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# TRANSCENDENTAL NUMBERS AS VALUES OF ELLIPTIC FUNCTIONS

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Abstract. As a by-product of [4], we give algebraic integers of certain values of quotients of Weierstrass $\wp'(\tau)$, $\wp''(\tau)$-functions. We also show that special values of elliptic functions are transcendental numbers.

## 0. Introduction

In [1] and [4], certain values of theta series are algebraic. In this article, we mainly deal with algebraic integer or transcendental number of values of elliptic functions.

In section 1, we write infinite product formula for $\wp(\tau)$, $g_2(\tau)$ and $g_3(\tau)$. We will show that $\frac{81\sqrt{3}}{\pi^3}\frac{\wp'(\frac{\tau}{3})}{\eta(\frac{2}{3}\tau)^6}$, $\frac{1}{2}\frac{\wp''(\frac{1}{2})}{\pi^4\eta(\tau)^8}$, $\frac{1}{2}\frac{\wp''(\frac{\tau}{2})}{\pi^4\eta(\tau)^8}$, and $\frac{1}{2}\frac{\wp''(\frac{\tau+1}{2})}{\pi^4\eta(\tau)^8}$ are algebraic integers (Theorem 1.5). In the final section, we will prove that certain values of Eisenstein series and modular discriminant are transcendental.

## 1. Algebraic integers as values of Weierstrass functions

Throughout this section, we shall fix the following notations: $k$ is an imaginary quadratic field, $\mathfrak{h}$ the complex upper half plane and $\tau \in \mathfrak{h}\cap k$.

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Let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ $(\tau \in \mathfrak{h})$ be a lattice and $z \in \mathbb{C}$. The *Weierstrass $\wp$-function* (*relative to* $\Lambda_\tau$ ) is defined by the series

$$\wp(z;\Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda_\tau \\ \omega\neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}$$

and the *Eisenstein series of weight* $2k$ (for $\Lambda_\tau$ and $k > 1$) is the series

$$G_{2k}(\Lambda_\tau) = \sum_{\substack{\omega\in\Lambda_\tau \\ \omega\neq 0}} \omega^{-2k}.$$

The infinite product

$$\sigma(z;\Lambda_\tau) = z \prod_{\substack{\omega\in\Lambda_\tau \\ \omega\neq 0}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}(\frac{z}{\omega})^2}$$

defines a holomorphic function on $\mathbb{C}$ with simple zeros on $\Lambda_\tau$ and no other zeros. It is called the *Weierstrass $\sigma$-function* (*associated to the lattice* $\Lambda_\tau$).

We shall use the notations $\wp(z)$, $G_{2k}$, and $\sigma(z)$ instead of $\wp(z;\Lambda_\tau)$, $G_{2k}(\Lambda_\tau)$, and $\sigma(z;\Lambda_\tau)$, respectively, when the lattice $\Lambda_\tau$ has been fixed. As is customary, by setting

$$g_2(\tau) = g_2(\Lambda_\tau) = 60G_4 \quad \text{and} \quad g_3(\tau) = g_3(\Lambda_\tau) = 140G_6,$$

the algebraic relation between $\wp(z)$ and $\wp'(z)$ becomes

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau). \text{ ([5], [9], [10], [11])}$$

The modular discriminant is the function $\Delta(\tau) := g_2(\tau)^3 - 27g_3(\tau)^2 = (2\pi)^{12}\eta(\tau)^{24}$, where $\eta(\tau)$ is the Dedekind $\eta$-function. Any elliptic functions can be factored as a product of Weierstrass $\sigma$-functions reflecting its zeros and poles. We write two important examples.

PROPOSITION 1.1 ([10], [11]).

(a) $\wp(z) - \wp(a) = -\dfrac{\sigma(z+a)\sigma(z-a)}{\sigma(z)^2\sigma(a)^2}$.

(b) $\wp'(z) = -\dfrac{\sigma(2z)}{\sigma(z)^4}$.

Moreover, we have the following proposition at hand which will be useful in extracting infinite product expressions.

PROPOSITION 1.2 ([5], [10]). *Let* $p = e^{\pi i\tau}$.

(1) $\wp\left(\dfrac{\tau}{2}\right) - \wp\left(\dfrac{1}{2}\right) = -\pi^2\prod_{n=1}^{\infty}(1-p^{2n})^4(1+p^{(2n-1)})^8$.

(2) $\wp\left(\dfrac{\tau+1}{2}\right) - \wp\left(\dfrac{1}{2}\right) = -\pi^2\prod_{n=1}^{\infty}(1-p^{2n})^4(1-p^{(2n-1)})^8$.

(3) $\wp\left(\dfrac{\tau+1}{2}\right) - \wp\left(\dfrac{\tau}{2}\right) = 16\pi^2 p\prod_{n=1}^{\infty}(1-p^{2n})^4(1+p^{2n})^8$.

In [4], using Proposition 1.2, we have the following results:

$$\wp\left(\frac{\tau}{2}\right) = -\frac{\pi^2}{3}\prod_{n=1}^{\infty}(1-p^{2n})^4\left(\prod_{n=1}^{\infty}(1+p^{(2n-1)})^8 + 16p\prod_{n=1}^{\infty}(1+p^{2n})^8\right),$$

$$\wp\left(\frac{\tau+1}{2}\right) = -\frac{\pi^2}{3}\prod_{n=1}^{\infty}(1-p^{2n})^4\left(\prod_{n=1}^{\infty}(1+p^{(2n-1)})^8 - 32p\prod_{n=1}^{\infty}(1+p^{2n})^8\right),$$

and

$$\wp\left(\frac{1}{2}\right) = \frac{\pi^2}{3}\prod_{n=1}^{\infty}(1-p^{2n})^4\left(2\prod_{n=1}^{\infty}(1+p^{(2n-1)})^8 - 16p\prod_{n=1}^{\infty}(1+p^{2n})^8\right).$$

It follows from above $p$-expansions of $\wp(\frac{\tau}{2})$, $\wp(\frac{\tau+1}{2})$, and $\wp(\frac{1}{2})$ that

$$g_2(\tau) = \frac{4\pi^4}{3}\prod_{n=1}^{\infty}(1-p^{2n})^8\left[\prod_{n=1}^{\infty}(1+p^{(2n-1)})^{16} - 16p\prod_{n=1}^{\infty}(1+p^n)^8 + 256p^2\prod_{n=1}^{\infty}(1+p^{2n})^{16}\right],$$

$$\begin{aligned} g_3(\tau) = \frac{8\pi^6}{27} \prod_{n=1}^{\infty} (1-p^{2n})^{12} \Big( & \prod_{n=1}^{\infty} (1+p^{2n-1})^{24} \\ & - 24p \prod_{n=1}^{\infty} (1+p^{2n-1})^{16} (1+p^{2n})^8 \\ & - 384p^2 \prod_{n=1}^{\infty} (1+p^{2n-1})^8 (1+p^{2n})^{16} \\ & + 4096p^3 \prod_{n=1}^{\infty} (1+p^{2n})^{24} \Big). \end{aligned}$$

Let $\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $b$ mod $d$ and $|\alpha|$ the determinant of $\alpha$, and let

$$\phi_\alpha(\tau) := |\alpha|^{12} \frac{\Delta\left(\alpha\binom{\tau}{1}\right)}{\Delta\left(\binom{\tau}{1}\right)} = |\alpha|^{12} d^{-12} \frac{\Delta(\alpha\tau)}{\Delta(\tau)}.$$

Then we recall the following well known fact.

Proposition 1.3 ([1], [3], [5], [12]). *For any $\tau \in k \cap \mathfrak{h}$, the value $\phi_\alpha(\tau)$ is an algebraic integer, which divides $|\alpha|^{12}$.*

From above infinite product formula for $\wp(\tau)$, $g_2(\tau)$, $g_3(\tau)$ and Proposition 1.3, we get the Proposition 1.4.

Proposition 1.4 ([4]). *Let $\tau \in k \cap \mathfrak{h}$. Then the following assertions hold:*

(a) $\sqrt{2} p^{\frac{1}{24}} \prod_{n=1}^{\infty} (1+p^n)$, $p^{-\frac{1}{24}} \dfrac{1}{\prod_{n=1}^{\infty}(1+p^n)}$, $p^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1-p^{2n-1})$, $\sqrt{2} \prod_{n=1}^{\infty}(1+p^n)\ (1-p^{2n-1})$, $p^{-\frac{1}{24}} \prod_{n=1}^{\infty}\ (1+p^{2n-1})$, and $\sqrt{2} \prod_{n=1}^{\infty}\ (1+p^n)\ (1+p^{2n-1})$ *are algebraic integers.*

(b) $\dfrac{3}{\pi^2} \dfrac{\wp(\frac{\tau}{2})}{\eta(\tau)^4}$, $\dfrac{3}{\pi^2} \dfrac{\wp(\frac{\tau+1}{2})}{\eta(\tau)^4}$, $\dfrac{3}{\pi^2} \dfrac{\wp(\frac{1}{2})}{\eta(\tau)^4}$, $\dfrac{3}{4\pi^4} \dfrac{g_2(\tau)}{\eta(\tau)^8}$, $\frac{27}{\pi^6} \frac{g_3(\tau)}{\eta(\tau)^{12}}$, $\frac{\wp(\frac{\tau}{2})-\wp(\frac{1}{2})}{\pi^2\eta(\tau)^4}$, $\frac{\wp(\frac{\tau+1}{2})-\wp(\frac{1}{2})}{\pi^2\eta(\tau)^4}$, *and* $\dfrac{\wp(\frac{\tau+1}{2})-\wp(\frac{\tau}{2})}{\pi^2\eta(\tau)^4}$ *are algebraic integers.*

THEOREM 1.5. *Let $\tau \in k \cap \mathfrak{h}$.*

(a) $\dfrac{81\sqrt{3}}{\pi^3}\dfrac{\wp'(\frac{\tau}{3})}{\eta(\frac{2}{3}\tau)^6}$, $\dfrac{1}{2}\dfrac{\wp''(\frac{1}{2})}{\pi^4\eta(\tau)^8}$, $\dfrac{1}{2}\dfrac{\wp''(\frac{\tau}{2})}{\pi^4\eta(\tau)^8}$, *and* $\dfrac{1}{2}\dfrac{\wp''(\frac{\tau+1}{2})}{\pi^4\eta(\tau)^8}$ *are algebraic integers.*

(b) $\dfrac{\sigma(\frac{2\tau}{3})^5}{\sigma(\frac{\tau}{3})^4\sigma(\frac{4\tau}{3})}$ *and* $\dfrac{\sigma(\frac{\tau+1}{2})^3\sigma(\frac{\tau-1}{2})}{\sigma(\frac{\tau}{2})^3\sigma(\frac{\tau+2}{2})}$ *are algebraic numbers.*

*Proof.* Let $\alpha_{\frac{3}{2}} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$. Then

$$\begin{aligned}\phi_{\alpha_{\frac{3}{2}}}(\frac{2}{3}\tau)^{\frac{1}{24}} &= \left(6^{12}\cdot\frac{1}{2^{12}}\frac{\eta(\tau)^{24}}{\eta(\frac{2}{3}\tau)^{24}}\right)^{\frac{1}{24}}\\ &= \sqrt{3}\frac{\eta(\tau)}{\eta(\frac{2}{3}\tau)}\end{aligned}$$

is an algebraic integer. Also, let $\alpha_{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Then

$$\frac{\eta(\frac{1}{2}\tau)}{\eta(\tau)}$$

is an algebraic integer. Moreover,

$$\wp'(\frac{1}{3}\tau)^2 = 4\wp(\frac{1}{3}\tau)^3 - g_2(\tau)\wp(\frac{1}{3}\tau) - g_3(\tau)$$

implies that

$$\begin{aligned}\frac{19683}{\pi^6}\frac{\wp'(\frac{1}{3}\tau)^2}{\eta(\frac{2}{3}\tau)^{12}} &= 2916\left(\frac{3}{\pi^2}\frac{\wp(\frac{1}{3}\tau)}{\eta(\frac{2}{3}\tau)^4}\right)^3\\ &\quad - 27\left(\frac{3}{\pi^4}\frac{g_2(\tau)}{\eta(\tau)^8}\right)\left(\frac{3}{\pi^2}\frac{\wp(\frac{1}{3}\tau)}{\eta(\frac{2}{3}\tau)^4}\right)\left(\sqrt{3}\frac{\eta(\tau)}{\eta(\frac{2}{3}\tau)}\right)^8\\ &\quad - \left(\frac{3}{\pi^6}\frac{g_3(\tau)}{\eta(\tau)^{12}}\right)\left(\sqrt{3}\frac{\eta(\tau)}{\eta(\frac{2}{3}\tau)}\right)^{12},\end{aligned}$$

and thus we have

$$\frac{81\sqrt{3}}{\pi^3}\frac{\wp'(\frac{\tau}{3})}{\eta(\frac{2}{3}\tau)^6}$$

is an algebraic integer. We know from [10] that

$$\begin{aligned}
\wp''(\frac{1}{2}) &= 2\left(\wp(\frac{1}{2})-\wp(\frac{\tau}{2})\right)\left(\wp(\frac{1}{2})-\wp(\frac{\tau+1}{2})\right),\\
(1-1)\qquad \wp''(\frac{\tau}{2}) &= 2\left(\wp(\frac{\tau}{2})-\wp(\frac{1}{2})\right)\left(\wp(\frac{\tau}{2})-\wp(\frac{\tau+1}{2})\right),\\
\wp''(\frac{\tau+1}{2}) &= 2\left(\wp(\frac{\tau+1}{2})-\wp(\frac{1}{2})\right)\left(\wp(\frac{\tau+1}{2})-\wp(\frac{\tau}{2})\right).
\end{aligned}$$

By Proposition 1.4 and (1-1), we derive that

$$\frac{1}{2}\frac{\wp''(\frac{1}{2})}{\pi^4\eta(\tau)^8},\quad \frac{1}{2}\frac{\wp''(\frac{\tau}{2})}{\pi^4\eta(\tau)^8},\quad \text{and } \frac{1}{2}\frac{\wp''(\frac{\tau+1}{2})}{\pi^4\eta(\tau)^8}$$

are algebraic integers. And we deduce from Proposition 1.4 that

$$\frac{\wp'(\frac{\tau}{3})}{\wp'(\frac{2\tau}{3})}=\frac{\frac{81\sqrt{3}}{\pi^3}\frac{\wp'(\frac{\tau}{3})}{\eta(\frac{2}{3}\tau)^6}}{\frac{81\sqrt{3}}{\pi^3}\frac{\wp'(\frac{2\tau}{3})}{\eta(\frac{4}{3}\tau)^6}}\left(\frac{\eta(\tau)}{\eta(\frac{4}{3}\tau)}\right)^6\left(\frac{\eta(\frac{2\tau}{3})}{\eta(\tau)}\right)^6$$

is an algebraic number, so

$$\frac{\sigma(\frac{2\tau}{3})^5}{\sigma(\frac{\tau}{3})^4\sigma(\frac{4\tau}{3})}=\frac{-\frac{\sigma(\frac{2\tau}{3})}{\sigma(\frac{\tau}{3})^4}}{-\frac{\sigma(\frac{4\tau}{3})}{\sigma(\frac{2\tau}{3})^4}}=\frac{\wp'(\frac{\tau}{3})}{\wp'(\frac{2\tau}{3})}$$

is an algebraic number. Also,

$$\frac{\sigma(\frac{\tau+1}{2})^3\sigma(\frac{\tau-1}{2})}{\sigma(\frac{\tau}{2})^3\sigma(\frac{\tau+2}{2})}=\frac{-\frac{\sigma(\frac{\tau+1}{2})\sigma(\frac{\tau-1}{2})}{\sigma(\frac{\tau}{2})^2\sigma(\frac{1}{2})^2}}{-\frac{\sigma(\frac{\tau+2}{2})\sigma(\frac{\tau}{2})}{\sigma(\frac{\tau+1}{2})^2\sigma(\frac{1}{2})^2}}=\frac{\wp(\frac{\tau}{2})-\wp(\frac{1}{2})}{\wp(\frac{\tau+1}{2})-\wp(\frac{1}{2})}$$

is an algebraic number. Therefore we get the theorem. □

## 2. Transcendental numbers

In 1916 ([7]), S. Ramanujan introduced the following functions:

$$P(z) = 1-24\sum_{n=1}^{\infty}\sigma_1(n)z^n,\ Q(z) = 1+240\sum_{n=1}^{\infty}\sigma_3(n)z^n,\ R(z) = 1-504\sum_{n=1}^{\infty}\sigma_5(n)z^n,$$

with $\sigma_k(n) = \sum_{d|n} d^k$ and $z \in \mathbb{C}$.

As is well known ([2], [5]),

$$Q(p^2) = \frac{3g_2(\tau)}{4\pi^4} \quad \text{and} \quad R(p^2) = \frac{27g_3(\tau)}{8\pi^6}.$$

PROPOSITION 2.1 ([6]). *For any complex $p$ with $0 < |p| < 1$, there are not less than three algebraically independent numbers over $\mathbb{Q}$ among the numbers $p$, $P(p)$, $Q(p)$, and $R(p)$.*

Many interesting results can be obtained as corollaries of this theorem. We list up one of them. The three numbers

$$(2-1) \qquad \pi,\ e^{\pi},\ \Gamma(1/4)$$

are algebraically independent.

THEOREM 2.2. *Let $\tau = \dfrac{ai+b}{d}$ with $a, b$ and $d \neq 0$ integers. Then $\Delta(\tau)$, $\wp(\frac{\tau}{2})$, $\wp(\frac{\tau+1}{2})$, $\wp(\frac{1}{2})$, $g_2(\tau)$, $\wp(\frac{\tau}{2}) - \wp(\frac{1}{2})$, $\wp(\frac{\tau+1}{2}) - \wp(\frac{1}{2})$, and $\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2})$ are transcendental numbers. More generally, for all integers $n \geq 1$, $G_{4n}(i)$ is a transcendental number.*

*Proof.* A theorem of Hurwitz ([3], [8]) says that

$$g_2(i) = 64\left(\int_0^1 \frac{1}{\sqrt{1-t^4}}\,dt\right)^4 = \frac{1}{16\pi^2}\Gamma\left(\frac{1}{4}\right)^8,$$

where $\Gamma(x)$ is the value $x$ of Gamma function. By (2-1), $g_2(i)$ is a transcendental number, and also $\Delta(i)$ is transcendental, since $\Delta(i) = g_2(i)^3 - 27g_3(i)^2 = g_2(i)^3$.

Let $\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. By Proposition 1.3, $\dfrac{\Delta(\alpha i)}{\Delta(i)}$ is an algebraic number, so $\Delta(\frac{ai+b}{d})$ is transcendental. Also, so is $\dfrac{\Delta(\frac{ai+b}{d})}{(2\pi)^{12}} = \eta(\frac{ai+b}{d})^{24}$. By (2-1) and Proposition 1.4, we get the first part of the proof of this theorem. As is well known ([3], [8]), for all $n \geq 1$, $G_{4n}(i)$ is a rational number multipled by $\left(\int_0^1 \frac{1}{\sqrt{1-t^4}}\,dt\right)^{4n}$. So we get the theorem. □

COROLLARY 2.3. *Let* $\tau = \dfrac{ai+b}{d}$ *with* $a, b$ *and* $d$ *integers and let* $\theta_3(v,\tau) = 1 + 2\sum_{n=1}^{\infty} p^{n^2}\cos(n\pi v)$. *Then* $\theta_3(0,\tau)$ *is a transcendental number.*

*Proof.* By Proposition 1.3, $\dfrac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau+1)}$ is an algebraic number, and $\eta(\dfrac{\tau+1}{2})$ is transcendental(Theorem 2.2), so $\theta_3(0,\tau) = \dfrac{\eta(\frac{\tau+1}{2})^2}{\eta(\tau+1)}$ is a transcendental number. Here we refer to [1] and [2] for the equality. □

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