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ON A KIND OF FUNCTION EQUATIONS AND THE FUNDAMENTAL THEOREM OF ALGEBRA

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Abstract. We define the terminologies for the special type of functions defined on the field $\mathbb{C}$ of complex numbers and consider the solutions of function equations containing such functions in $\mathbb{C}$.

Generally, it is not simple to determine whether function equations in the field $\mathbb{C}$ of complex numbers have a solution. For example, analytic or algebraic proofs for the fundamental theorem of algebra [1], which says the field $\mathbb{C}$ is algebraically closed, are much long. In this paper, we introduce a sequence of special type of continuous functions defined on the field $\mathbb{C}$, called a power-type sequence, with properties of the solutions of function equations containing such functions in $\mathbb{C}$. In particular, as an application, we easily obtain an analogue for the fundamental theorem of algebra.

We also define a rotary function for the special type of complex valued function and give some properties of the solution of a function equation containing a power-type sequence of rotary functions and their applications.

A sequence of continuous functions $\{\varphi_k\} \subseteq \mathbb{C}^{\mathbb{C}}$ is said to be *power-type* if $\varphi_k(0) = 0, \varphi_k(z) \neq 0$ for $z \neq 0$, $\lim_{|z|\to\infty} |\varphi_k(z)| = \infty$, and

$$\lim_{|z|\to\infty} \left|\frac{\varphi_n(z)}{\varphi_k(z)}\right| = \infty \quad \text{for } k \leq n.$$

Examples. (1) $\varphi_k(z) = z^k, \quad z \in \mathbb{C}, \quad k \in \mathbb{N}$.

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(2) $\varphi_k(z) = z^{\frac{1}{k}} e^{k|z|}, \quad z \in \mathbb{C}, \quad k \in \mathbb{N}.$
(3) $\varphi_k(z) = e^{\sqrt{z}} \ln(|z^k| + 1), \quad z \in \mathbb{C}, \quad k \in \mathbb{N}.$

THEOREM 1. *Let $a_0, a_1, \cdots, a_n \in \mathbb{C}$ with $a_0, a_n \neq 0$, and let $\{\varphi_k\} \subseteq \mathbb{C}^{\mathbb{C}}$ be power-type. Then there exists a function $\theta : [0, 2\pi] \longrightarrow [0, 2\pi]$ such that for each $\alpha \in [0, 2\pi]$ the equation*

$$a_n\varphi_n(z) + a_{n-1}\varphi_{n-1}(z) + \cdots + a_1\varphi_1(z) + e^{i\theta(\alpha)}a_0 = 0$$

has a solution of the form $z = re^{i\alpha}$.

Proof. Let

$$f(z) = |a_n\varphi_n(z) + a_{n-1}\varphi_{n-1}(z) + \cdots + a_1\varphi_1(z)|.$$

Then

$$f(z) \geq |a_n\varphi_n(z)| - \sum_{k=1}^{n-1} |a_k\varphi_k(z)| = |a_n\varphi_n(z)| \left(1 - \frac{\sum_{k=1}^{n-1} |a_k\varphi_k(z)|}{|a_n\varphi_n(z)|}\right)$$

for $z \neq 0$ and

$$\lim_{|z|\to\infty} \frac{\sum_{k=1}^{n-1} |a_k\varphi_k(z)|}{|a_n\varphi_n(z)|} = \sum_{k=1}^{n-1} \lim_{|z|\to\infty} \frac{|a_k\varphi_k(z)|}{|a_n\varphi_n(z)|} = 0,$$

which implies $\lim\limits_{|z|\to\infty} f(z) = \infty$. Hence there is an $r_0 > 0$ such that

$$f(z) > |a_0| \quad \text{if} \quad |z| \geq r_0.$$

Now let $\alpha \in [0, 2\pi]$. Then f is continuous on the half line

$$L_\alpha = \{re^{i\alpha} : r \geq 0\}$$

and

$$f(0) = 0, \; f(r_0e^{i\alpha}) > |a_0|.$$

By the intermediate value theorem, there is an $re^{i\alpha} \in L_\alpha$ such that $0 < r < r_0$ and

$$|a_n\varphi_n(re^{i\alpha}) + a_{n-1}\varphi_{n-1}(re^{i\alpha}) + \cdots + a_1\varphi_1(re^{i\alpha})| = f(re^{i\alpha}) = |a_0|$$

and hence there is a $\theta(\alpha) \in [0, 2\pi]$ for which

$$a_n\varphi_n(re^{i\alpha}) + a_{n-1}\varphi_{n-1}(re^{i\alpha}) + \cdots + a_1\varphi_1(re^{i\alpha}) + e^{i\theta(\alpha)}a_0 = 0.$$

This completes the proof. □

Taking $\varphi_k(z) = z^k$ in Theorem 1, we have the following corollary, which is an analogue for the fundamental theorem of algebra because there are extremely many coefficient combinations.

COROLLARY 2. *Let $a_1, a_2, \cdots, a_n \in \mathbb{C}$ with $a_n \neq 0$. Then for every $r \geq 0$ there are uncountably many $\theta \in [0, 2\pi]$ such that the equation*

$$a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + re^{i\theta} = 0 \tag{1}$$

has a solution.

Proof. If $r = 0$, then (1) has a solution $z = 0$. Let $r > 0$ be arbitrary. By Theorem 1, for each $\alpha \in [0, 2\pi]$ there is a $\theta(\alpha) \in [0, 2\pi]$ such that the equation

$$a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + re^{i\theta(\alpha)} = 0 \tag{2}$$

has a solution $z_\alpha = |z_\alpha|e^{i\alpha}$. However, the factorization theorem [1] shows that the number of solutions of the equation (2) does not exceed n.

For each $\alpha \in [0, 2\pi]$, choose a $\theta(\alpha) \in [0, 2\pi]$ such that the equation (2) has a solution and let

$$A_\alpha = \{\beta \in [0, 2\pi] : z_\beta = |z_\beta|e^{i\beta} \text{ is a solution of (2)}\}.$$

Then A_α contains at most n different members and hence, if $\gamma \in [0, 2\pi] - A_\alpha$, then $\theta(\alpha) \neq \theta(\gamma)$. Therefore, the set

$$\{\theta(\alpha) \in [0, 2\pi] : \text{(2) has a solution } |z_\alpha|e^{i\alpha},\ 0 \leq \alpha \leq 2\pi\}$$

is uncountable. □

REMARK. Corollary 2 is a partial result of the fundamental theorem of algebra since the fundamental theorem of algebra says for every $r \geq 0$ and $\theta \in [0, 2\pi]$ the equation (1) has a solution. However, this result is an immediate consequence of Theorem 1 which is proved quite simply as shown before.

We now give the other definition for the special type of complex valued function and obtain properties of the solutions of function equations containing such functions.

A function $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ is said to be *rotary* if for every $\alpha \in [0, 2\pi]$ there is a $\beta \in [0, 2\pi]$ such that

$$e^{i\alpha}\varphi(z) = \varphi(e^{i\beta}z), \quad z \in \mathbb{C}.$$

Note that

$$e^{i\alpha}z^k = \left(e^{i\frac{\alpha}{k}}z\right)^k \text{ and } e^{i\alpha}z^{\frac{1}{k}}e^{k|z|} = \left(e^{ik\alpha}z\right)^{\frac{1}{k}}e^{k|e^{ik\alpha}z|},$$

so, in Examples (1) and (2), $\{\varphi_k\}$ is a power-type sequence of rotary functions.

THEOREM 3. *Let $a_0, a_1, \cdots, a_n \in \mathbb{C}$ with $a_0, a_n \neq 0$, and let $\{\varphi_k\} \subseteq \mathbb{C}^{\mathbb{C}}$ be a power-type sequence of rotary functions. Then the equation*

$$a_n\varphi_n(z_n) + a_{n-1}\varphi_{n-1}(z_{n-1}) + \cdots + a_1\varphi_1(z_1) + a_0 = 0$$

has a solution $(u_1, u_2, \cdots, u_n)$ such that $|u_1| = |u_2| = \cdots = |u_n|$.

Proof. By Theorem 1, there is a $\theta \in [0, 2\pi]$ and a non-zero $z_0 \in \mathbb{C}$ such that

$$a_n\varphi_n(z_0) + a_{n-1}\varphi_{n-1}(z_0) + \cdots + a_1\varphi_1(z_0) + e^{i\theta}a_0 = 0,$$

i.e.,

$$a_ne^{-i\theta}\varphi_n(z_0) + a_{n-1}e^{-i\theta}\varphi_{n-1}(z_0) + \cdots + a_1e^{-i\theta}\varphi_1(z_0) + a_0 = 0.$$

But $e^{-i\theta} = e^{i(-\theta+2\pi)}$, $-\theta + 2\pi \in [0, 2\pi]$ and

$$a_ne^{i(2\pi-\theta)}\varphi_n(z_0) + a_{n-1}e^{i(2\pi-\theta)}\varphi_{n-1}(z_0) + \cdots + a_1e^{i(2\pi-\theta)}\varphi_1(z_0) + a_0 = 0.$$

Since each φ_k is rotary,

$$e^{i(2\pi-\theta)}\varphi_k(z_0) = \varphi_k(e^{i\alpha_k}z_0) \text{ for some } \alpha_k \in [0, 2\pi]$$

so
$$a_n\varphi_n(e^{i\alpha_n}z_0) + a_{n-1}\varphi_{n-1}(e^{i\alpha_{n-1}}z_0) + \cdots + a_1\varphi_1(e^{i\alpha_1}z_0) + a_0 = 0$$
and
$$|e^{i\alpha_1}z_0| = |e^{i\alpha_2}z_0| = \cdots = |e^{i\alpha_n}z_0| = |z_0|. \qquad \square$$

COROLLARY 4. *For every* $a_0, a_1, \cdots, a_n \in \mathbb{C}$ *with* $a_0, a_n \neq 0$, *there are* $u_1, u_2, \cdots, u_n \in \mathbb{C}$ *such that*
$$|u_1| = |u_2| = \cdots = |u_n|$$
and
$$a_n u_n{}^n + a_{n-1}u_{n-1}{}^{n-1} + \cdots + a_1 u_1 + a_0 = 0.$$
In particular, for every $a_0, a_1, \cdots, a_n \in \mathbb{C}$ *with* $a_0, a_n \neq 0$, *there are* $z_1, z_2, \cdots, z_n \in \mathbb{C}$ *such that*
$$|z_k| = |z_1|^k, \quad 1 \leq k \leq n$$
and
$$a_1 z_1 + a_2 z_2 + \cdots + a_n z_n = a_0.$$

COROLLARY 5. *For every* $a_1, \cdots, a_n \in \mathbb{C}$ *with* $a_n \neq 0$, *there are* $u_1, \cdots, u_n, v_1, \cdots, v_n \in \mathbb{C}$ *such that*
$$|u_k| = |u_1|^k, \quad |v_k| = |v_1|^k, \quad 1 \leq k \leq n$$
and
$$\sum_{k=1}^{n} a_k(u_k + v_k) = 0.$$

Proof. By Corollary 4, there are $u_1, \cdots, u_n, v_1, \cdots, v_n \in \mathbb{C}$ such that
$$|u_k| = |u_1|^k, \quad |v_k| = |v_1|^k, \quad 1 \leq k \leq n$$
and
$$\sum_{k=1}^{n} a_k u_k = 1, \quad \sum_{k=1}^{n} a_k v_k = -1. \qquad \square$$

REMARK. All the statements in this paper which contain the interval $[0, 2\pi]$ are true when the interval $[0, 2\pi]$ is replaced by the set $\mathbb{R}$.

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