = C_0 \oplus \cdots \oplus C_m$ be an optimal linear extension with chains $C_0, \cdots, C_m$ in $P[x]$. Then $x'' = \min C_i$ and $x' = \max C_j$ for some $i \leq j$. If $x'' < a$ in C_i and $b < x'$ in C_j with $i \neq j$, then $b < x < a$ in P, which is impossible. So at least one of $\{x'\}$, $\{x''\}$ and $\{x'', x'\}$ is a (P, L)-chain. Since $L - \{x', x''\}$ is a linear extension of $P - \{x\}$ with at most m chains, we have $s(P) = s(P - \{x\}) + 1 \leq (m - 1) + 1 = m < s(P)$, which is a contradiction. Thus $s(P) \leq s(P[x])$. □

COROLLARY. *Let P be an ordered set and S a subset of P. Then*

$$s(P) \leq s(P[S]) \leq s(P) + 2|S|.$$

EXAMPLES. In each row of the following figures, we split the elements x_1 and x_2 consecutively. The number below each diagram is the jump number of the ordered set.



THEOREM 2. *Let P be an ordered set and $x \in P$. Then $s(P[x]) = s(P)$ if and only if $\{x\}$ is a (P, L)-chain for some optimal linear extension L of P.*

Proof. Let $s(P) = n$. If $\{x\}$ is a (P, L)-chain for some optimal linear extension L of P, then there exists an optimal linear extension $C_0 \oplus \cdots \oplus C_n$ with chains $C_0, \cdots, C_n$ in P and $C_i = \{x\}$ for some i and so $s(P) = s(P - \{x\}) + 1$. By replacing $\{x\}$ with $\{x'', x'\}$, we obtain a linear extension of $P[x]$ of $n + 1$ chains. Hence, $s(P[x]) \leq n$. Now the sufficiency of the condition follows from Theorem 1.

Conversely, suppose that $s(P[x]) = n$. Then there exists a linear extension L of $(P[x])$ with $n+1$ (P, L)-chains. As in the proof of Theorem 1, at least one of $\{x'\}$, $\{x''\}$ and $\{x'', x'\}$ is a (P, L)-chain. In any case an optimal linear extension L' of P can be obtained from L by removing $\{x'', x'\}$ and adding $\{x\}$ as a (P, L')-chain. □

Let P be an ordered set. A nonempty subset S of P is called a *down-set* [an *up-set*] if $x \in S$ and $y \leq x$ [$y \geq x$] imply $y \in S$.

THEOREM 3. *Let S be a down-set or an up-set of an ordered set P. Then, for $S \neq P$,*

$$s(P[S]) = s(P - S) + |S|$$

and

$$s(P[P]) = |P| - 1.$$

Proof. If $S = \{x_1, ..., x_k\}$ is a down-set of P, then we may assume that each x_i is minimal in $P - \{x_1, ..., x_{i-1}\}$ for $1 \leq i \leq k$. Now it suffices to show that if x is minimal in P, then $s(P[x]) = s(P - \{x\}) + 1$. Since a linear extension L can be constructed such that $\{x'', x'\}$ is the first (P, L)-chain, it follows that $s(P[x]) \leq s(P - \{x\}) + 1$. On the other hand, let $L = C_0 \oplus \cdots \oplus C_n$ be an optimal linear extension of $P[x]$. Then $\{x'\}$ or $\{x'', x'\}$ is a $(P[x], L)$-chain. In any case we obtain the linear extension $L' = L - \{x\}$ of $P' = P - \{x\}$ with n (P, L')-chains. The other cases can be verified similarly. □

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