Application of the Equivalent Frequency Response Method to Runoff Analysis

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ABSTRUCT: This paper introduces the equivalent frequency response method (EFRM) into runoff analysis. This EFRM originally had been developed to analyze dynamic behavior of nonlinear elements such as threshold and saturation in control engineering . Many runoff models are described by nonlinear ordinary or partial differential equations. This paper presents that these nonlinear differential equations can be converted into semi-linear ones based on EFRM. The word of " a semi-linear equation" means that the coefficients of derived equations depend on average rainfall.

1 INTRODUCTION

It is well known that several runoff models such as diffusion and kinematic equation stem from Saint Venant equation. R.Hamouda and M. Fujita (2000) propose an application of EFRM to these equations. In this paper, we present an example of the application of EFRM to runoff analysis.

2. Nonlinear Runoff System and Equivalent Frequency Response Method

2.1 Lumped Parameter Runoff Model

First of all, consider a simple runoff model described by

$$\frac{ds}{dt} + q = r \tag{1}$$

$$s = k_1 q^{p_1} + k_2 \frac{dq^{p_2}}{dt} \qquad p_1 \neq 1, p_2 \neq 1 \tag{2}$$

$$q(0) = 0 \quad , \quad \left[\frac{dq}{dt}\right]_{t=0} = 0 \tag{3}$$

s : storage q : depth of runoff r : effective rainfall k_1, k_2, p_1, p_2 : constants

The following nonlinear equation is derived from eqs. (1) and (2).

$$k_2 p_2 q^{p_2-1} \frac{d^2q}{dt^2} + k_2 p_2 (p_2 - 1) q^{p_2-2} \left(\frac{dq}{dt} \right)^2 + k_1 p_1 q^{p_1-1} \frac{dq}{dt} + q = r \tag{4}$$

In order to derive the equivalent frequency transfer function, we assume that

$$r = \bar{r} + Ae^{j\omega t} \tag{5}$$

$$q = \bar{q} + Ce^{j\omega t} \qquad j : \text{imaginary unit} \tag{6}$$

$\bar{r}, \bar{q}$: average rainfall and discharge (constants) ω : frequency A : constant C : complex number

The following approximations are adopted to express the nonlinear terms in eq.(2)

$$k_1 q^{p_1} \approx k_1 \left(\bar{q}^{p_1} + p_1 \bar{q}^{p_1-1} Ce^{j\omega t} \right) \qquad k_2 q^{p_2} \approx k_2 \left(\bar{q}^{p_2} + p_2 \bar{q}^{p_2-1} Ce^{j\omega t} \right) \tag{7}$$

By substituting eqs. (5),(6) and (7) into eqs. (1) and (2), eq. (8) is obtained.

$$\bar{q} = \bar{r} \qquad \left(1 - \omega^2 k_2 p_2 \bar{q}^{p_2-1} + j\omega k_1 p_1 \bar{q}^{p_1-1} \right) C = A \tag{8}$$

C / A in eq. (8) denotes the equivalent frequency transfer function $Z(j\omega)$ between $r(t)$ and $q(t)$.

$$Z(j\omega) = \frac{1}{1 - \omega^2 k_2 p_2 \bar{r}^{p_2-1} + j\omega k_1 p_1 \bar{r}^{p_1-1}} = R_e(\omega) + jI_m(\omega) \tag{9}$$

$$R_e(\omega) = \frac{1 - \omega^2 k_2 p_2 \bar{r}^{p_2-1}}{\left(1 - \omega^2 k_2 p_2 \bar{r}^{p_2-1}\right)^2 + \left(\omega k_1 p_1 \bar{r}^{p_1-1}\right)^2} \qquad I_m(\omega) = -\frac{\omega k_1 p_1 \bar{r}^{p_2-1}}{\left(1 - \omega^2 k_2 p_2 \bar{r}^{p_2-1}\right)^2 + \left(\omega k_1 p_1 \bar{r}^{p_1-1}\right)^2} \tag{10}$$

Gain and time lag functions are defined by

$$G(\omega) = |Z(j\omega)| = \sqrt{R_e^2(Z) + I_m^2(Z)} \qquad T_L(\omega) = \frac{Arg\{Z(j\omega)\}}{\omega} \tag{11}$$

Eq. (9) suggests the following differential equation as $r(t)$ ~ $q(t)$ relation.

$$k_2 p_2 \bar{r}^{p_2-1} \frac{d^2q}{dt^2} + k_1 p_1 \bar{r}^{p_1-1} \frac{dq}{dt} + q = r \tag{12}$$

The nonlinear differential equation (4) is converted into a semi-linear differential equation (12). The word " semi-linear differential equation" means that eq. (12) is not a completely linear differential equation since the coefficients of eq. (12) are affected by rainfall input. As eq.(7) is an approximation, let us check its accuracy. K.Hoshi and I. Yamaoka (1982) gave coefficients of eq. (2) by lumping non-dimensional kinematic equations (23).

$$k_1 = \frac{p}{p+1} \quad p_1 = \frac{1}{P} \quad k_2 = 0.1 p_1^{-0.2} \quad p_2 = p_1^{1.5} \quad p = \begin{cases} 5/3 & \textit{Manning formula} \\ 3/2 & \textit{Chezy formula} \end{cases} \tag{13}$$

We assume that

$$r(t) = \bar{r} + A\sin(\omega t) \tag{14}$$

Fig. 1 shows a schematic relationship between sinusoidal rainfall input and discharge. The gain function is defined by

$$G(\omega) = \frac{D}{A} \qquad (15)$$

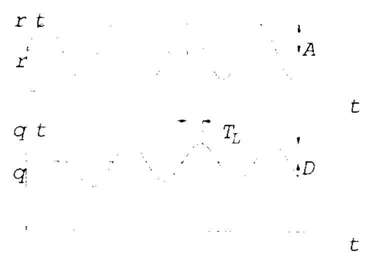


Fig. 1 Schematic relationship between sinusoidal rainfall input and discharge output

Time lag is defined by the time interval between the two sinusoidal peaks as shown in Fig. 1. Fig. 2 (A), (B), (C) show the vector locus (relationship between $R_e(Z)$ and $I_m(Z)$ in eq. (9)), gain and time lag function. The solid lines denote eq. (11). The black circles show the numerical results obtained from eq. (4) and (14). The numerical agrees with the theoretical one. Eq. (12) is obtained on condition that the forcing term in eq. (4) is a sinusoidal function such as that in eq. (5). Therefore, the input term in eq. (12) is limited to a sinusoidal one. However, any time function can be expanded in the Fourier series expressed by

$$r(t) = \bar{r} + \sum_{n=1}^{\infty} \{a_n \cos(n\omega_0 t) - b_n \sin(n\omega_0 t)\} \qquad (16)$$

$$\omega_0 = \frac{2\pi}{T_p} \qquad \bar{r} = \frac{1}{T_p}\int_0^{T_p} r(t)\,d\tau \qquad T_p\text{: period} \qquad (17)$$

$$a_n = \frac{2}{T_p}\int_0^{T_p} r(\tau)\cos(n\omega_0\tau)\,d\tau \qquad b_n = \frac{2}{T_p}\int_0^{T_p} r(\tau)\sin(n\omega_0\tau)\,d\tau \qquad (18)$$

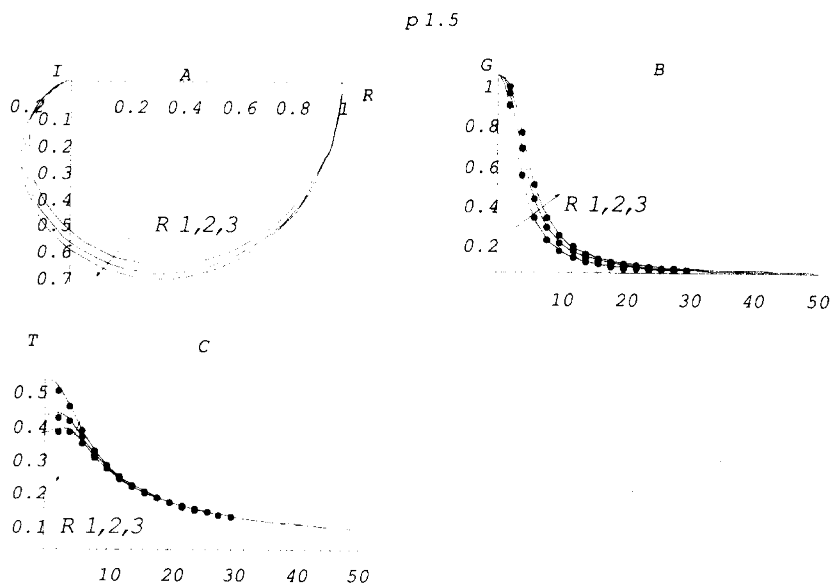


Fig. 2 Second order storage function model

We obtain eq. (19) from eqs. (11) and (16).

$$q(t) = \bar{r} + \sum_{n=1}^{\infty} G(n\omega_0)\left[a_n \cos\{n\omega_0(t - T_L(n\omega_0))\} + b_n \sin\{n\omega_0(t - T_L(n\omega_0))\}\right] \tag{19}$$

Fig. 3 shows a comparison of the solution of eq. (4) (solid line) and that of eq. (12) (dashed line) using rainfall input as shown in Fig. 3 (A). The solutions are almost the same. T_p in eqs. (17) and (18) equalizes to a duration time of runoff ($T_p = 3$). We add that discharge obtained from eq. (19) completely coincides with the solution of eq. (12).

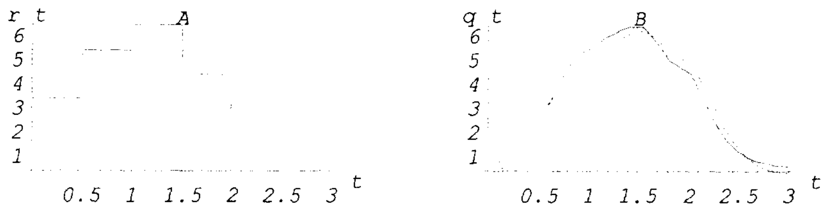


Fig. 3 Comparison of eq. (4) with eq. (12)

Solid line: eq. (4) Dashed line: eq. (12)

2.2 Distributed Parameter Runoff Model

One of the runoff model describing propagation of shallow waves over a plane surface is the kinematic wave equation.

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = r \qquad 0 \le x \le l \qquad\qquad q = \alpha h^p \tag{20}$$

$$h(0,x) = 0 \ , \ q(0,x) = 0 \quad , \quad h(t,0) = 0 \quad , \quad q(t,0) = 0 \tag{21}$$

Eq. (20) may be rewritten by introducing the following normalizing parameters. The capital letters such as Q and H indicate a dimensional quantity corresponding to the small letter such as q and h.

$$h = h_* H \quad q = q_* Q \quad t = t_* T \quad x = x_* X \quad r = r_* R$$

$$r_* = \bar{r} \qquad x_* = l \qquad q_* = \bar{r}l \qquad t_* = (\bar{r}^{-1-p}\, l/\alpha)^{1/p} \qquad \bar{r}\text{: average rainfall} \tag{22}$$

Eq. (20) is replaced by

$$\frac{\partial H}{\partial T} + \frac{\partial Q}{\partial X} = R \qquad 0 \le X \le 1 \qquad\qquad Q = H^p \tag{23}$$

$$H(0,X) = 0 \ , \ Q(0,X) = 0 \ , \ H(T,0) = 0 \ , \ Q(T,0) = 0 \tag{24}$$

In order to derive the equivalent frequency transfer function between $R(T)$ and $Q(T,1)$, we assume that

$$R(T) = \bar{R} + Ae^{j\Omega T} \tag{25}$$

$$H(T,X) = \bar{H}(X) + B(X)e^{j\Omega T} \tag{26}$$

$$Q(T,X) = \bar{Q}(X) + C(X)e^{j\Omega T} \tag{27}$$

$\bar{H}(X)$ and $\bar{Q}(X)$ denote the stationary components of $H(T,X)$ and $Q(T,X)$, respectively and satisfy eq.

(28).

$$\overline{H}(0)=0 \quad , \quad \overline{Q}(0)=0 \tag{28}$$

Eq. (29) is derived from eqs. (24), (26) ,(27) and (28).

$$B(0)=0 \quad , \quad C(0)=0 \tag{29}$$

The nonlinear term of eq. (23) is approximated by

$$Q(T,X)=\overline{Q}(X)+C(X)e^{j\Omega T}=\overline{H}^{p}+p\overline{H}^{p-1}B(X)e^{j\Omega T} \tag{30}$$

$\overline{H}(X)$ and $\overline{Q}(X)$ are

$$\overline{Q}(X)=\overline{R}X \qquad\qquad \overline{H}(X)=(\overline{R}X)^{1/p} \tag{31}$$

Eqs. (23) and (25) give

$$\frac{dC}{dX}+j\Omega B=A \tag{32}$$

Eq. (33) is derived from eqs. (30),(31) and (32).

$$\frac{dC}{dX}+\frac{j\Omega}{p}(\overline{R}X)^{(1-p)/p}C=A \qquad\qquad C(0)=0 \tag{33}$$

The solution of eq. (33) is

$$C(X)=Ae^{-\int^{X}\frac{j\Omega}{p}(\overline{R}X_1)^{(1-p)/p}dX_1}\int_0^{X}e^{\int^{X_2}\frac{j\Omega}{p}(\overline{R}X_3)^{(1-p)/p}dX_3}dX_2 \tag{34}$$

The equivalent frequency transfer function between $R(T)$ and $Q(T,1)$ is defined by

$$Z(j\Omega)=\frac{C(1)}{AL}=R_e(Z)+jI_m(Z) \tag{35}$$

$$R_e(Z)=I_1\cos(\Omega\overline{R}^{(1-p)/p})+I_2\sin(\Omega\overline{R}^{(1-p)/p}) \qquad I_m(Z)=I_2\cos(\Omega\overline{R}^{(1-p)/p})-I_1\sin(\Omega\overline{R}^{(1-p)/p}) \tag{36}$$

$$I_1=\int_0^1\cos(\Omega\overline{R}^{(1-p)/p}X^{1/p})dX=\frac{1}{2}\left\{{}_1F_1(p,p+1,j\Omega\overline{R}^{(1-p)/p})+{}_1F_1(p,p+1,-j\Omega\overline{R}^{(1-p)/p})\right\} \tag{37}$$

$$I_2=\int_0^1\sin(\Omega\overline{R}^{(1-p)/p}X^{1/p})dX=-\frac{j}{2}\left\{{}_1F_1(p,p+1,j\Omega\overline{R}^{(1-p)/p})-{}_1F_1(p,p+1,-j\Omega\overline{R}^{(1-p)/p})\right\} \tag{38}$$

L : non-dimensional slope length $(L=1)$

where ${}_1F_1(a,b,z)$ denotes Kummer's confluent hypergeometric function defined by

$${}_1F_1(a,b,z)=1+\frac{az}{b}+\frac{a(a+1)z^2}{2!b(b+1)}+\frac{a(a+1)(a+2)z^3}{3!b(b+1)(b+2)}+ \tag{39}$$

If $p=1$ in eq. (23), eq. (40) is obtained.

$$R_e(Z)=\frac{\sin(\Omega)}{\Omega} \qquad I_m(Z)=\frac{\cos(\Omega)-1}{\Omega} \tag{40}$$

Fig. 4 shows the vector locus of eq. (35). It is noticed that there are great differences between fig. 2 and fig. 4. Fig. 5 shows the gain and time lag functions of eq. (35). The black circles in fig. 5 show the computational results ($\bar{R}=1$) based on eqs. (14) and (23). The theoretically obtained gain and time functions agree with the computational ones. It is impossible to obtain a differential equation from eq. (35) because $R_e(Z)$ and $I_m(Z)$ are numerically calculated. Therefore, we define an equivalent impulse response function $J(T)$ calculated by

$$J(T)=\frac{2}{\pi}\int_0^{\infty} R_e(Z)\cos(\Omega T)\,d\Omega \tag{41}$$

Fig. 6 shows the equivalent impulse response function. The impulse response function of a linear kinematic equation ($p=1$) has a rectangular shape as shown in fig. 6. This means discharge is expressed by a moving average of rainfall input.

$$Q(T)=\int_{T-1}^{T} R(\tau)\,d\tau \tag{42}$$

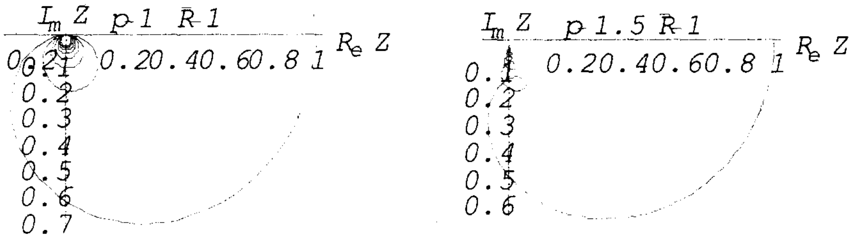
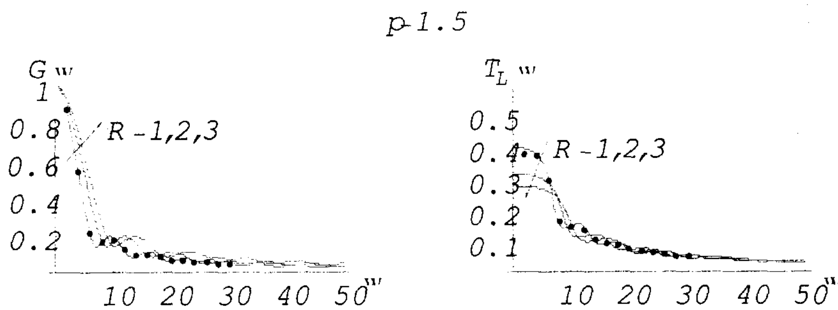


Fig. 4 Vector locus of eq. (35)



Eq. (42) leads to a fluctuated gain and time lag functions as shown in fig. 7. This tendency is preserved in the gain and time lag functions for a nonlinear kinematic wave equation ($p = 1.5$).

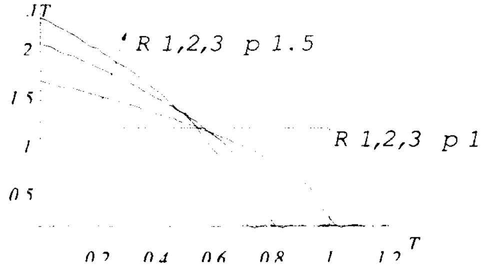
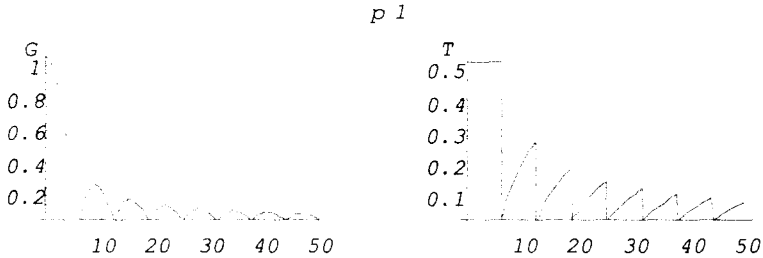


Fig. 6 Impulse response function



3. Application of the Equivalent Frequency Response method to Runoff Analysis

It has been proved that nonlinear runoff equations, even those that are partial differential equations, can be converted into a semi linear equations. It is possible to estimate the model's equations and their coefficients using the presented theory if observed hydrological data are available. The frequency transfer function between effective rainfall and direct runoff is defined by

$$Z(j\omega) = \frac{Q(j\omega)}{R(j\omega)} = R_e(Z) + jI_m(Z) \tag{43}$$

$$R(j\Omega) = \int_0^\infty r(t)e^{-j\omega t}dt = R_e(R) + jI_m(R) \qquad Q(j\Omega) = \int_0^\infty q(t)e^{-j\omega t}dt = R_e(Q) + jI_m(Q) \tag{44}$$

$r(t), q(t)$: effective rainfall and direct runoff

$R(j\omega), Q(j\omega)$: Fourier transformed functions of $r(t)$ and $q(t)$, respectively

$R_e(Y)$, $I_m(Y)$: real and imaginary parts of $Y(j\omega)$

Eq. (45) can be rewritten as

$$R_e(Z) = \frac{R_e(Q)R_e(R) + I_m(Q)I_m(R)}{R_e^2(R) + I_m^2(R)} \qquad I_m(Z) = -\frac{R_e(Q)I_m(R) - I_m(Q)R_e(R)}{R_e^2(R) + I_m^2(R)} \tag{45}$$

It is possible to calculate the vector locus and the gain and time lag functions using eq. (45). Figs 8 and 9 (C), (D) and (E) show the computed results using runoff data (Sept. 16, 1998) obtained at Maruseppu and Engaru ,Yubetsu River , Hokkaido. (A) and (B) are runoff data. We assume that a second order differential equation can be used to describe the runoff system with reference to fig. 2.

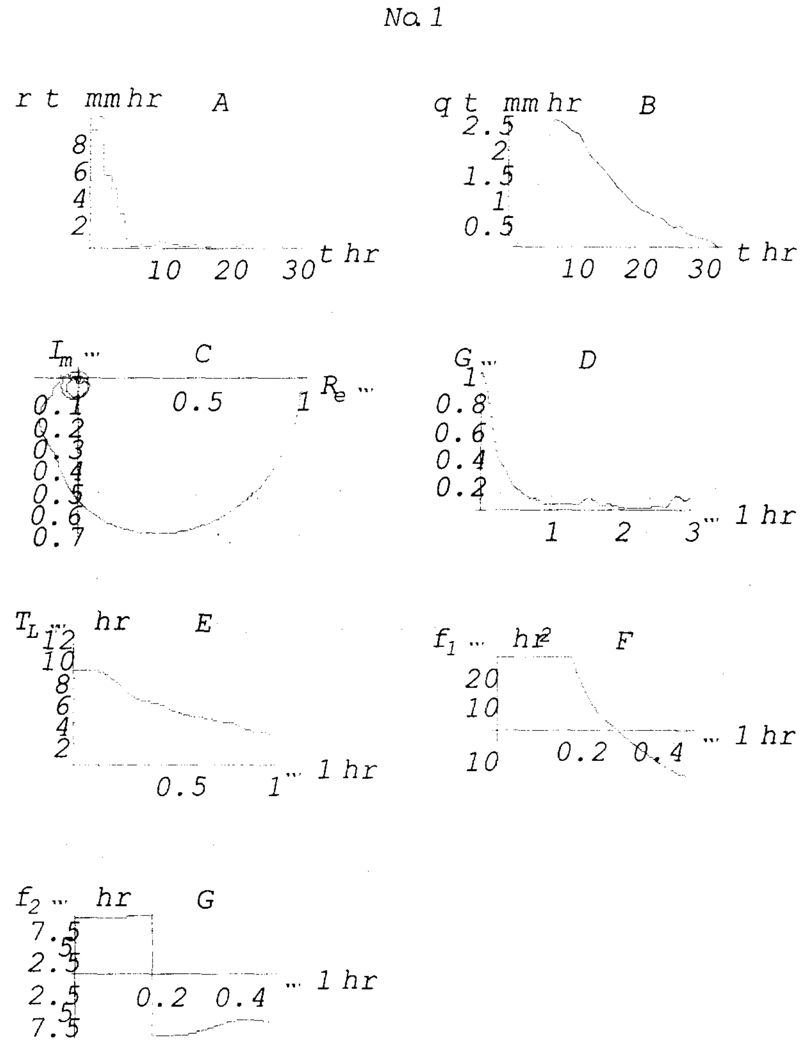


Fig. 8 Computed results at Maruseppu, Yubetsu River

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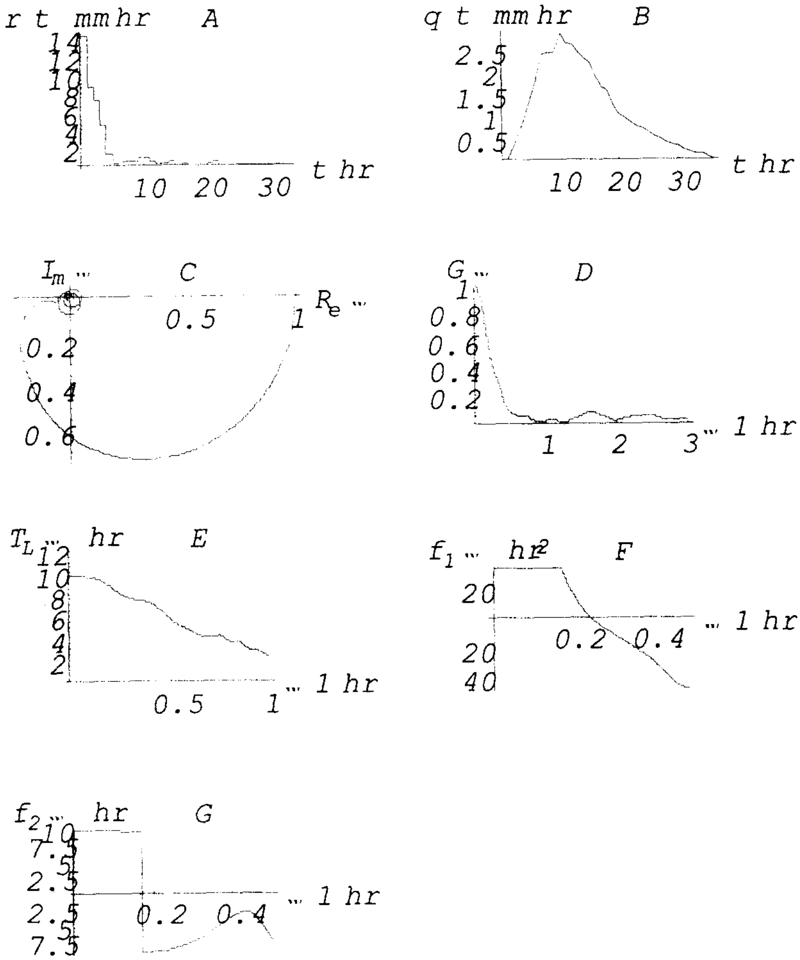

Fig. 9 Computed results at Engraru, Yubetsu River

$$f_1(\bar{r})\frac{d^2q}{dt^2} + f_2(\bar{r})\frac{dq}{dt} + q = r \tag{46}$$

Eq. (47) is derived from eq. (46).

$$G(\omega) = \frac{1}{\sqrt{\{1-\omega^2 f_1(\bar{r})\}^2 + \{\omega f_2(\bar{r})\}^2}} \qquad T_L(\omega) = \frac{1}{\omega}\tan^{-1}\left\{\frac{\omega f_2(\bar{r})}{1-\omega^2 f_1(\bar{r})}\right\} \tag{47}$$

$f_1(\bar{r})$ and $f_2(\bar{r})$ are

$$f_1(\bar{r}) = \frac{1}{\omega^2}\left\{1 - \frac{1}{G(\omega)\sqrt{1+\tan^2\{\omega T_L(\omega)\}}}\right\} \qquad f_2(\bar{r}) = \frac{\tan\{\omega T_L(\omega)\}}{\omega G(\omega)\sqrt{1+\tan^2\{\omega T_L(\omega)\}}} \tag{48}$$

(F) and (G) in Figs. 8 and 9 (F) show $f_1(\bar{r})$ and $f_2(\bar{r})$ ~ ω relation. Eq. (49) describes only the low frequency runoff component. The same tendency is obtained at Kaise and Naka Yubetsu, Yubetus River. Table 1 indicates obtained parameter values. Fig. 10 shows a comparison of observed discharge (solid line) and computed one (dashed line) obtained from eq. (46). The simple runoff model described by eq. (46) approximates the observed discharge. However, improvements are required as eq. (46) covers only a narrow band of frequency range.

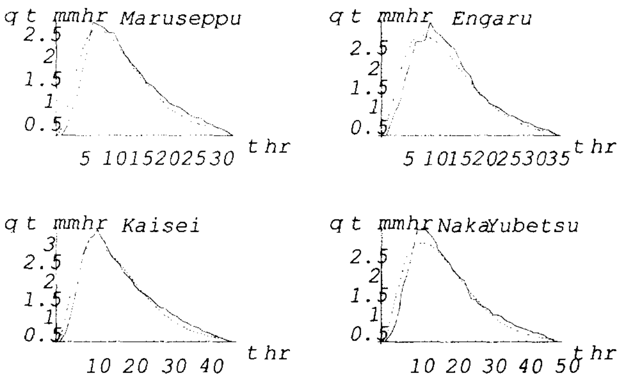

Fig. 10 Comparison of observed discharge with computed one

Table 1 Coefficients of eq.(46)

Location	f_1 (hr^2)	f_2 (hr)	Drainage Area
Maruseppu	26.1	8.9	802(km^2)
Engaru	33.0	9.5	958
Kaisei	48.3	12.8	1375
Naka Yubetsu	56.1	13.1	1452

4. Conclusion

This paper shows that EFRM is a useful tool for analyzing a nonlinear runoff system. Nonlinear differential equations are converted into semi linear equations. This paper deals with a storage function model and a kinematic wave equation. We have already derived equivalent impulse response functions of more complex runoff models such as the Saint Venant equation and diffusion equation. It is possible to use these models through EFRM.

References

R.,Hamouda, and M.,Fujita: Application of Equivalent Frequency Response to Nonlinear Runoff System under contribution

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