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Fuzzy r-convergent nets

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ABSTRACT

In this paper, we investigate some properties of fuzzy r-cluster points and fuzzy r-limit points in smooth fuzzy topological spaces. We define fuzzy r-convergent nets and investigate some of their properties.

1. Introduction and preliminaries

Pu and Liu [13] introduced the notions of Q-neighborhoods and fuzzy nets Q-neighborhoods and established the convergence theory in fuzzy topological spaces. Chen and Cheng [3] introduced the concepts of fuzzy clusterand fuzzy limit points in fuzzy topological spaces with respect to R-neighborhoods instead of Q-neighborhoods. The convergence theory in fuzzy topological spaces has been developed in many directions [4,5,7,15]. A.P. Sostak [14] introduced the smooth fuzzy topology as an extension of Chang's fuzzy topology [1]. In [11], it was introduced the concepts of fuzzy r-cluster and fuzzy r-limit points in smooth fuzzy topological spaces.

In this paper, we investigate some properties of fuzzy r-cluster points and fuzzy r-limit points in smooth fuzzy topological spaces. We define fuzzy r-convergent nets and investigate some of their properties.

Throughout this paper, let X be a nonempty set, $I=[0,1]$ and $I_0=(0,1]$. A *fuzzy point* x_t for $t\in I_0$ is an element of I^X such that, for $y\in X$,

$$x_t(y)=\begin{cases} t \text{ if } y=x,\\ 0 \text{ if } y\neq x.\end{cases}$$

The set of all fuzzy points in X is denoted by $Pt(X)$. For $x_t\in Pt(X)$, $x_t\in\lambda$ iff $t\leq\lambda(x)$. For λ, $\mu\in I^X$, λ is *quasi-coincident* with μ, denoted by $\lambda\ q\ \mu$, if there exists $x\in X$ such that $\lambda(x)+\mu(x)>1$. If λ is not quasi-coincident with μ, we denote $\lambda\ \bar{q}\ \mu$.

All the other notations and the other definitions are standard in fuzzy set theory.

Lemma 1.1 [12] Let $f:X\to Y$ be a function. Let λ, μ, ρ, $\lambda_i\in I^X$ for each $i\in\Gamma$, $x_t\in Pt(X)$ and $\nu\in I^Y$. Then the following properties hold:
(1) If $\lambda\ q\ \mu$ and $\mu\leq\rho$, then $\lambda\ q\ \rho$.
(2) $x_t\ q\bigvee_{i\in\Gamma}\lambda_i$ iff there exists $j\in\Gamma$ such that $x_t\ q\ \lambda_j$.
(3) $\lambda\leq\mu$ iff $x_t\in\mu$ for all $x_t\in\lambda$ iff $x_t\ q\ \lambda$ implies $x_t\ q\ \mu$.
(4) $\lambda\ q\ f^{-1}(\nu)$ iff $f(\lambda)\ q\ \nu$.

Definition 1.2 [14] A function $\tau: I^X\to I$ is called a *smooth fuzzy topology* on X if it satisfies the following conditions:
(O1) $\tau(\tilde{0})=\tau(\tilde{1})=1$, where $\tilde{0}(x)=0$ and $\tilde{1}(x)=1$ for all $x\in X$.
(O2) $\tau(\mu_1\wedge\mu_2)\geq\tau(\mu_1)\wedge\tau(\mu_2)$, for any $\mu_1,\mu_2\in I^X$.
(O3) $\tau(\bigvee_{i\in\Gamma}\mu_i)\geq\bigwedge_{i\in\Gamma}\tau(\mu_i)$, for any $\{\mu_i\}_{i\in\Gamma}\subset I^X$.
The pair (X,τ) is called a *smooth fuzzy topological space.*

Theorem 1.3 [2] Let (X,τ) be a smooth fuzzy topological space. For each $r\in I_0$ and $\lambda\in I^X$, we define a fuzzy closure operator $C_\tau: I^X\times I_0\to I^X$ as follows:

$$C_\tau(\lambda,r)=\bigwedge\{\rho\in I^X \mid \lambda\leq\rho,\ \tau(\tilde{1}-\rho)\geq r\}.$$

For λ, $\mu\in I^X$ and r, $s\in I_0$, it satisfies the following properties:
(1) $C_\tau(\tilde{0},r)=\tilde{0}$.
(2) $\lambda\leq C_\tau(\lambda,r)$.
(3) $C_\tau(\lambda,r)\vee C_\tau(\mu,r)=C_\tau(\lambda\vee\mu,r)$.
(4) $C_\tau(\lambda,r)\leq C_\tau(\lambda,s)$, if $r\leq s$.
(5) $C_\tau(C_\tau(\lambda,r),r)=C_\tau(\lambda,r)$.

Definition 1.4 [6] Let (X,τ) be a smooth fuzzy topological space, $\mu\in I^X$, $x_t\in Pt(X)$ and $r\in I_0$. μ is called a r-open Q-neighborhood of x_t if $x_t\ q\ \mu$ with $\tau(\mu)\geq r$.

We denote

$$\mathcal{N}(x_t,r)=\{\mu\in I^X \mid x_t\ q\ \mu,\ \tau(\mu)\geq r\}.$$

Definition 1.5 [10] Let (X,τ) be a smooth fuzzy topological space, $\lambda\in I^X$, $x_t\in Pt(X)$ and $r\in I_0$. x_t is called a *fuzzy r-adherent point* of λ if for every $\mu\in\mathcal{N}(x_t,r)$, we have $\mu\ q\ \lambda$.

Theorem 1.6 [10] Let (X,τ) be a smooth fuzzy topological space. For each $\lambda\in I^X$ and $r\in I_0$, we have

$C_\tau(\lambda,r)=\bigvee\{x_t\in Pt(X) \mid x_t$ is a fuzzy r-adherent point of $\lambda\}$.

Definition 1.7 [13] Let D be a directed set. A function $S: D \to Pt(X)$ is called a *fuzzy net*. Let $\lambda \in I^X$. We say S is a *fuzzy net in* λ if $S(n) \in \lambda$ for every $n \in D$. A fuzzy net S is *increasing*(resp. decreasing) if $S(m) \leq S(n)$ (resp. $S(n) \leq S(m)$) for every $m \leq n$ with $m, n \in D$.

Definition 1.8 [11] Let (X, τ) be a smooth fuzzy topological space, $\mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$.

(1) x_t is called a *fuzzy r-cluster point* of S, denoted by $S \overset{r}{\infty} x_t$, if for every $\mu \in \mathcal{N}(x_t, r)$, S is frequently quasi-coincident with μ, that is, for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $S(n_0)\ q\ \mu$.

(2) x_t is called a *fuzzy r-limit point* of S, denoted by $S \overset{r}{\rightarrow} x_t$, if for every $\mu \in \mathcal{N}(x_t, r)$, S is eventually quasi-coincident with μ, that is, there exists $n_0 \in D$ such that for each $n \in D$ with $n \geq n_0$, we have $S(n)\ q\ \mu$.

We denote

$clu_\tau(S, r) = \bigvee \{x_t \in Pt(X) \mid x_t$ is a fuzzy r-cluster point of $S\}$,

$lim_\tau(S, r) = \bigvee \{x_t \in Pt(X) \mid x_t$ is a fuzzy r-limit point of $S\}$.

Definition 1.9 [13] Let $S: D \to Pt(X)$ and $T: E \to Pt(X)$ be two fuzzy nets. A fuzzy net T is called a *subnet* of S if there exists a function $N: E \to D$, called by a *cofinal selection* on S, such that

(1) $T = S \circ N$;

(2) For every $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geq n_0$ for $m \geq m_0$.

Theorem 1.10 [11] Let (X, τ) be a smooth fuzzy topological space. Let $S: D \to Pt(X)$ be a fuzzy net and T $E \to Pt(X)$ a subnet of S. For $r, s \in I_0$, the following properties hold:

(1) If $S \overset{r}{\rightarrow} x_t$, then $S \overset{r}{\infty} x_t$.
(2) $lim_\tau(S, r) \leq clu_\tau(S, r)$.
(3) If $S \overset{r}{\infty} x_t$ and $x_t \geq x_s$, then $S \overset{r}{\infty} x_s$.
(4) If $S \overset{r}{\rightarrow} x_t$ and $x_t \geq x_s$, then $S \overset{r}{\rightarrow} x_s$.
(5) $S \overset{r}{\infty} x_t$ iff $x_t \in clu_\tau(S, r)$.
(6) $S \overset{r}{\rightarrow} x_t$ iff $x_t \in lim_\tau(S, r)$.
(7) If $S \overset{r}{\rightarrow} x_t$, then $T \overset{r}{\rightarrow} x_t$.
(8) $lim_\tau(S, r) \leq lim_\tau(T, r)$.
(9) If $T \overset{r}{\infty} x_t$, then $S \overset{r}{\infty} x_t$.
(10) $clu_\tau(T, r) \leq clu_\tau(S, r)$.

Theorem 1.11 [11] Let (X, τ) be a smooth fuzzy topological space and $x_t \in Pt(X)$ and $r \in I_0$. For every fuzzy net S, $S \overset{r}{\rightarrow} x_t$ iff $T \overset{r}{\infty} x_t$, for every subnet T of S.

Theorem 1.12 [11] Let (X, τ) be a smooth fuzzy topological space and $x_t \in Pt(X)$ and $r \in I_0$. For every fuzzy net $S: D \to Pt(X)$, $S \overset{r}{\infty} x_t$ iff S has a subnet T such that $T \overset{r}{\rightarrow} x_t$.

Theorem 1.13 [11] Let (X, τ) be a smooth fuzzy topological space and $x_t \in Pt(X)$ and $r \in I_0$. Then the following statements are equivalent.

(1) $x_t \in C_\tau(\lambda, r)$.
(2) There exists a fuzzy net $S \in \lambda$ such that $S \overset{r}{\infty} x_t$.
(3) There exists a fuzzy net $S \in \lambda$ such that $S \overset{r}{\rightarrow} x_t$.

2. The properties of fuzzy r-cluster and fuzzy r-limit points

Theorem 2.1 Let (X, τ) be a smooth fuzzy topological space and $S: D \to Pt(X)$ a fuzzy net. For $r \in I_0$, the following properties hold:

(1) $C_\tau(clu_\tau(S, r), r) = clu_\tau(S, r)$.
(2) $clu_\tau(S, r) \leq C_\tau(\bigvee_{n \in D} S(n), r)$.

Proof. (1) From Theorem 1.3(2), we have

$C_\tau(clu_\tau(S, r), r) \geq clu_\tau(S, r)$.

Suppose $C_\tau(clu_\tau(S, r), r) \not\leq clu_\tau(S, r)$. From Theorem 1.6, there exists a fuzzy r-adherent point x_t of $clu_\tau(S, r)$ such that

$C_\tau(clu_\tau(S, r), r)(x) \geq t > clu_\tau(S, r)(x)$.

Since x_t is a fuzzy r-adherent point of $clu_\tau(S, r)$, for each $\mu \in \mathcal{N}(x_t, r)$, we have

$\mu\ q\ clu_\tau(S, r)$.

Since $\mu\ q\ clu_\tau(S, r)$, there exists $y \in X$ such that

$\mu(y) + clu_\tau(S, r)(y) > 1$.

From the definition of $clu_\tau(S, r)$, there exists a fuzzy r-cluster point y_p of S such that

$\mu(y) + clu_\tau(S, r)(y) \geq \mu(y) + p > 1$.

Thus $\mu \in \mathcal{N}(y_p, r)$. Since $S \overset{r}{\infty} y_p$ and $\mu \in \mathcal{N}(y_p, r)$, for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $S(n_0)$ $q\ \mu$. Hence x_t is a fuzzy r-cluster point of S. So, $clu_\tau(S, r)(x) \geq t$. It is a contradiction. Hence

$C_\tau(clu_\tau(S, r), r) \leq clu_\tau(S, r)$.

(2) Suppose $clu_\tau(S, r) \not\leq C_\tau(\bigvee_{n \in D} S(n), r)$. Then there exists a fuzzy r-cluster point x_t of S such that

$$clu_\tau(S, r)(x) \geq t > C_\tau(\bigvee_{n \in D} S(n), r)(x). \qquad \text{(I)}$$

Since x_t is a fuzzy r-cluster point of S, for each $\mu \in \mathcal{N}(x_t, r)$, for each $n \in D$, there exists $n_0 \geq n$ with $S(n_0)\ q$ μ. Since $S(n_0) \leq \bigvee_{n \in D} S(n)$, by Lemma 1.1(1), we have $\bigvee_{n \in D} S(n)\ q\ \mu$. Hence x_t is a fuzzy r-adherent point of $\bigvee_{n \in D} S(n)$. Therefore $C_\tau(\bigvee_{n \in D} S(n), r)(x) \geq t$. It is a contradiction for (I). Hence

$clu_\tau(S, r) \leq C_\tau(\bigvee_{n \in D} S(n), r)$. □

Theorem 2.2 Let (X, τ) be a smooth fuzzy topological space and S, $U : D \to Pt(X)$ fuzzy nets such that $S(n) \vee U(n)$, $S(n) \wedge U(n) \in Pt(X)$ for each $n \in D$. Define fuzzy nets $S \vee U$, $S \wedge U : D \to Pt(X)$ by, for each $n \in D$,

$$(S \vee U)(n) = S(n) \vee U(n),\ (S \wedge U)(n) = S(n) \wedge U(n).$$

For each $r \in I_0$, the following properties hold:
(1) If $S(n) \leq U(n)$ for all $n \in D$, then

$$clu_\tau(S, r) \leq clu_\tau(U, r),\ lim_\tau(S, r) \leq lim_\tau(U, r).$$

(2) $clu_\tau(S \vee U, r) = clu_\tau(S, r) \vee clu_\tau(U, r)$.
(3) $clu_\tau(S \wedge U, r) \leq clu_\tau(S, r) \wedge clu_\tau(U, r)$.
(4) $lim_\tau(S \vee U, r) \leq lim_\tau(S, r) \vee lim_\tau(U, r)$.
(5) $lim_\tau(S \wedge U, r) \leq lim_\tau(S, r) \wedge lim_\tau(U, r)$.

Proof. (1) Let x_t is a fuzzy r-cluster point of S. For each $\mu \in \mathcal{M}(x_t, r)$ and for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $S(n_0)\ q\ \mu$. Since $S(n) \leq U(n)$ for all $n \in D$, by Lemma 1.1(1), $U(n_0)\ q\ \mu$. Thus x_t is a fuzzy r-cluster point of U. Hence $clu_\tau(S, r) \leq clu_\tau(U, r)$.
Similarly, we have $lim_\tau(S, r) \leq lim_\tau(U, r)$.
(2) Since $S \leq S \wedge U$ and $T \leq S \vee U$, by (1), we have

$$clu_\tau(S \vee U, r) \geq clu_\tau(S, r) \vee clu_\tau(U, r).$$

Suppose $clu_\tau(S \vee U, r) \not\leq clu_\tau(S, r) \vee clu_\tau(U, r)$. Then there exists a fuzzy r-cluster point x_t of $S \vee U$ such that

$$clu_\tau(S \vee U, r)(x) \geq t > clu_\tau(S, r)(x) \vee clu_\tau(U, r)(x).$$

Hence $x_t \notin clu_\tau(S, r)$ and $x_t \notin clu_\tau(U, r)$.
Since x_t is not a fuzzy r-cluster point of S, there exist $\mu_1 \in \mathcal{M}(x_t, r)$ and $n_1 \in D$ such that $S(n)\ \bar{q}\ \mu_1$ for every $n \in D$ with $n \geq n_1$.
Since x_t is not a fuzzy r-cluster point of U, there exist $\mu_2 \in \mathcal{M}(x_t, r)$ and $n_2 \in D$ such that $U(n)\ \bar{q}\ \mu_2$ for every $n \in D$ with $n \geq n_2$.
Let $\mu = \mu_1 \wedge \mu_2$ and $n_3 \in D$ such that $n_3 \geq n_1$ and $n_3 \geq n_2$. Since $\mu_1 \leq \tilde{1} - S(n)$ and $\mu_2 \leq \tilde{1} - U(n)$ for $n \geq n_3$, we have $\mu_1 \wedge \mu_2 \leq \tilde{1} - (S(n) \vee U(n))$. So, $\mu \in \mathcal{M}(x_t, r)$ and $n_3 \in D$ such that $(S \vee U)(n)\ \bar{q}\ \mu$ for every $n \in D$ with $n \geq n_3$. Thus x_t is not a fuzzy r-cluster point of $S \vee U$. It is a contradiction. Hence we have

$$clu_\tau(S \vee U, r) \leq clu_\tau(S, r) \vee clu_\tau(U, r).$$

(3),(4) and (5) are easily proved.

Theorem 2.3 Let (X, τ) be a smooth fuzzy topological space and $S : D \to Pt(X)$ a fuzzy net. Then we have

$$clu_\tau(S, r) = \bigwedge_{n_0 \in D} C_\tau(\bigvee_{n \geq n_0} S(n), r).$$

Proof. Let $x_t \in clu_\tau(S, r)$. From Theorem 1.10 (5), since x_t is a fuzzy r-cluster point of S, for each $\mu \in \mathcal{M}(x_t, r)$ and for each $n_0 \in D$, there exists $n \in D$ such that $n \geq n_0$ and $S(n)\ q\ \mu$. Since $S(n) \leq \bigvee_{n \geq n_0} S(n)$, by Lemma 1.1(1), we have $\bigvee_{n \geq n_0} S(n)\ q\ \mu$. Hence x_t is a fuzzy r-adherent point of $\bigvee_{n \geq n_0} S(n)$, for all $n_0 \in D$, that is,

$$x_t \in \bigwedge_{n_0 \in D} C_\tau(\bigvee_{n \geq n_0} S(n), r).$$

From Lemma 1.1 (3), we have

$$clu_\tau(S, r) \leq \bigwedge_{n_0 \in D} C_\tau(\bigvee_{n \geq n_0} S(n), r).$$

Suppose

$$clu_\tau(S, r) \not\geq \bigwedge_{n_0 \in D} C_\tau(\bigvee_{n \geq n_0} S(n), r).$$

There exists a fuzzy r-adherent point x_t of $\bigvee_{n \geq n_0} S(n)$, for all $n_0 \in D$, such that

$$clu_\tau(S, r)(x) < t \leq C_\tau(\bigvee_{n \geq n_0} S(n), r)(x).$$

Since x_t is a fuzzy r-adherent point of $\bigvee_{n \geq n_0} S(n)$, for each $n_0 \in D$, for each $\mu \in \mathcal{M}(x_t, r)$, we have

$$\bigvee_{n \geq n_0} S(n)\ q\ \mu.$$

Since $\bigvee_{n \geq n_0} S(n)\ q\ \mu$, there exists $y \in X$ such that

$$\bigvee_{n \geq n_0} S(n)(y) + \mu(y) > 1.$$

Then there exists $n \in D$ such that $n \geq n_0$ and

$$\bigvee_{n \geq n_0} S(n)(y) + \mu(y) \geq S(n)(y) + \mu(y) > 1.$$

It implies $S(n)\ q\ \mu$. Hence x_t is a fuzzy r-cluster point of S, that is,

$$x_t \in clu_\tau(S, r).$$

It is a contradiction. Hence

$$clu_\tau(S, r) \geq \bigwedge_{n_0 \in D} C_\tau(\bigvee_{n \geq n_0} S(n), r). \qquad \square$$

Theorem 2.4 Let (X, τ) be a smooth fuzzy topological space and $S : D \to Pt(X)$ a fuzzy net. For $r \in I_0$, the following properties hold:
(1) $C_\tau(lim_\tau(S, r), r) = lim_\tau(S, r)$.
(2) $\bigwedge_{n \in D} S(n) \leq lim_\tau(S, r)$.
(3) $\bigvee_{n_0 \in D}(\bigwedge_{n \geq n_0} S(n)) \leq lim_\tau(S, r)$.

Proof. (1) It is similarly proved as Theorem 2.1(1).
(2) Suppose $\bigwedge_{n \in D} S(n) \not\leq lim_\tau(S, r)$. Then there exist $x \in X$ and $t \in I_0$ such that

$$\bigwedge_{n \in D} S(n)(x) > t > lim_\tau(S, r)(x).$$

Since $t > lim_\tau(S, r)(x)$, by Theorem 1.10(6), x_t is not a fuzzy r-limit point of S. So, there exists $\mu \in \mathcal{M}(x_t, r)$ such that for each $n \in D$, there exists $n_0 \in D$ satisfying $n_0 \geq n$ and $\mu\ \bar{q}\ S(n_0)$. Since $x_t\ q\ \mu$, we have

$$S(n_0)(x) + 1 - t < S(n_0)(x) + \mu(x) \leq 1.$$

Thus $S(n_0)(x) < t$ implies $\bigwedge_{n \in D} S(n)(x) < t$. It is a contradiction. Hence we have

$$\bigwedge_{n \in D} S(n) \leq lim_\tau(S, r).$$

(3) Suppose $\bigvee_{n_0 \in D}(\bigwedge_{n \geq n_0} S(n)) \not\leq lim_\tau(S, r)$. Then

there exist a $x\in X$ and $t\in I_0$ such that

$$\vee_{n_0\in D}(\wedge_{n\geq n_0} S(n))(x) > t > lim_\tau(S, r)(x).$$

Since $t < \vee_{n_0\in D}(\wedge_{n\geq n_0} S(n))(x)$, there exists $n_0\in D$ such that

$$x_t\in \wedge_{n\geq n_0} S(n).$$

It implies $t\leq S(n)(x)$ for all $n\geq n_0$. Hence for each $\mu \in \mathcal{M}(x_t, r)$, $t+\mu(x)>1$ implies $S(n)(x)+\mu(x)>1$, for all $n\geq n_0$. So, x_t is a fuzzy r-limit point of S. It is a contradiction. Hence we have

$$\vee_{n_0\in D}(\wedge_{n\geq n_0} S(n))\leq lim_\tau(S, r).$$ □

Example 2.5 Let $X=\{a, b\}$ be a set, N a natural number set and $\mu\in I^X$ as follows:

$$\mu(a)=0.3,\ \mu(b)=0.4.$$

We define a smooth fuzzy topology $\tau : I^X\to I$ as follows:

$$\tau(\lambda)=\begin{cases}1, & \text{if } \lambda=\tilde{0} \text{ or } \tilde{1},\\ \frac{1}{2}, & \text{if } \lambda=\mu,\\ 0, & \text{otherwise.}\end{cases}$$

(1) In general, $clu_\tau(S, r)\neq C_\tau(\vee_{n\in D}S(n), r)$.
Define a fuzzy net $S : N\to Pt(X)$ by

$$S(n)=x_{a_n},\ a_n=0.6+0.2/n.$$

Then $\vee_{n\in N} S(n)=x_{0.8}$. From Theorem 1.3, we have for all $r\in I_0$,

$$C_\tau(x_{0.8}, r)=\tilde{1}.$$

But $x_{0.8}$ is not a fuzzy 1/2-cluster point of S, because there exist $\mu\in\mathcal{M}(x_{0.8}, 1/2)$ and $2\in N$, for all $n\geq 2$, we have $S(n)\ \bar{q}\ \mu$. It follows

$$clu_\tau(S, 1/2)(x)<0.8 \text{ but } C_\tau(\vee_{n\in D}S(n), 1/2)(x)=1.$$

(2) In general, $clu_\tau(S\wedge U, r)\neq clu_\tau(S, r)\wedge clu_\tau(U, r)$.
Define fuzzy nets $S, U : N\to Pt(X)$ by

$$S(n)=x_{a_n}\ a_n=0.8+(-1)^n 0.2.$$

$$U(n)=x_{b_n}\ b_n=0.8+(-1)^{n+1}0.2.$$

From Theorem 2.2, $(S\wedge U)(n)=x_{0.6}$ is a fuzzy net. For $\mu\in\mathcal{M}(x_{0.8}, 1/2)$ and for all $n\in N$, we have $(S\wedge U)(n)\ \bar{q}\ \mu$. Thus $x_{0.8}$ is not a fuzzy 1/2-cluster point of $S\wedge U$.

On the other hand, for $\tilde{1}, \mu\in\mathcal{M}(x_{0.8}, 1/2)$ and for each $n\in N$, there exists $2n\geq n$ such that $S(2n)\ q\ \mu$ and there exists $2n+1\geq n$ such that $U(2n+1)\ q\ \mu$. It implies

$$x_{0.8}\in clu_\tau(S, 1/2),\ x_{0.8}\in clu_\tau(U, 1/2).$$

Hence we have

$$clu_\tau(S\wedge U, 1/2)(x)<0.8\leq clu_\tau(S, 1/2)(x) \wedge clu_\tau(U, 1/2)(x).$$

(3) In general, $lim_\tau(S\vee U, r)\neq lim_\tau(S, r)\vee lim_\tau(U, r)$.
Define fuzzy nets $S, U : N\to Pt(X)$ by

$$S(n)=x_{a_n}\ a_n=0.6+(-1)^n 0.2.$$

$$U(n)=x_{b_n}\ b_n=0.6+(-1)^{n+1}0.2.$$

From Theorem 2.2, $(S\vee U)(n)=x_{0.8}$ is a fuzzy net. For $\tilde{1}, \mu\in\mathcal{M}(x_{0.8}, 1/2)$ and for each $n\in N$, $(S\vee U)(n)\ q\ \mu$ and $(S\vee U)(n)\ q\ \tilde{1}$. Hence $x_{0.8}$ is a fuzzy 1/2-limit point of $S\vee U$.

On the other hand, for $\mu\in\mathcal{M}(x_{0.8}, 1/2)$ and for each $n\in N$, there exists $2n+1\geq n$ such that $S(2n+1)\ \bar{q}\ \mu$ and there exists $2n\geq n$ such that $U(2n)\ \bar{q}\ \mu$. Thus

$$x_{0.8}\notin lim_\tau(S, 1/2),\ x_{0.8}\notin lim_\tau(U, 1/2).$$

So,

$$lim_\tau(S\vee U, 1/2)(x)\geq 0.8>lim_\tau(S, 1/2)(x) \vee lim_\tau(U, 1/2)(x).$$

(4) In general, $\wedge_{n\in D} S(n)\neq lim_\tau(S, r)$ and $\vee_{n_0\in D}(\wedge_{n\geq n_0} S(n))\neq lim_\tau(S, r)$.
Define a fuzzy net $S : N\to Pt(X)$ by

$$S(n)=x_{a_n},\ a_n=0.8+(-1)^n 0.2.$$

Then we have

$$\wedge_{n\in D} S(n)=\vee_{n_0\in D}(\wedge_{n\geq n_0} S(n))=x_{0.6}.$$

Then $x_{0.7}$ is a fuzzy 1/2-limit point of S, for $\tilde{1}\in\mathcal{M}(x_{0.7}, 1/2)$ and for all $n\in N$, we have $S(n)\ q\ \tilde{1}$.

$$lim_\tau(S, 1/2)(x)\geq 0.7.$$

Hence

$$\wedge_{n\in D} S(n)=\vee_{n_0\in D}(\wedge_{n\geq n_0} S(n))\neq lim_\tau(S, 1/2).$$

Theorem 2.6 Let (X, τ) be a smooth fuzzy topological space and $S : D\to Pt(X)$ a decreasing fuzzy net. Then, for each $r\in I_0$, we have

$$clu_\tau(S, r)=\wedge_{n\in D} C_\tau(S(n), r).$$

Proof. Suppose

$$clu_\tau(S, r)\not\leq \wedge_{n\in D} C_\tau(S(n), r).$$

There exists a fuzzy r-cluster point x_t of S such that

$$clu_\tau(S, r)(x)\geq t>\wedge_{n\in D}C_\tau(S(n), r)(x).$$

Since x_t is a fuzzy r-cluster point of S, for each $\mu\in\mathcal{M}(x_t, r)$ and $n\in D$, there exists $n_0\in D$ such that $n_0\geq n$ and $S(n_0)\ q\ \mu$. Since S is a decreasing fuzzy net, for $n_0\geq n$, by Lemma 1.1(1), $S(n_0)\ q\ \mu$ implies $S(n)\ q\ \mu$. Hence x_t is a fuzzy r-adherent point of $S(n)$, for each $n\in D$, that is,

$x_t \in \bigwedge_{n\in D} C_\tau(S(n), r)$.

It is a contradiction. Hence

$clu_\tau(S, r) \leq \bigwedge_{n\in D} C_\tau(S(n), r)$.

Suppose

$clu_\tau(S, r) \ngeq \bigwedge_{n\in D} C_\tau(S(n), r)$.

There exists $x\in X$ such that

$clu_\tau(S, r)(x) < \bigwedge_{n\in D} C_\tau(S(n), r)(x)$.

There exists a fuzzy r-adherent point x_t of $S(n)$, for all $n\in D$, such that

$clu_\tau(S, r)(x) < t \leq \bigwedge_{n\in D} C_\tau(S(n), r)(x)$.

Since x_t is a fuzzy r-adherent point of $S(n)$, for all $n \in D$, for each $\mu\in \mathcal{M}(x_t, r)$ and for $n\in D$, there exists $n \in D$ such that $n \geq n$ and $S(n)\ q\ \mu$. Hence x_t is a fuzzy r-cluster point of S, that is,

$x_t \in clu_\tau(S, r)$.

It is a contradiction. Hence

$clu_\tau(S, r) \geq \bigwedge_{n\in D} C_\tau(S(n), r)$. □

Theorem 2.7 Let (X, τ) be a smooth fuzzy topological space and $S : D \to Pt(X)$ an increasing fuzzy net. Then, for each $r\in I_0$, we have

$lim_\tau(S, r) = C_\tau(\bigvee_{n\in D} S(n), r)$.

Proof. Suppose

$lim_\tau(S, r) \nleq C_\tau(\bigvee_{n\in D} S(n), r)$.

There exists a fuzzy r-limit point x_t of S such that

$lim_\tau(S, r)(x) \geq t > C_\tau(\bigvee_{n\in D} S(n), r)(x)$.

Since x_t is a fuzzy r-limit point of S, for each $\mu \in \mathcal{M}(x_t, r)$, there exists $n_0\in D$ such that for all $n \geq n_0$, $S(n)\ q\ \mu$. It implies $\bigvee_{n\in D} S(n)\ q\ \mu$. Hence x_t is a fuzzy r-adherent point of $\bigvee_{n\in D} S(n)$. It is a contradiction. Hence

$lim_\tau(S, r) \leq C_\tau(\bigvee_{n\in D} S(n), r)$.

Suppose

$lim_\tau(S, r) \ngeq C_\tau(\bigvee_{n\in D} S(n), r)$.

There exists a fuzzy r-adherent point x_t of $\bigvee_{n\in D} S(n)$ such that

$lim_\tau(S, r)(x) < t \leq C_\tau(\bigvee_{n\in D} S(n), r)(x)$.

Since x_t is a fuzzy r-adherent point of $\bigvee_{n\in D} S(n)$, for each $\mu\in \mathcal{M}(x_t, r)$, we have $\bigvee_{n\in D} S(n)\ q\ \mu$. By Lemma 1.1(2), there exists $n_0\in D$ such that $S(n_0)\ q\ \mu$. Since S is an increasing fuzzy net, for $n \geq n_0$, $S(n_0)\ q\ \mu$ implies $S(n)\ q\ \mu$. Hence x_t is a fuzzy r-limit point of S, that is,

$x_t \in lim_\tau(S, r)$.

It is a contradiction. Hence

$lim_\tau(S, r) \geq C_\tau(\bigvee_{n\in D} S(n), r)$. □

Definition 2.8 Let (X, τ_1) and (Y, τ_2) be smooth fuzzy topological spaces. A function $f : (X, \tau_1) \to (Y, \tau_2)$ is *fuzzy continuous* if for all $v\in I^Y$, $\tau_1(f^{-1}(v)) \geq \tau_2(v)$.

Theorem 2.9 Let (X, τ_1) and (Y, τ_2) be smooth fuzzy topological spaces. For every fuzzy net S, $x_t\in Pt(X)$, $r\in I_0$ and $\lambda\in I^X$, the following statements are equivalent.

(1) $f : (X, \tau_1) \to (Y, \tau_2)$ is fuzzy continuous.
(2) If $S \overset{r}{\infty} x_t$, then $f(S) \overset{r}{\infty} f(x)_t$.
(3) If $S \xrightarrow{r} x_t$, then $f(S) \xrightarrow{r} f(x)_t$.
(4) $f(C_{\tau_1}(\lambda, r)) \leq C_{\tau_2}(f(\lambda), r)$.

Proof. (1)⇒(2) Let $\mu\in \mathcal{M}(f(x)_t, r)$. Since f is fuzzy continuous, then $\tau_1(f^{-1}(\mu)) \geq \tau_2(\mu) \geq r$ and $f(x)_t\ q\ \mu$ implies $x_t\ q\ f^{-1}(\mu)$ from Lemma 1.1(4). Hence $f^{-1}(\mu)\in \mathcal{M}(x_t, r)$. Since $S \overset{r}{\infty} x_t$, for $f^{-1}(\mu)\in\mathcal{M}(x_t, r)$ and for each $n \in D$, there exists $n_0\in D$ such that $n_0 \geq n$ and $S(n_0)\ q\ f^{-1}(\mu)$. By Lemma 1.1(4), it implies $f(S(n_0))\ q\ \mu$. Hence $f(S) \overset{r}{\infty} f(x)_t$.

(2)⇒(3) Let $S \xrightarrow{r} x_t$. Every subnet $U : E \to Pt(Y)$ of $f(S)$, there exists a cofinal selection $N : E \to D$ such that $U = f(S) \circ N = f \circ (S \circ N)$. Put $T = S \circ N$. Then T is a subnet of S. We can prove it from the followings:

$S \xrightarrow{r} x_t \Rightarrow T \xrightarrow{r} x_t$ (by Theorem 1.10(7))
$\Rightarrow T \overset{r}{\infty} x_t$ (by Theorem 1.10(1))
$\Rightarrow f(T) = U \overset{r}{\infty} f(x)_t$ (by (2))
$\Rightarrow f(S) \overset{r}{\infty} f(x)_t$. (by Theorem 1.11)

(3)⇒(4) Suppose there exist λ and $r\in I_0$ such that

$f(C_{\tau_1}(\lambda, r)) \nleq C_{\tau_2}(f(\lambda), r)$.

Then there exists $y\in Y$ such that

$f(C_{\tau_1}(\lambda, r))(y) > C_{\tau_2}(f(\lambda), r)(y)$. (II)

So, there exists $x\in f^{-1}(y)$ such that

$f(C_{\tau_1}(\lambda, r))(y) \geq C_{\tau_1}(\lambda, r)(x) > C_{\tau_2}(f(\lambda), r)(y)$.

From Theorem 1.6, there exist a fuzzy r-adherent point x_t of λ on (X, τ_1) such that

$C_{\tau_1}(\lambda, r)(x) \geq t > C_{\tau_2}(f(\lambda), r)(f(x))$.

Since $x_t\in C_{\tau_1}(\lambda, r)$, by Theorem 1.13, there exists a fuzzy net $S\in\lambda$ such that $S \xrightarrow{r} x_t$. By (3), $f(S) \xrightarrow{r} f(x)_t$ with $f(S)$ in $f(\lambda)$. From Theorem 1.13, we have $f(x)_t = y_t \in C_{\tau_2}(f(\lambda), r)$. It is a contradiction for (II). Hence, for all $\lambda \in I^X$ and $r\in I_0$, we have

$f(C_{\tau_1}(\lambda, r)) \leq C_{\tau_2}(f(\lambda), r)$.

(4)⇒(1) It is similar to Theorem 2.12 of [10]. □

From Theorem 2.9, we can easily obtain the following corollary.

Corollary 2.10 Let (X, τ_1) and (Y, τ_2) be smooth fuzzy topological spaces. For each fuzzy net S, $\lambda \in I^X$ and $r \in I_0$, the following statements are equivalent.

(1) $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is fuzzy continuous.

(2) $f(clu_{\tau_1}(S, r)) \leq clu_{\tau_2}(f(S), r)$.

(3) $f(lim_{\tau_1}(S, r)) \leq lim_{\tau_2}(f(S), r)$.

(4) $f(C_{\tau_1}(\lambda, r)) \leq C_{\tau_2}(f(\lambda), r)$.

3. Fuzzy r-convergent nets

Definition 3.1 Let (X, τ) be a smooth fuzzy topological space, $\mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. A fuzzy net S is said to be *fuzzy r-convergent* to μ, denoted by $con_\tau(S, r) = \mu$, if $clu_\tau(S, r) = lim_\tau(S, r) = \mu$.

Theorem 3.2 Let (X, τ) be a smooth fuzzy topological space and S, $U : D \rightarrow Pt(X)$ fuzzy r-convergent nets such that $S(n) \vee U(n) \in Pt(X)$ for each $n \in D$. Then for each $r \in I_0$,

$$con_\tau(S \vee U, r) = con_\tau(S, r) \vee con_\tau(U, r).$$

Proof. From Theorem 2.2, $S \vee U$ is a fuzzy net. We easily proved it from the followings:

$clu_\tau(S \vee U, r) = clu_\tau(S, r) \vee clu_\tau(U, r)$ (by Theorem 2.2(2))

(since S and U are fuzzy r-convergent nets,)

$= lim_\tau(S, r) \vee lim_\tau(U, r)$

$\leq lim_\tau(S \vee U, r)$ (by Theorem 2.2(4))

$\leq clu_\tau(S \vee U, r)$. (by Theorem 1.10(2)) □

Theorem 3.3 Let (X, τ) be a smooth fuzzy topological space. Let S be a fuzzy net and $\mathcal{H} = \{T \mid T$ is a subnet of $S\}$. Then the following statements hold:

(1) $lim_\tau(S, r) = \wedge_{T \in \mathcal{H}} clu_\tau(T, r)$.

(2) $clu_\tau(S, r) = \vee_{T \in \mathcal{H}} lim_\tau(T, r)$.

(3) If $con_\tau(S, r) = \mu$, then $con_\tau(T, r) = \mu$ for each $T \in \mathcal{H}$.

Proof. (1) For each $T \in \mathcal{H}$, by Theorem 1.10 (2,8,10), we have

$$lim_\tau(S, r) \leq lim_\tau(T, r) \leq clu_\tau(T, r) \leq clu_\tau(S, r). \quad \text{(III)}$$

Hence

$$lim_\tau(S, r) \leq \wedge_{T \in \mathcal{H}} clu_\tau(T, r).$$

Suppose

$$lim_\tau(S, r) \not\geq \wedge_{T \in \mathcal{H}} clu_\tau(T, r).$$

Then there exist $x \in X$ and $t \in I_0$ such that

$$lim_\tau(S, r)(x) < t < \wedge_{T \in \mathcal{H}} clu_\tau(T, r)(x). \quad \text{(IV)}$$

Since $lim_\tau(S, r)(x) < t$, by Theorem 1.10(6), x_t is not a fuzzy r-limit point of S, that is, there exists $\mu \in \mathcal{N}(x_t, r)$ such that for each $n \in D$ there exists $N(n) \in D$ with for $N(n) \geq n$ and $S(N(n)) \bar{q} \mu$. Hence there exists a cofinal selection $N : E \rightarrow D$ such that $T = S \circ N$. Thus T is a subnet of S. Moreover, x_t is not a fuzzy r-cluster point of T. By Theorem 1.10(5), $clu_\tau(T, r)(x) < t$. It is a contradiction for (IV). Hence

$$lim_\tau(S, r) \geq \wedge_{T \in \mathcal{H}} clu_\tau(T, r).$$

(2) From (III) of (1), we have

$$\vee_{T \in \mathcal{H}} lim_\tau(T, r) \leq clu_\tau(S, r).$$

Suppose

$$\vee_{T \in \mathcal{H}} lim_\tau(T, r) \not\geq clu_\tau(S, r).$$

Then there exist $x \in X$ and $t \in I_0$ such that

$$\vee_{T \in \mathcal{H}} lim_\tau(T, r)(x) < t < clu_\tau(S, r)(x). \quad \text{(V)}$$

Since $x_t \in clu_\tau(S, r)$, by Theorem 1.10(5), we have $S \overset{r}{\infty} x_t$. By Theorem 1.12, there existsa subnet T of S such that $T \overset{r}{\rightarrow} x_t$. Thus

$$x_t \in lim_\tau(T, r) \leq \vee_{T \in \mathcal{H}} lim_\tau(T, r).$$

It is a contradiction for (V). Hence

$$\vee_{T \in \mathcal{H}} lim_\tau(T, r) \geq clu_\tau(S, r).$$

(3) From (III) of (1), we easily prove it. □

Theorem 3.4 Let (X, τ) be a smooth fuzzy topological space. Let S be a fuzzy net. If every subnet of S has a subnet which is r-convergent to μ, then $con_\tau(S, r) = \mu$.

Proof. Let $\mathcal{H} = \{T \mid T$ is a subnet of $S\}$. For each $T \in \mathcal{H}$, since T has a subnet K with $con_\tau(K, r) = \mu$, by Theorem 1.10(8), we have

$$lim_\tau(T, r) \leq lim_\tau(K, r) = clu_\tau(K, r) = \mu.$$

Hence, by Theorem 3.3(2),

$$clu_\tau(S, r) = \vee_{T \in \mathcal{H}} lim_\tau(T, r) \leq \mu. \quad \text{(VI)}$$

Conversely, by Theorem 1.10(10),

$$\mu = lim_\tau(K, r) = clu_\tau(K, r) \leq clu_\tau(T, r).$$

Hence, by Theorem 3.3(1),

$$\mu \leq \wedge_{T \in \mathcal{H}} clu_\tau(T, r) = lim_\tau(S, r). \quad \text{(VII)}$$

By (VI) and (VII), $clu_\tau(S, r) \leq lim_\tau(S, r)$. Since $lim_\tau(S, r) \leq clu_\tau(S, r)$ from Theorem 1.10(2), $clu_\tau(S, r) = lim_\tau(S, r)$, that is, $con_\tau(S, r) = \mu$. □

Example 3.5 We define a smooth fuzzy topology τ

as Example 2.6. Let N be a natural number set. Define a fuzzy net $S : N \to Pt(X)$ by

$$S(n) = x_{a_n},\ a_n = 0.6 + (-1)^n 0.2.$$

We can show $clu_\tau(S, 1/2) = \widetilde{1}$ from (1) to (2)

(1) x_t for $t \le 0.7$ or y_s for $s \le 0.6$ is a fuzzy 1/2-cluster point of S because, for $\widetilde{1} \in \mathcal{M}(p, 1/2)$ with $p = x_t$ or y_s and for all $n \in N$, we have $S(n)\ q\ \widetilde{1}$.

(2) x_t for $t > 0.7$ or y_s for $s > 0.6$ is a fuzzy 1/2-cluster point of S because, for $\widetilde{1}, \mu \in \mathcal{M}(p, 1/2)$ with $p = x_t$ or y_s and for all $n \in N$, there exists $2n \in N$ such that $2n \ge n$, $S(2n) = x_{0.8}\ q\ \mu$.

We can show $lim_\tau(S, 1/2) = \widetilde{1} - \mu$ from (3) to (4).

(3) x_t for $t \le 0.7$ or y_s for $s \le 0.6$ is a fuzzy 1/2-limit point of S because, for $\widetilde{1} \in \mathcal{M}(p, 1/2)$ with $p = x_t$ or y_s and for all $n \in N$, we have $S(n)\ q\ \widetilde{1}$.

(4) x_t for $t > 0.7$ or y_s for $s > 0.6$ is not a fuzzy 1/2-limit point of S because, for $\mu \in \mathcal{M}(p, 1/2)$ such that for all $n \in N$, there exists $2n+1 \in N$ such that $2n+1 \ge n$ and $S(2n+1) = x_{0.4}\ \bar{q}\ \mu$.

Since $clu_\tau(S, 1/2) \ne lim_\tau(S, 1/2)$, S is not fuzzy 1/2-convergent.

In a similar method, we show for $0 < r \le 1/2$,

$$\widetilde{1} = clu_\tau(S, r) \ne lim_\tau(S, r) = \widetilde{1} - \mu$$

and for $r > 1/2$,

$$\widetilde{1} = clu_\tau(S, r) = lim_\tau(S, r).$$ □

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