볼록 다면체 단위 법선 벡터의 구면 보로노이 다이아그램을 계산하기 위한 선형시간 알고리즘

(A Linear-time Algorithm for Computing the Spherical Voronoi Diagram of Unit Normal Vectors of a Convex Polyhedron)

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요 약 보로노이 다이아그램은 계산기하학에서 다양한 형태의 근접 문제를 해결함에 있어 중요한 역할을 하고 있다. 일반적으로 평면상의 n 개의 점에 의한 평면 보로노이 다이아그램은 $O(n\log n)$ 시간에 생성할 수 있으며 이 알고리즘의 시간 복잡도가 최적임이 밝혀져 있다. 본 논문에서는 특별한 관계를 갖는 단위 구면상의 점들에 대한 구면 상에서 정의되는 보로노이 다이아그램을 $O(n)$에 생성하는 알고리즘을 제시한다. 이때 주어진 구면상의 점들은 볼록 다면체의 단위 법선 벡터들의 종점에 해당되며, 구면 보로노이 다이아그램의 선분은 구면상의 geodesic으로 이루어진다.

Abstract The Voronoi diagrams play a central role for solving a variety of proximity problems. It is well-known that the Voronoi diagram of n sites in the plane can be computed in $O(n\log n)$ time, and this bound is optimal. In this paper, we show that a special Voronoi diagram named as a spherical Voronoi diagram, of n sites on the unit sphere can be computed in $O(n)$ time, where these sites correspond to the outward unit normal vectors of the faces of a convex polyhedron.

1. Introduction

The Voronoi diagram is a natural and intuitively appealing structure to solve a variety of proximity problems appearing in several fields. Within the domain of the computer science, it is applied to solve geometric problems such as finding Euclidean minimum spanning trees and closest points. The planar Voronoi diagram of a set of points - called *sites* - is a partition of the plane that assigns a surrounding convex polygon of nearby points to each of the sites. The straight-line dual of the Voronoi diagram is called the *Delaunay triangulation* of *S*. In favor of the extensive applications of the Voronoi diagrams and Delaunay triangulations, the problem of computing these structures has received considerable attention in recent decades. Shamos [5] showed that the Voronoi diagram of n sites in the plane can be computed in $O(n\log n)$ time, and this bound is optimal. As the specialization of a general problem to a restricted class of data, Aggarwal et al. [1] solved one of the outstanding open problems: construct the Voronoi diagram of S in $O(n)$ time, where S is the set of vertices of a convex polygon. They also showed that the technique can be used to reduce the time complexity of several other problems [2].

The surface of a sphere has a different topology from the plane, so it is hard to directly apply Shamos's algorithm to the surface. Brown presented an $O(n\log n)$ algorithm for computing the spherical Voronoi diagram of n points on the surface of a sphere by using a $3D$ convex hull algorithm [3].

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The $\Omega(n\log n)$ lower bound in ref. [2, 3] does not apply when the input sites are spherical points which correspond to the outward unit normal vectors of the faces of a convex polyhedron. The set of sites may be regarded as a dimensional extension of the set of vertices of a convex polygon in ref. [1]. In this paper, we present an $O(n)$ algorithm for constructing a special Voronoi diagram of the n spherical points, named as a spherical Voronoi diagram. The spherical Voronoi diagram of the set can be used to find the face of a convex polyhedron which has the best reflexibility for a given light source.

The rest of this paper is organized in the following manner : In Section 2, we first review the point-plane duality which is the main idea of this paper and then show that the spherical Voronoi diagram of S can be computed in $O(n)$ time, where S is the set of n outward unit normal vectors of faces in a convex polyhedron. Finally, we conclude this paper with some remarks in Section 3.

2. Algorithm and Analysis

We start by reviewing some standard geometric constructions. Suppose that we are given a set S of n sites on the surface of the unit sphere S^2. For two distinct sites $p, q \in S$, the *spherical dominance* of p to q, denoted by $sdom(p,q)$, is defined as the subset of points on the surface of the sphere being at least as close to p as to q. Formally, $sdom(p,q)=\{x \in S^2 \mid \delta(x,p) \le \delta(x,q)\}$, where $\delta(x,y)$ is the geodesic between x and y. Clearly, $sdom(p,q)$ is a closed half sphere divided by the perpendicular bisector of p and q. This bisector separates all points on the surface of the sphere closer to p from those closer to q and will be termed the *separator* of p and q. The *spherical region*, $sreg(p)=\bigcap_{q \in S-\{p\}} sdom(p,q)$ of a site $p \in S$, is the portion of the surface of the sphere lying in all of the spherical dominances of p over the remaining sites in S. The n spherical regions form a partition of the surface of the unit sphere. This partition is called the *Spherical Voronoi diagram*, $SV(S)$, of the finite point set S.

Brown presented an $O(n\log n)$ algorithm for computing the spherical Voronoi diagram of n points on the unit sphere [3]. His algorithm consists of three major steps: The first step is to construct the convex hull of n points in $O(n\log n)$ time. The next step is to compute the spherical Voronoi vertices. Let u_i be the point on the sphere that is equi-distant from the vertices of F_i, where F_i is a face of the convex hull. Then, u_i is a spherical Voronoi vertex. It takes $O(n)$ time to compute all of the Voronoi vertices u_i's. The final step is to connect the Voronoi vertices in $O(n)$ time : Two Voronoi vertices, u_i and u_j are connected by an arc on a great circle (geodesic arc) if and only if F_i an F_j share an edge. Clearly, the time complexity of the first step dominates those of the others. Hence, if the first step can be done in $O(n)$ time, then we can construct the spherical Voronoi diagram in $O(n)$ time. We exploit the point-plane duality to achieve this time bound when S is the set of points on the Gauss sphere which correspond to the outward unit normal vectors, denoted by nF_i's, of faces F_i's of a convex polyhedron.

Consider a transformation that maps a point $p=(p_1,p_2,p_3)$ to a plane $\langle p,x\rangle = p_1x_1+p_2x_2+, p_3x_3=1$ and vice versa [4]. This transformation gives a dual D of a polyhedron P : Every vertex v_h of P corresponds to a face Dv_h of D, and every face F_i of P does to a vertex DF_i of D. Without loss of generality, we may assume that P contains the origin in its interior. Otherwise, we can always translate P to satisfy this assumption, since the topology of P is translation-invariant. It is well-known that D is also a convex polyhedron containing the origin in its interior. Moreover, Dv_h is a convex polygon for all $v_h \in P$.

For convenience, we relabel the faces containing $v_h=(x_h, y_h, z_h)$ in P so that they form a cycle $(F_{h,0}, F_{h,1}, \cdots, F_{h,k-1})$, where each $F_{h,j}$ is adjacent to $F_{h,j+1}$ and the second subscripts are taken modulo k. v_h is transformed to a plane

$$\langle v_h, x\rangle = x_h x_1 + y_h x_2 + z_h x_3 = 1.$$

Since $nF_{h,j}$ is the unit normal vector of $F_{h,j}$ of P, the plane containing $F_{h,j}$ is given by

$$\langle nF_{h,j},\ x\rangle = d_{h,j}, \text{ or } \langle \frac{nF_{h,j}}{d_{h,j}},\ x\rangle = 1.$$

Therefore, $DF_{h,j} = nF_{h,j}/d_{h,j}$ is a vertex of Dv_h corresponding to $F_{h,j}$ of P. Furthermore,

$$\langle v_h, x\rangle = \|v_h\| \langle \frac{v_h}{\|v_h\|}, x\rangle = 1, \text{ or } \langle \frac{v_h}{\|v_h\|}, x\rangle = \frac{1}{\|v_h\|}.$$

That is, the convex polygon Dv_h corresponding to a vertex v_h of P is contained in the plane $\langle v_h, x\rangle = 1$ whose distance from the origin is $1/\|v_h\|$.

Due to the point-plane duality, $DF_{h,j}$'s are the vertices of the convex polygon Dv_h, and the line segment joining two vertices, $DF_{h,j}$ and $DF_{h,j+1}$, is an edge of Dv_h. As shown in Fig. 1, the ray from the origin to $DF_{h,j}$ intersects the Gauss sphere at $nF_{h,j}$. Therefore, Dv_h is projected onto the sphere as a spherical region whose boundary can be represented by a sequence of points $(nF_{h,0}, nF_{h,1}, \cdots, nF_{h,k-1})$.

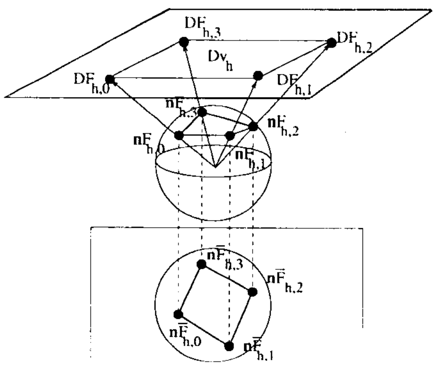


Fig. 1 Two projections: a projection of Dv_h onto the Gauss sphere (solid arrows) and an orthographic projection of the closed piecewise curve onto a plane parallel to Dv_h (dotted arrows)

Let $H_{h,j}$ be the plane containing the origin, $nF_{h,j}$, and $nF_{h,j+1}$, $0 \le j < k$. $H_{h,j}$ divides the spaceinto two half-spaces, $H^+_{h,j}$ and $H^-_{h,j}$. Let $H^+_{h,j}$ be the half-space containing Dv_h. The intersection of half-spaces $H^+_{h,j}$, $0 \le j < k$, is a cone with the apex at the origin. Therefore, the spherical region, that is the projection of Dv_h onto the sphere, is the intersection of the sphere and the cone. The boundary of this region is represented by a sequence of points $(nF_{h,0}, nF_{h,1}, \cdots, nF_{h,k-1})$ such that $nF_{h,j}$ and $nF_{h,j+1}$ are joined by a geodesic arc for all $0 \le j < k$. Moreover, the spherical region is a simple spherical polygon on the sphere. Hence, the sequence of the line segment joining $nF_{h,j}$ and $nF_{h,j+1}$, $0 \le j < k$, forms a simple closed piecewise linear curve. We show that its orthographic projection onto a plane parallel to Dv_h is a convex polygon:

Lemma 1 *The orthographic projection of the closed piecewise linear curve* $(nF_{h,0}, nF_{h,1}, \cdots, nF_{h,k-1})$ *onto a plane parallel to* Dv_h *is a convex polygon.*

[Proof] We denote the projected image of the closed piecewise linear curve by $(\overline{nF_{h,0}}, \overline{nF_{h,1}}, \cdots, \overline{nF_{h,k-1}})$, where $\overline{nF_{h,j}}$, $0 \le j < k$, is the projected image of $nF_{h,j}$. Suppose that the projection is not convex. Since it is a simple closed curve, there would exist one or more vertices of the projection contained in the interior of the convex hull of the projection. Take any of such vertices, say $\overline{nF_{h,j}}$ for some $0 \le j < k$. Then, it must be contained in the interior of the triangle $(\overline{nF_{h,a}}, \overline{nF_{h,b}}, \overline{nF_{h,c}})$, where $0 \le a < b < c < k$. The inverse projection of the triangle onto the sphere gives a spherical triangle $(nF_{h,a}, nF_{h,b}, nF_{h,c})$, that contains $nF_{h,j}$ as an interior point. When the spherical triangle is transformed back onto the plane containing Dv_i, $DF_{h,j}$ lies in the interior of the triangle $(DF_{h,a}, DF_{h,b}, DF_{h,c})$, that is completely contained in Dv_i. Thus, $DF_{h,j}$ is an interior point of Dv_i, which contradicts that Dv_i is a convex polygon. Hence, the result holds true. □

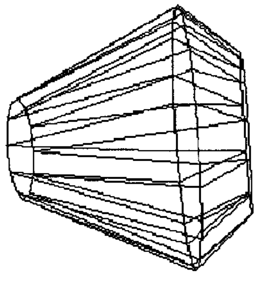
Aggarwal et. al [1] showed that the convex hull of n points can be found in $O(n)$ time if their projections onto a plane are the vertices of a convex polygon. By Lemma 1, the points $nF_{h,j}$, $0 \le j < k$, satisfy this condition. Therefore, their convex hull can be constructed in $O(k)$ time. The piecewise linear curve enables for us to construct the convex hull of nF_i's in $O(n)$ time.

Lemma 2 *The closed piecewise linear curve* $(nF_{h,0}, nF_{h,1}, \cdots, nF_{h,k-1})$ *consists of a subset of edges of the convex hull of* nF_i *for all* $i = 1, 2, \cdots, n$.

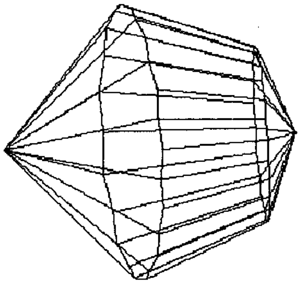
[**Proof**] Since nF_i's lie on the Gauss sphere, all of them are extreme points, i.e., the vertices of the convex hull of nF_i's. Since $nF_{h,j} \in \{nF_1, nF_2, \cdots, nF_n\}$ for all $0 \le j < k$, $nF_{h,j}$'s are also extreme points. We will be done if we show that the line segment joining $nF_{h,j}$ and $nF_{h,j+1}$ is an edge of the convex hull of nF_i's.

Note that the projection of each face Dv_h of the dual D of the convex polyhedron P is a spherical region $(nF_{h,0}, nF_{h,1}, \cdots, nF_{h,k-1})$. This region is the intersection of the Gauss sphere and the cone bounded by the planes $H_{h,j}$ for all $0 \le j < k$. The geodesic arc connecting $nF_{h,j}$ and $nF_{h,j+1}$ lies on $H_{h,j}$, and so does the line segment joining them. For every $v_h \in P$, the cone CO_h is well-defined, i.e., $CO_h = \bigcap_j H^+_{h,j}$. The set of all these cones partitions the sphere into n disjoint spherical regions. Any of the spherical regions does not contain a spherical image nF_i, $1 \le i \le n$ in its interior.

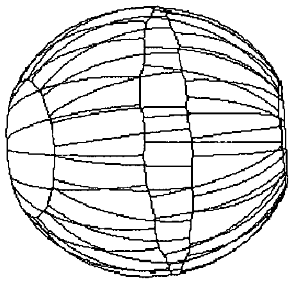
Suppose, for a contradiction, that a line segment of the curve is not an edge of the convex hull of nF_i's, say the line segment joining $nF_{h,l}$ and $nF_{h,l+1}$ for some $0 \le l < k$. Then, it must be a diagonal. Therefore, the line segment excepting its end points $nF_{h,l}$ and $nF_{h,l+1}$ is completely contained in the interior of the cone CO_h. If we project the line segment back to Dv_h, it becomes a diagonal of Dv_h, which is a contradiction since the inverse projection of the line segment is an edge of the convex polyhedron Dv_h. Hence, the result follows immediately. □



(a)



(b)



(c)

Fig. 2 (a) A pot, (b) its corresponding dual and (c) its corresponding spherical Voronoi diagram

Now, we are ready to describe how to construct the convex hull of the Gauss images nF_i's of the normal vectors of the faces of P in $O(n)$ time. By Lemma 2, the set of all such curves, that result from the faces of D, partitions the boundary of the convex hull into n disjoint regions. Lemma 1 guarantees that the convex hull of each of these regions can be found in linear time. Therefore, the convex hull of nF_i's can be constructed in $O(n)$ time once all such curves are identified. The curves are obtained in $O(n)$ time by simply projecting the faces of D onto the Gauss sphere with the origin as the projection center. Given the convex hull of nF_i's, the second and third steps of Brown's algorithm take care of the remainder to construct the spherical Voronoi diagram of nF_i's in $O(n)$ time. Fig. 2 shows a pot, its corresponding dual and spherical Voronoi diagram.

Theorem 1 *The convex hull of* nF_i*'s,* $1 \le i \le n$*, can be found in* $O(n)$ *time. Moreover, their spherical Voronoi diagram can be constructed in* $O(n)$.

3. Concluding Remarks

We presented a linear-time algorithm for computing the Voronoi diagram of the set of spherical points which correspond to outward unit normal vectors of faces in a convex polyhedron. We applied a point-plane dual transformation which is the main idea of our algorithm. The dual transformation is simple and powerful for computing the Voronoi diagram of the set of points possessing convexity. Moreover, based on the point-plane duality and convexity, we are going to find linear-time algorithm computing the 3*D* inner and outer Voronoi diagrams of a convex polyhedron.

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