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A Study on the Generation of Capillary Waves on Steep Gravity Waves

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Abstract

A formal solution method using the complex analysis is given for the problems derived by Longuet-Higgins(1963). The same method is applied to a new perturbation problem of higher approximation. Interpretation of its solution made it possible to confirm that the rough agreement of Longuet-Higgins' prediction with the experimental data of Cox(1958) was mainly due to the fact that the gravity effect in the perturbation problem was neglected for the case when the basic gravity wave was not sufficiently steep.

Keywords: steep gravity waves, capillary waves, zero-order solution, perturbation solution

1 Introduction

Cox(1958) presented an experimental report, in which he described capillary waves generated on the forward face of free steep gravity waves of wavelengths 5~50 cm. Following this, Longuet-Higgins(1963)(hereafter will be denoted as LH)gave a theory for the phenomenon, and obtained an approximate expression for the ripple steepness. The purpose of the present paper is to give for the problems derived by LH a formal solution procedure, which can also provide more information on characteristics of the solution, and to consider other possible solutions of the problems of higher approximation. Developing his theory, LH made use of the wave theory proposed first by Levi-Civita(1925) and further advanced by Davies(1951). The former devised a way of approximation suitable for waves of small steepness and the latter for those of finite steepness. Later, Tulin(1982) also used the similar theory for describing the generation of waves by moving bodies. Following LH, we consider two-dimensional, irrotational waves in a perfect fluid moving horizontally with velocity -c, and the uniform velocity c is superposed to make the whole flow steady. The x-axis is taken horizontally, and y-axis vertically upward and we write $z = x + iy$. Let's denote the velocity potential ϕ, the stream function ψ, the complex potential χ, and the complex velocity w, then the following relations hold.

$$w = \frac{d\chi}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv \tag{1.1}$$

where u and v are the components of velocity. Let q and θ be the magnitude and direction of the velocity, and we define τ by $q = ce^{\tau}$, then we have

$$w = u - iv = qe^{-i\theta} = ce^{\tau - i\theta} = ce^{\zeta} \tag{1.2}$$

where we set

$$\zeta = \tau - i\theta \tag{1.3}$$

We may then formulate a problem for which ζ and z are described in terms of χ.

From Bernoulli's equation on the free surface we have

$$\frac{1}{2}q^2 + gy + \frac{p}{\rho} = const., \quad \text{on} \quad \psi = 0 \tag{1.4}$$

where g is the gravitational acceleration, p the pressure, and ρ the density of the fluid, respectively. Here, we take $\phi = 0$ at the crest of the wave. The water depth is assumed as infinite, and we have $\psi \longrightarrow -\infty$, and $\zeta \longrightarrow 0$ as $y \longrightarrow -\infty$. On the free surface the pressure is given by $p = const - T\kappa$, where T is the surface tension and κ the curvature. Since on the free surface

$$\kappa = \frac{\partial \theta}{\partial s} = q\frac{\partial \theta}{\partial \phi} \tag{1.5}$$

where s is the arc length, and from the Cauchy-Riemann condition for $\zeta(\chi)$ to be analytic, we have

$$\kappa = q\frac{\partial \tau}{\partial \psi} = \frac{\partial q}{\partial \psi} \tag{1.6}$$

Then (1.4) can be rewritten as

$$\frac{1}{2}q^2 + gy - T'\frac{\partial q}{\partial \psi} = const., \quad \text{on} \quad \psi = 0 \tag{1.7}$$

where we denote

$$T' = \frac{T}{\rho} \tag{1.8}$$

Differentiating (1.7) with respect to ϕ , we get

$$q\frac{\partial q}{\partial \phi} + g\frac{\partial y}{\partial \phi} - T'\frac{\partial^2 q}{\partial \phi \partial \psi} = 0, \quad \text{on} \quad \psi = 0 \tag{1.9}$$

On the free surface

$$\sin\theta = \frac{\partial y}{\partial s} = q\frac{\partial y}{\partial \phi} \tag{1.10}$$

and substitution of this in (1.9) gives

$$q\frac{\partial q}{\partial \phi} + \frac{g}{q}\sin\theta - T'\frac{\partial^2 q}{\partial \phi \partial \psi} = 0, \quad \text{on} \quad \psi = 0 \tag{1.11}$$

Neglecting the surface tension from (1.11) entirely, and using the subscript 0 for representing the basic flow, we obtain the zero-order problem of LH as follows,

$$q_0\frac{\partial q_0}{\partial \phi} + \frac{g}{q_0}\sin\theta_0 = 0, \quad \text{on} \quad \psi = 0 \tag{1.12}$$

In deriving the perturbation problem, LH first let $\zeta = \zeta_0 + \zeta_1$, $q = q_0 + q_1$, etc., where the subscript 1 is used for perturbation terms, then from (1.7) by neglecting the second order term he got

$$(\frac{1}{2}q_0^2 + q_0 q_1) + g(y_0 + y_1) - T'(\frac{\partial q_0}{\partial\psi} + \frac{\partial q_1}{\partial\psi}) = const., \quad \text{on} \quad \psi = 0 \tag{1.13}$$

If we subtract from this the boundary condition for the basic flow, we obtain the perturbation problem of LH as follows,

$$q_0 q_1 + g y_1 - T'\frac{\partial q_1}{\partial\psi} = T'\frac{\partial q_0}{\partial\psi} + const., \quad \text{on} \quad \psi = 0 \tag{1.14}$$

In the sequel, we first solve the zero-order problem (1.12), then the perturbation problem (1.14), and give a discussion in the last section.

2 Zero-order problem

If we momentarily delete the subscript 0 for the basic flow in this section, (1.12) is rewritten as

$$q\frac{\partial q}{\partial\phi} + \frac{g}{q}\sin\theta = 0, \quad \text{on} \quad \psi = 0 \tag{2.1}$$

In approximating the term $\sin\theta$ in the equation above, Levi-Civita(1925) used $\sin\theta \cong \theta$ for waves of small steepness, and Davies(1951) $\sin\theta \cong \frac{1}{3}\sin 3\theta$ for those of finite steepness. Furthermore, LH used

$$\sin\theta \cong \frac{1}{2}\sin 3\theta \tag{2.2}$$

and justified its use by showing that this approximation in a way satisfies the exact boundary condition on the free surface for the Stokes' limiting wave of 120° angle. Substituting (2.2) in (2.1) we have

$$q\frac{\partial q}{\partial\phi} + \frac{g}{2q}\sin 3\theta = 0, \quad \text{on} \quad \psi = 0 \tag{2.3}$$

Solving this, we follow the fashion of Tulin(1982). If we define $G = w^3$, and multiply (2.3) by $3/q^2$, we get

$$Re\{\frac{1}{G}(\frac{dG}{d\chi} - \frac{3ig}{2})\} = 0, \quad \text{on} \quad \psi = 0 \tag{2.4}$$

where Re stands for 'the real part of.' Now, in order to extend this into the region $\psi < 0$, we make use of the facts that

$$\frac{dG}{d\chi} \longrightarrow 0, \quad G \longrightarrow c^2, \quad \text{as} \quad \psi \longrightarrow -\infty \tag{2.5}$$

Then we obtain a first order differential equation for G as follows,

$$\frac{dG}{d\chi} + ikG = \frac{3ig}{2}, \quad \text{for} \quad \psi \leq 0 \tag{2.6}$$

where we set

$$k = \frac{3g}{2c^3} \tag{2.7}$$

If we introduce a new dependent variable defined by $F = G - c^3$, in terms of which we may rewrite (2.6) as

$$\frac{dF}{d\chi} + ikF = 0, \quad \text{for} \quad \psi \le 0 \tag{2.8}$$

Its solution is given by

$$F = Ce^{-ik\chi} \tag{2.9}$$

where C is a complex constant. Considering that $\theta = 0$ at the crest where $\chi = 0$ for symmetric waves, we may set $C = -c^3 A$, where A is a real positive constant. Substitution of this in (2.9), and the definitions of F and G render

$$e^{3\zeta} = 1 - Ae^{-ik\chi} \tag{2.10}$$

From the fact that $e^{3\tau}$, which is the left-hand side of (2.10) at the crest, must be positive, $A \in (0, 1)$ can be seen. Certainly, (2.10) is the same as the solution that was obtained by LH. If the subscript 0 is revived for denoting the basic flow, (2.10) can be rewritten as follows,

$$\begin{aligned} q_0 &= c(1 + A^2 e^{2k\psi} - 2Ae^{k\psi} \cos k\phi)^{1/6} \\ \theta_0 &= -\frac{1}{3} \tan^{-1} \frac{Ae^{k\psi} \sin k\phi}{1 - Ae^{k\psi} \cos k\phi} \end{aligned} \tag{2.11}$$

For almost-highest waves, $A \approx 1$, and we may let $A = 1 - \delta$, where $\delta$is a small positive quantity. If we set

$$(\xi, \eta) = \frac{k}{\delta}(\phi, \psi) \tag{2.12}$$

approximations of (2.10) near the crest, where the relation $e^{-ik\chi} \cong 1 - ik\chi$ holds, can be given as

$$q_0 \cong c\delta^{1/3}\{(1-\eta)^2 + \xi^2\}^{1/6}, \quad \theta \cong -\frac{1}{3} \tan^{-1}(\frac{\xi}{1-\eta}) \tag{2.13}$$

The curvature of the free surface is given by

$$\kappa_0 = \frac{\partial q_0}{\partial \psi} = -\frac{ckA}{3} \frac{\cos k\phi - A}{(1 + A^2 - 2A\cos k\phi)^{5/6}} \tag{2.14}$$

and thus near the crest we have approximately

$$\kappa_0 \cong -\frac{K}{(1+\xi^2)^{5/6}} \tag{2.15}$$

where we let the magnitude of the curvature at the crest K as

$$K = \frac{ck}{3\delta^{2/3}} = \frac{g}{2c^2\delta^{2/3}} \tag{2.16}$$

3 Perturbation problem

In order to get a solution for the perturbation problem (1.14), LH neglected the term representing the effect of gravity, and solved the following equation.

$$q_0 q_1 - T' \frac{\partial q_1}{\partial \psi} = T' \frac{\partial q_0}{\partial \psi}, \quad \text{on} \quad \psi = 0 \tag{3.1}$$

where the constant term is also neglected without losing generality. Using the definitions of perturbed terms given below (1.12), we can easily see $q_1 \approx q_0 \tau_1$, and substitution of this in (3.1) gives

$$q_0^2 \tau_1 - T' \frac{\partial (q_0 \tau_1)}{\partial \psi} = T' \frac{\partial q_0}{\partial \psi}, \quad \text{on} \quad \psi = 0 \tag{3.2}$$

If we divide this by $T' q_0$ and rearrange the resulting equation, we obtain

$$\frac{\partial \tau_1}{\partial \psi} - P(\phi) \tau_1 = -Q(\phi), \quad \text{on} \quad \psi = 0 \tag{3.3}$$

where we set

$$P(\phi) = \frac{q_0}{T'} - \frac{\partial \tau_0}{\partial \psi}, \quad Q(\phi) = \frac{\partial \tau_0}{\partial \psi} \tag{3.4}$$

Introducing a new complex variable defined by

$$\sigma = \alpha + i\beta = \int P(\chi) d\chi \tag{3.5}$$

we can see that

$$\frac{\partial \alpha}{\partial \phi} = P(\phi), \quad \text{on} \quad \psi = 0 \tag{3.6}$$

and also that $\beta = 0$, $\frac{\partial \beta}{\partial \phi} = 0$, on $\psi = 0$. Using the Cauchy-Riemann condition for $\sigma(\chi)$ to be analytic, we can also show that on the free surface

$$\frac{\partial \tau_1}{\partial \psi} = \frac{\partial \tau_1}{\partial \beta} \frac{\partial \beta}{\partial \psi} = \frac{\partial \tau_1}{\partial \beta} \frac{\partial \alpha}{\partial \phi} = P(\phi) \frac{\partial \tau_1}{\partial \beta} \tag{3.7}$$

If we substitute this in (3.3), divide by $P(\phi)$, and set

$$R(\phi) = \frac{Q(\phi)}{P(\phi)} \tag{3.8}$$

we have

$$\frac{\partial \tau_1}{\partial \beta} - \tau_1 = -R(\alpha), \quad \text{on} \quad \beta = 0 \tag{3.9}$$

Since $\zeta_1 = \tau_1 - i\theta_1$, (3.9) can be rewritten as

$$Re\{i \frac{d\zeta_1}{d\sigma} - \zeta_1\} = -Re\{R(\sigma)\}, \quad \text{on} \quad \beta = 0 \tag{3.10}$$

Now, in order to extend this into the region $\beta < 0$, we make use of the facts that

$$\frac{d\zeta_1}{d\sigma} \longrightarrow 0, \quad \zeta_1 \longrightarrow 0, \quad R(\sigma) \longrightarrow 0, \quad \text{as} \quad \psi \longrightarrow -\infty, \quad \text{i.e.} \quad \text{as} \quad \beta \longrightarrow -\infty \tag{3.11}$$

Then again we obtain a first order differential equation for ζ_1 as follows

$$\frac{d\zeta_1}{d\sigma} + i\zeta_1 = iR(\sigma), \quad \text{for} \quad \beta \leq 0 \tag{3.12}$$

for which the solution is given by

$$\zeta_1(\sigma) = ie^{-i\sigma} \int_{-i\infty}^{\sigma} R(a)e^{ia}da \tag{3.13}$$

where a is a complex variable. If we let $r(t)$ be the Fourier transform of $R(\alpha)$, which is a real even function of ϕ, and so of α, we have

$$r(t) = 2\int_0^\infty R(\alpha)\cos\alpha t d\alpha \tag{3.14}$$

Then from the Fourier theorem, we also have

$$R(\alpha) = \frac{1}{\pi}\int_0^\infty r(t)e^{-i\alpha t}dt \tag{3.15}$$

Using the analytic continuation of (3.15) in (3.13), we obtain

$$\zeta_1(\sigma) = ie^{-i\sigma} \int_{-i\infty}^{\sigma} \frac{1}{\pi}\int_0^\infty r(t)e^{-iat}dte^{ia}da \tag{3.16}$$

After we change the order of integration, we may carry out the integration with respect to a to get

$$\zeta_1(\sigma) = -\frac{1}{\pi}\int_0^\infty \frac{r(t)e^{-iat}}{t-1}dt \tag{3.17}$$

where the path of integration in the complex t-plane should pass below the pole at $t = 1$. Since, for large values of $|\alpha|$, the major contribution comes from the residue at $t = 1$, ζ_1 is approximately given by

$$\zeta_1 \approx \begin{cases} -2ir(1)e^{-i\sigma}, & \alpha < 0 \\ 0, & \alpha > 0 \end{cases} \tag{3.18}$$

On the forward(backward) face, $\alpha < 0 (> 0)$, thus the perturbed effects will be shown up mainly on the forward face of the gravity wave. Again, (3.18) is the same as the solution given by LH except the negative sign. We note that the perturbation solution (3.18) is in fact the homogeneous solution of (3.12) and its amplitude is determined by the Fourier Transform of the forcing term. Substituting (3.8) in (3.14) and using (3.4) and (3.6), we obtain

$$r(1) = 2\int_0^\infty \frac{Q}{P}\cos\alpha\frac{\partial\alpha}{\partial\phi}d\phi = 2\int_0^\infty \left.\frac{\partial\tau_0}{\partial\psi}\right|_{\psi=0}\cos\alpha d\phi \tag{3.19}$$

Near the crest where ξ is small, α can be approximated by $\lambda\xi$ as will be shown in (4.16), and by using (2.12), (2.13), (2.15) and (2.16), we can obtain the following approximation of $r(1)$, as shown by LH,

$$r(1) \approx -\frac{2}{3}\int_0^\infty \frac{\cos\lambda\xi}{1+\xi^2}d\xi = -\frac{\pi}{3}e^{-\lambda} \tag{3.20}$$

Here, λ is defined and can be expressed as follows,

$$\lambda \equiv \frac{c\delta^{4/3}}{kT'} = \frac{2}{3}\frac{c^4\delta^{4/3}}{gT'} = \frac{g}{6K^2T'} \tag{3.21}$$

Substituting (3.20) in (3.18), we get on the forward face of the basic gravity wave

$$\tau_1 - i\theta_1 \approx b(\sin\alpha - i\cos\alpha) \tag{3.22}$$

where b is given by

$$b = \frac{2\pi}{3}e^{-\lambda} \tag{3.23}$$

We note that the solution (3.18) was obtained for large $|\alpha|$, but the solution (3.22) for small ξ . We will get back to this point in the next section.

4 Discussion

4.1 Zero-order solution

Taking $\psi = 0$ in (2.10), we have on the free surface

$$q_0 = c(1 + A^2 - 2A\cos k\phi)^{1/6} \tag{4.1}$$

from which we see that the wave is indeed periodic with the period given by $k\phi = 2\pi$. q_0 takes its maximum $c(2-\delta)^{1/3}$ at the trough where $k\phi \pm \pi$, thus the magnitude of the fluid velocity cannot be larger than $2^{1/3}c = 1.260c$, which corresponds to the limiting wave($A = 1$). At the wave crest where $\xi = \eta = 0$, from (2.13) we see that $q_0 = c\delta^{1/3}$, thus the fluid velocity at the wave crest is larger than zero in general and equal to zero only for the limiting wave.

The curvature of the free surface is negative only when $|k\phi| < \cos^{-1} A$, as can be seen from (2.14). Furthermore, when $\delta \ll 1$, approximately speaking, we have the point of zero curvature at $|k\phi| = \sqrt{2\delta}$. Therefore, the limiting wave has no region of negative curvature. The steeper the basic gravity wave gets, the narrower the region of negative curvature becomes. We also note that the approximation of the curvature (2.15) holds only where it is negative.

As mentioned below (2.2), LH gave a justification for using the approximation (2.2). However, as Tulin(1982) pointed out, the solution (2.10) satisfies the Stokes' limiting wave regardless of the value of k. Since the use of the approximation (2.2) changes the value of k in (2.8) only, we see that it is not a pre-condition for obtaining the limiting solution. At the moment, it is acceptable simply because (2.2) is a good approximation near the crest of highly steep waves where $\theta \approx \pm\pi/6$.

4.2 Perturbation solution

An assumption for the whole theory of LH is that the capillary effect is smaller than the other terms in (1.7), and in particular the following relation should hold,

$$T'\left|\frac{\partial q_0}{\partial \psi}\right| \ll \frac{1}{2}q_0^2 \tag{4.2}$$

This inequality gives two important relations. First, since $\frac{\partial q_0}{\partial \psi} = q_0 \frac{\partial \tau_0}{\partial \psi}$, substituting this in (4.2) we have $\left|\frac{\partial \tau_0}{\partial \psi}\right| \ll \frac{1}{2}\frac{q_0}{T'}$. Thus, for the function $P(\phi)$ defined in (3.4), we get the following approximation,

$$P(\phi) \approx \frac{q_0}{T'} \tag{4.3}$$

Second, since at the wave crest where $\xi = \eta = 0$ we have from (2.14) and (2.16) $\left|\frac{\partial q_0}{\partial \psi}\right| = K = \frac{g}{2c^2\delta^{2/3}}$, and from (2.13) $q_0 = c\delta^{1/3}$, substitution of these in (4.2) gives $gT' \ll c^4\delta^{4/3}$, for which we use the second equality of (3.21) to obtain

$$\lambda \gg \frac{2}{3} \tag{4.4}$$

LH obtained $\lambda = 2.92$ by using the experimental data of Cox(1958) for K in (3.21), and argued that since (4.4) is only marginally satisfied we can expect no more than rough agreement. If we make use of the following values given by LH, $c = 0.3m/s$, $g = 9.8m/s^2$, $T' = 7.4 \times 10^{-5}m^3/s^2$, $\lambda = 2.92$, in the second equality of (3.21), we get $\delta = 0.50$, which is much greater than originally assumed. Since λ is the ratio of the inertia(or gravity) effect and the surface tension effect, the fact that λ is only marginally large means that the basic gravity wave is not sufficiently steep. Furthermore, unacceptably large δ implies the basic gravity wave with insufficient steepness. Although this may explain the rough agreement between LH's theoretical prediction and the experimental data, there may be other ways of explaining the rough agreement.

First, as mentioned at the end of the last section, the solution (3.22) was obtained under the two assumptions that $|\alpha|$ is large, and ξ is small. Using (4.3) and (3.21), for small ξ we have

$$\alpha = \int P(\phi)d\phi \approx \int \frac{q_0}{T'}d\phi \approx \frac{c\delta^{1/3}}{T'}\phi = \frac{c\delta^{4/3}}{kT'}\xi = \lambda\xi \tag{4.5}$$

Thus, if $|\alpha|$ is to be large for small ξ, λ should be really very large. In LH's case, λ was surely not large enough, and certainly this fact can be accounted as a source of the rough agreement.

LH neglected the effect of gravity in the perturbation equation, as stated at the beginning of the section 3. For justifying the neglect, he also obtained a solution of higher approximation for the homogeneous equation with the gravity term included, and showed that the steepness of the capillary wave is proportional to q_0. He then argued that q_0 varies little compared to the surface tension forcing near the crest, thus the capillary wave amplitude there also changes little. As (3.22) shows, the amplitude of capillary waves when the gravity is neglected is constant b, and thus he established the justification by saying that the neglect of gravity affects the solution little. It is likely that his higher approximation is valid only for the region where the capillary waves are fully formed. In the following let us consider the possibility of finding other higher approximation.

5 A new perturbation solution

Starting from (1.14), which we differentiate with respect to ϕ to eliminate y_1, we get

$$\frac{\partial}{\partial\phi}(q_0 q_1 - T'\frac{\partial q_1}{\partial\psi}) + \frac{g}{q_0}(\theta_1\cos\theta_0 - \tau_1\sin\theta_0) = T'\frac{\partial^2 q_0}{\partial\phi\partial\psi}, \quad \text{on} \quad \psi = 0 \tag{5.1}$$

Hereafter, since the right-hand side acts as a forcing, we may consider the homogeneous equation only, as LH did for his higher approximation. Substituting $q_1 = q_0\tau_1$ in (5.1), and using (3.7) and (3.4), we have

$$\frac{\partial}{\partial\phi}\{T' q_0 P(\phi)(\frac{\partial\tau_1}{\partial\beta} - \tau_1)\} - \frac{g}{q_0}(\theta_1\cos\theta_0 - \tau_1\sin\theta_0) = 0, \quad \text{on} \quad \psi = 0 \tag{5.2}$$

Making use of the approximation (4.3), and carrying out differentiation for the first term, we obtain

$$\frac{\partial q_0^2}{\partial\phi}(\frac{\partial\tau_1}{\partial\beta} - \tau_1) + q_0^2\frac{\partial}{\partial\phi}(\frac{\partial\tau_1}{\partial\beta} - \tau_1) - \frac{g}{q_0}(\theta_1\cos\theta_0 - \tau_1\sin\theta_0) = 0, \quad \text{on} \quad \psi = 0 \tag{5.3}$$

In the vicinity of the wave crest, we see from (2.13) that $\theta_0 \approx -\xi/3 \ll 1$, thus the second term is much smaller than the first in the third parenthesis of (5.3), and we may neglect the second term and approximate the first term as θ_1. Furthermore, if we substitute $\partial q_0^2/\partial\phi = -2g\sin\theta_0/q_0$, which can be seen easily from (1.12), in the first term of (5.3), then the first term is of the same order as the term neglected already. For his higher approximation, LH assumed a solution, which had slowly varying amplitude, and with this assumption the first term was also neglected. Therefore, for further development we may neglect the first term as a whole, and we are left with

$$q_0^2\frac{\partial}{\partial\phi}(\frac{\partial\tau_1}{\partial\beta} - \tau_1) - \frac{g}{q_0}\theta_1 = 0 \quad \text{on} \quad \psi = 0 \tag{5.4}$$

Using (3.6) and (4.3), we can show that

$$\frac{\partial}{\partial\phi} = \frac{\partial\alpha}{\partial\phi}\frac{\partial}{\partial\alpha} = P(\phi)\frac{\partial}{\partial\alpha} \approx \frac{q_0}{T'}\frac{\partial}{\partial\alpha} \tag{5.5}$$

If we substitute this in (5.4), and divide the result by q_0^3/T', and then use the approximation $q_0 \approx c\delta^{1/3}$, we have

$$\frac{\partial}{\partial\alpha}(\frac{\partial\tau_1}{\partial\beta} - \tau_1) - \varepsilon\theta_1 = 0, \quad \text{on} \quad \psi = 0 \tag{5.6}$$

Here, ε is defined and can be expressed as

$$\varepsilon \equiv \frac{gT'}{c^4\delta^{4/3}} = \frac{2}{3}\lambda \ll 1 \tag{5.7}$$

for which we have used (3.20) and (4.4). Extending (5.6) into the lower-half plane while (3.11) is again taken into account, we get a second order complex differential equation

$$\frac{d^2\zeta_1}{d\sigma^2} + i\frac{d\zeta_1}{d\sigma} - \varepsilon\zeta_1 = 0, \quad \text{for} \quad \beta \le 0 \tag{5.8}$$

Its solution is obtained as

$$\zeta_1 = C_1 e^{i\gamma_1\sigma} + C_2 e^{i\gamma_2\sigma} \tag{5.9}$$

where C_1, C_2 are complex constants, and γ_1 and γ_2 are given by

$$\gamma_{1,2} = -\frac{1}{2}(1 \pm \sqrt{1-4\varepsilon}) \tag{5.10}$$

It should be reminded that (5.10) represents free waves and that their amplitudes have to be determined in relation with the forcing term. Furthermore, during the process of determining the amplitude, it will be apparent that this solution is valid only on the forward face of the basic gravity wave. Related to (5.6) and (5.7), it is clear that the gravity effect is indeed smaller than the capillary effect as was assumed by LH. However, if we neglected the gravity term, (5.6) were a first order differential equation, which is in fact the homogeneous version of (3.9). Without the gravity term, the second term of the solution (5.9) would not have been obtained. We note that if ε is larger than $1/4$ in (5.10), the square-root term becomes imaginary and consequently, the corresponding solution will damp out(or amplify) and the steepness be halved. Thus $\varepsilon < 1/4$ can be regarded as an important criterion for the generation of capillary waves. In other words, if $\lambda > 8/3$, $\gamma_{1,2}$ are all real and (5.9) represents free waves without damping or amplification. In LH's case, λ was 2.92, and indeed the condition $\lambda > 8/3$ is only marginally satisfied and in consequence only a rough agreement may be expected. Since, as shown in (5.7), when ϵ is much smaller than one, we may use the approximation $\sqrt{1-4\varepsilon} \approx 1 - 2\varepsilon$ in (5.10) to get

$$\zeta_1 = C_1 e^{-i(1-\varepsilon)\sigma} + C_2 e^{-i\varepsilon\sigma} \tag{5.11}$$

The first term corresponds to the LH's perturbation solution, and the second is entirely due to the gravity effect. As mentioned above, if ε is sufficiently small, the second term is negligible. However, in LH's case, $\lambda = 2.92$ gives $\varepsilon = 0.228$, and with this ε, the first term is significantly changed from $e^{-i\alpha}$ due to the gravity effect, and the second certainly not negligible.

It is hoped that the argument so far offers a more reasonable explanation of the rough agreement between LH's theoretical prediction and the Cox's experimental data.

6 Conclusion

We have applied the formal solution procedure to the zero-order and the perturbation problem to obtain a complex differential equation for each case, on the other hand LH employed different method for each problem. Interpreting the solution so attained, we could explain why LH's solution could have only a rough agreement with the experimental data of Cox(1958). In conclusion, we note that LH's perturbation solution is good only when the basic gravity wave is sufficiently steep. If the steepness of the basic gravity wave is not large enough, as we have shown, the gravity effect upon the perturbation solution is not negligible.

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