A NOTE ON THE DISPERSION RELATION OF THE MODIFIED BOUSSINESQ EQUATIONS

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Abstract: Optimal values of α characterizing the linear dispersion property in the modified Boussinesq equations are determined by minimizing the combined relative errors of the phase and group velocities. The value of α is fixed in previous studies, whereas it is varying in the present study. The phase and group velocities are calculated by using variable α and compared to those of the linear Stokes wave theory and previous studies. It is found that the present study produces the best match to the linear Stokes theory.

Key words: Boussinesq equations, dispersion, phase velocity, group velocity

1. INTRODUCTION

Many theoretical studies have been carried out to extend the applicable range of the governing equations for water waves propagating from deep water to shallow water depths. It is widely known that the conventional Boussinesq equations give reasonable results in shallow water, whereas they may not give favorable solutions in deep water. The major limitation of the conventional Boussinesq equations is the problem in the model's applicability to relatively deep water. The search for the governing equations describing the wave propagation from a deeper water depth to a shallower water depth is still an active area of research (Witting, 1984; Madsen *et al.*, 1991; Madsen and Sørensen, 1992; Nwogu, 1993; Chen and Liu, 1995). Most of these studies have tried to improve the linear dispersion property of the Boussinesq equations.

Madsen *et al.* (1991) used the conventional Boussinesq equations in terms of the horizontal volume flux instead of the depth-averaged velocity. And, they included some correction terms in the momentum equations so that the resulting equations yield a linearized dispersion relation close to the dispersion relation of the linear Stokes waves. Nwogu (1993) derived a modified form of the conventional Boussinesq equations in terms of the horizontal velocity on an arbitrary water level. The linearized dispersion relation induced from the modified Boussinesq equations becomes close to that of the linear Stokes waves if the velocities are chosen near mid-depth. Nwogu's derivation of the modified Boussinesq equations is much more systematic than Madsen *et al.*'s. Chen and Liu (1995) also derived the modified Boussinesq equations in terms of the velocity potential on an arbitrary elevation. The optimal elevation providing the velocity potential is estimated by comparing the dispersion and shoaling properties of linearized modified Boussinesq equations with those of the linear Stokes wave theory.

In this paper, we first test the applicable range of the modified Boussinesq equations derived by Nwogu (1993). The role of parameter α determining an optimal elevation where the velocity potential should be evaluated is then investigated in detail. In section 2, the phase and group velocities of the modified Boussinesq equations are derived. In section 3, Comparisons between thephase velocity and group velocities of the modified Boussinesq equations and those of the linear Stokes waves are made. The optimal value of parameter α determining the linear dispersion property is also discussed. Finally, concluding remarks are drawn in section 4.

2. PHASE AND GROUP VELOCITIES OF MODIFIED BOUSSINESQ EQUATIONS

The conventional Boussinesq equations are widely and frequently used to describe the weakly nonlinear and weakly dispersive water waves (Peregrine, 1972). However, there is a limitation in the applicability of the Boussinesq equationsto relatively deep water depth.

In this study, Nwogu's (1993) modified Boussinesq equations are employed to investigate the characteristics of the phase speed and group velocity of water wave. Nwogu's modified Boussinesq equations are written as

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot [(h+\zeta)\mathbf{u}] + \nabla \cdot \left[\left(\frac{z_\alpha^2 h}{2} - \frac{h^3}{6} \right) \nabla (\nabla \cdot \mathbf{u}) + \left(z_\alpha h + \frac{h^2}{2} \right) \nabla (\nabla \cdot (h\mathbf{u})) \right] = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + g \nabla \zeta + z_\alpha \nabla \left[\nabla \cdot \left(h \frac{\partial \mathbf{u}}{\partial t} \right) \right] + \frac{z_\alpha^2}{2} \nabla \left(\nabla \cdot \frac{\partial \mathbf{u}}{\partial t} \right) = 0 \tag{2}$$

in which ζ is the free surface displacement, **u** is the horizontal velocity vector on an arbitrary water level $z=z_\alpha$, g is the gravitational acceleration, ▽ is the horizontal gradient operator, and α is the tuning parameter defined as

$$\alpha = \frac{z_\alpha}{h} + \frac{1}{2}\left(\frac{z_\alpha}{h} \right)^2 \tag{3}$$

Equations (1) and (2) represent the continuity and momentum equations, respectively. It should be noted that the continuity equation of the conventional Boussinesq equations is an exact one, whereas equation (1) is no longer exact.

For horizontally one-dimensional case, the linearized modified Boussinesq equations over a constant water depth are expressed as

$$\frac{\partial \zeta}{\partial t} + h \frac{\partial u}{\partial x} + \left(\alpha + \frac{1}{3} \right) h^3 \frac{\partial^3 u}{\partial x^3} = 0 \tag{4}$$

$$\frac{\partial u}{\partial t} + g \frac{\partial \zeta}{\partial x} + \alpha h^2 \frac{\partial^3 u}{\partial x^2 \partial t} = 0 \tag{5}$$

To derive the phase and group velocities, the following forms have been assumed (Mei, 1989)

$$\zeta = A_0 e^{i(kx - \varpi t)}, \qquad u = U_0 e^{i(kx - \varpi t)} \tag{6}$$

By substituting equation (6) into equations (4) and (5) and cancelling the exponential factors, the following relations can be derived

$$\varpi A_0 - \left[kh - \left(\alpha + \frac{1}{3} \right) k^3 h^3 \right] U_0 = 0 \tag{7}$$

$$kgA_0 - \varpi\left(1 - \alpha k^2 h^2\right)U_0 = 0 \qquad (8)$$

Equations (7) and (8) are homogeneous for A_0 and U_0. To have a nontrivial solution the determinant of equations (7) and (8) should be vanished.

Then, the following phase velocity can be derived

$$C = \left[gh \frac{1 - (\alpha + 1/3)(kh)^2}{1 - \alpha (kh)^2} \right]^{1/2} \qquad (9)$$

By taking derivative of the frequency with respect to the wavenumber the group velocity can also be derived as

$$C_g = C\left[1 - \frac{(kh)^2}{3[1 - \alpha (kh)^2][1 - (\alpha + 1/3)(kh)^2]}\right] \qquad (10)$$

The phase and group velocities of the linear Stokes wave are given as

$$C_l = \left(gh \frac{1}{k_l h} \tanh k_l h\right)^{1/2} \qquad (11)$$

$$C_{gl} = \frac{C_l}{2}\left(1 + \frac{2k_l h}{\sinh 2k_l h}\right) \qquad (12)$$

It shoude be noted that the wavenumber k obtained from equation (9) is different from the wavenumber k_l obtained from (11).

3. COMPARISON AND DISCUSSION

In this section, the optimal value of α is estimated by comparing the characteristic properties of the modified Boussinesq equations with those of the linear Stokes wave.

Nwogu (1993) estimated an optimal value of α by minimizing the equared relative error of the phase velocity for the range of $0 < k_0 h/\pi < 1$ where k_0 ($=\omega^2/g$) is the wavenumber of a deep water wave. That is, the equared relative error is defined as

$$I_1 = \left(\frac{C}{C_l} - 1\right)^2 \qquad (13)$$

The obtained optimal value is α=-0.3900 corresponding to the water level of z_α=-0.5310h.

Although no attempt has been made in previous studies, the squared relative error of the group velocity is also tested in this study. That is,

$$I_2 = \left(\frac{C_g}{C_{gl}} - 1\right)^2 \qquad (14)$$

By minimizing the combined squared relative errors of the phase and group velocities for the range of $0 < k_0 h/\pi < 1$, Chen and Liu (1995) obtained α = -0.3855. The value of α = -0.3855 corresponds to the water depth of z_α= -0.5215h.

$$I_3 = \left(\frac{C}{C_l} - 1\right)^2 + \left(\frac{C_g}{C_{gl}} - 1\right)^2 \qquad (15)$$

Different from previous studies the value of α is no longer a constant in this study. In other words, the parameter α is varying as water waves propagate over a varying topography. The squared relative errors expressed by I_1, I_2 and I_3 are employed to calculate and compare optimal values of α for a wide range of water depths.

Firstly, by minimizing the squared relative error of the phase velocity expressed as I_1 the value of α can be obtained. Secondly, by minimizing the squared relative error of the group velocity expressed as I_2 the analytical expression of α can also be obtained. Thirdly, following

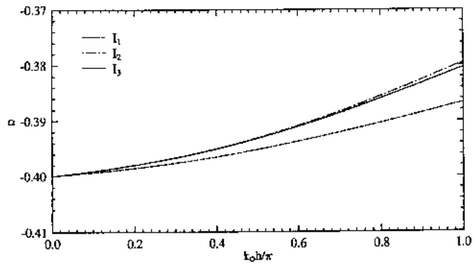

Fig. 1. Variation of the optimal value of α with respect to the relative water depth

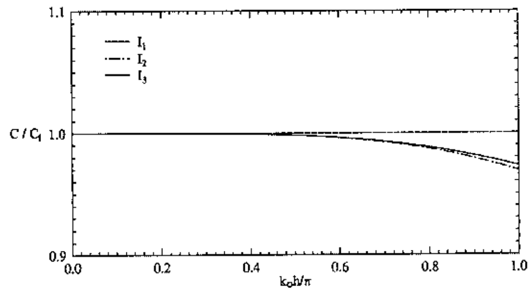

Fig. 2. Ratio of the phase velocity with optimal values of α to that of the linear Stokes wave

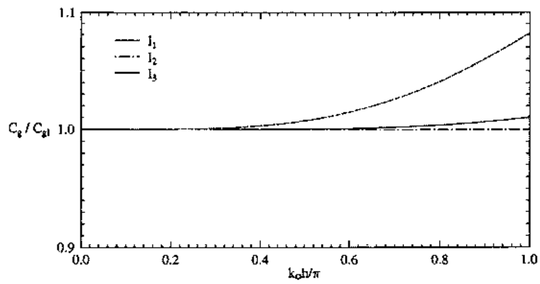

Fig. 3. Ratio of the group velocity with optimal values of α to that of the linear Stokes wave

Chen and Liu (1995) the value of α is determined by minimizing the sum of squared relative errors of the phase and group velocities as given by I_3. However, it should be noted again that α is no longer a constant in this study.

In Fig. 1, optimal values of α obtained from I_1, I_2 and I_3 are plotted. In very shallow water, α asymptotically approaches to -0.4000 for all cases. The discrepancy between values of α obtained from I_1 and I_2 or I_1 and I_3 increases very rapidly as k_0h increases. However, the discrepancy between obtained values by I_2 and I_3 is negligibly small. The optimal value of α increases as a wave propagates to deep water.

The ratios of the phase and group velocities to the respective velocities of the linear Stokes wave are calculated and compared in Fig. 2 and Fig. 3, respectively. As expected, I_1 produces no visible error of the phase velocity. On the other hand, I_2 and I_3 give errors of the phase velocity

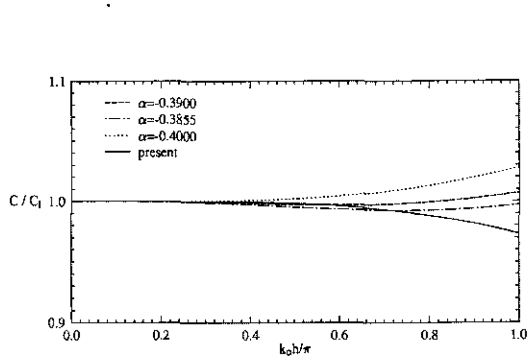

Fig. 4. Ratios of phase velocities with different values of α to that of the linear Stokes wave

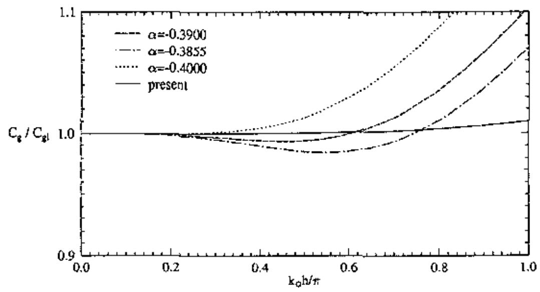

Fig. 5. Ratios of group velocities with different values of α to that of the linear Stokes wave

with maxima of -3.1% and -2.7 % at k_0h/π=1, respectively. The overall difference between the calculated phase velocity and that of the linear Stokes wave is small. Similarly, I_2 yields no visible error of the group velocity. The maximum relative error of the group velocity induced by I_1 is +8.2 %, whereas that induced by I_3 is only +1.0 %. Therefore, I_3 gives more agreeable phase and group velocities than I_1 and I_2.

We calculate and compare the relative errors of the phase and group velocities for different values of α=-0.4000 obtained by Witting (1984), α=-0.3900 obtained by Nwogu (1993), α=-0.3855 obtained by Chen and Liu (1995), and α with minimum combined errors of the phase and group velocities. It is remarked that, if α =-0.4000 is used, the phase velocity of the linear modified Boussinesq equations becomes the [2,2] Pad'e approximation to ghtanh kh/kh.

In Fig. 4, the ratios of the phase velocity to that of the linear Stokes wave are compared for different values of α. The present study yields the best match to the phase velocity in the depth range of $0 < k_0h/\pi \leqq 0.570$, while α=-0.3900 does in the depth range of $k_0h/\pi > 0.570$ except near k_0h/π=1 where α =-0.3855 produces better results. The maximum errors of the phase velocity for α =-0.4000, -0.3900, -0.3855, and the present study are +2.8 %, +0.7 %, -0.7 %, and -2.7 %, respectively. It is noted that the maximum error always occurs at k_0h/π=1 except α =-0.3855 where it does at k_0h/π =0.740.

In Fig. 5, the ratios of the group velocity to that of the linear Stokes wave are compared for different values of α. The present study yields the group velocity closest to that of the linear Stokes wave. The maximum errors of the group velocity are +17.0 %, +10.4 %, +7.1 %, and

+1.0 % for α=-0.4000, -0.3900, -0.3855, and the present study, respectively.

4. CONCLUDING REMARKS

In this study, the optimal values of α characterizing the dispersive property of the modified Boussinesq equations are determined by minimizing the combined squared relative errors of the phase and group velocities on the whole water depth. Different from previous studies the value of α is not fixed in this study.

It is shown that the optimal values of α produces the phase and group velocities closer to those of the linear Stokes wave than previous studies. The waves in nearshore region is irregular with narrow frequency band. Thus, if an optimal value of α is chosen for the peak frequency, the phase and group velocities of the linear Boussinesq equations would be very close those of the linear Stokes wave even in very deep water. This implies that the choice of optimal values of α determined in this study is necessary for accurate prediction of real sea state.

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