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SADDLE POINTS OF VECTOR-VALUED FUNCTIONS IN TOPOLOGICAL VECTOR SPACES

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Abstract. We give a new saddle point theorem for vector-valued functions on an admissible compact convex set in a topological vector space under weak condition that is the semicontinuity of two function scalarization and acyclicity of the involved sets. As application, we obtain the minimax theorem.

1. Introduction

In 1983, Nieuwenhuis [10] introduced the concept of saddle point for vector-valued functions in finite dimensional spaces. Tanaka [16–18] obtained various existence results on cone saddle points of vector-valued functions in infinite dimensional spaces. Recently, some existence theorems of cone saddle points on H-spaces are proved in [2].

In this paper, we first provide sufficient conditions for a multimap to be upper semicontinuous and then give a new saddle point theorem for vector-valued functions in topological vector spaces under the semicontinuity of scalarized functions whose proof is based on a fixed point theorem [11] due to Park instead of Fan-Glicksberg's fixed point theorem [5] for locally convex topological vector spaces, where the admissibility in the sense of Klee [7] plays a fundamental role. The main result is a generalization of [6]. Moreover, it is remarkable that convexity of the involved sets in the main theorem can be replaced by acyclicity. As application, we present the minimax theorem which reduces to von Neumann's minimax theorem [9]. For minimax problems relative to vector-valued functions, see [2–4, 14].

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A *multimap* $T : X \multimap Y$ is a function from a set X into the set of all nonempty subsets of a set Y. For topological spaces X and Y, a multimap $T : X \multimap Y$ is said to be *upper semicontinuous* if the set $\{x \in X : Tx \subset A\}$ is open in X for each open set A in Y. A multimap $T : X \multimap Y$ is said to be *compact* if the set $T(X)$ is relatively compact in Y; and *closed* if T has a closed graph.

A function $f : X \to \mathbb{R}$ on a topological space X is said to be *lower semicontinuous* if the set $\{x \in X : f(x) > \alpha\}$ is open in X for every real number α; and *upper semicontinuous* if the set $\{x \in X : f(x) < \alpha\}$ is open in X for every real number α.

Let Z be a real topological vector space with a partial order $\leq$; that is, a reflexive transitive binary relation. Let A be a nonempty set in Z. A point $a_0 \in A$ is said to be a *minimal point* of A if for any $a \in A$, $a \leq a_0$ implies $a = a_0$. It is said to be a *maximal point* of A if for any $a \in A$, $a_0 \leq a$ implies $a = a_0$. The set of minimal [resp. maximal] points of A is denoted by $\min A$ [resp. $\max A$].

Let f be a vector-valued function from a product $X \times Y$ to Z. For $x \in X$ and $y \in Y$ set $f(X, y) := \{f(x, y) : x \in X\}$ and $f(x, Y) := \{f(x, y) : y \in Y\}$. A point $(x_0, y_0) \in X \times Y$ is said to be a *saddle point* of f on $X \times Y$ if $f(x_0, y_0) \in \min f(X, y_0) \cap \max f(x_0, Y)$.

Let f, f_1 and f_2 be real-valued functions defined on the Cartesian product $X \times Y$ of sets X and Y. A point (x_0, y_0) is said to be a *semi-saddle point* of (f_1, f_2) on $X \times Y$ if $f_1(x_0, y_0) \leq f_1(x, y_0)$ and $f_2(x_0, y) \leq f_2(x_0, y_0)$ for all $x \in X$ and $y \in Y$. It is said to be a *saddle point* of f on $X \times Y$ if $f(x_0, y) \leq f(x_0, y_0) \leq f(x, y_0)$ for all $x \in X$ and $y \in Y$. See [17].

Let Z be a real topological vector space with a partial order $\leq$. A real-valued function $g : Z \to \mathbb{R}$ is said to be *strictly monotone* if $g(a) < g(b)$ for $a < b$, where $a < b$ means $a \leq b$ and $a \neq b$. See [8].

A nonempty subset X of a topological vector space E is said to be *admissible* (in the sense of Klee [7]) provided that, for every compact subset K of X and every neighborhood V of the origin 0 in E, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

It is well-known that every nonempty convex subset of a locally convex topological vector space is admissible. The spaces $L^p(0, 1)$ for $0 < p < 1$ and $S(0, 1)$ are admissible topological vector spaces, see [12, 13].

A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. In particular, any nonempty convex or star-shaped subset of a topological vector space is acyclic.

2. Saddle points of vector-valued functions

We give a new saddle point theorem for vector-valued functions in topological vector spaces under weaker conditions than known results. To this end, the following observation is necessary. See [1].

LEMMA 2.1. *Let X and Y be Hausdorff topological spaces and $f : X \times Y \to \mathbb{R}$ a real-valued function on the product space $X \times Y$. Then the following statements hold:*

(1) *If X is compact and if $f(\cdot, y)$ is lower semicontinuous on X for each $y \in Y$ and $f(x, \cdot)$ is upper semicontinuous on Y for each $x \in X$, then a function $h : Y \to \mathbb{R}$ defined by*

$$h(y) := \min_{x \in X} f(x, y) \qquad \text{for } y \in Y$$

is upper semicontinuous.

(2) *If Y is compact and if $f(x, \cdot)$ is upper semicontinuous on Y for each $x \in X$ and $f(\cdot, y)$ is lower semicontinuous on X for each $y \in Y$, then a function $k : X \to \mathbb{R}$ defined by*

$$k(x) := \max_{y \in Y} f(x, y) \qquad \text{for } x \in X$$

is lower semicontinuous.

(3) *If f is lower semicontinuous on $X \times Y$ and $f(x, \cdot)$ is upper semicontinuous on Y for each $x \in X$ and if X is compact, then a multimap $T : Y \multimap X$ defined by*

$$Ty := \{x \in X : f(x, y) = \min_{x \in X} f(x, y)\} \quad \text{for } y \in Y$$

is upper semicontinuous.

(4) *If f is upper semicontinuous on $X \times Y$ and $f(\cdot, y)$ is lower semicontinuous on X for each $y \in Y$ and if Y is compact, then a multimap $S : X \multimap Y$ defined by*

$$Sx := \{y \in Y : f(x, y) = \max_{y \in Y} f(x, y)\} \quad \text{for } x \in X$$

is upper semicontinuous.

Proof. (1) The function $h : Y \to \mathbb{R}$ is well-defined since $f(\cdot, y)$ is lower semicontinuous on the compact set X. We claim that h is upper semicontinuous on Y. Let $y_0 \in Y$ and $r \in \mathbb{R}$ such that $h(y_0) < r$. Then there is a point $x_0 \in X$ such that $f(x_0, y_0) = h(y_0) < r$. Since $f(x_0, \cdot)$ is upper semicontinuous on Y, there exists a neighborhood V of y_0 in Y such that $f(x_0, y) < r$ for all $y \in V$ and so $h(y) \le f(x_0, y) < r$ for all $y \in V$. Hence h is upper semicontinuous on Y.

(2) A similar argument establishes the result for the lower semicontinuity of k.

(3) We show that T has a closed graph. Let (x_α, y_α) be a net in the graph $\mathrm{Gr}(T)$ of T such that $(x_\alpha, y_\alpha) \to (x_0, y_0)$. Since f is lower semicontinuous on $X \times Y$, $(x_\alpha, y_\alpha) \in \mathrm{Gr}(T)$, and h is upper semicontinuous on Y, we have

$$\begin{aligned} f(x_0, y_0) &\le \liminf_{\alpha} f(x_\alpha, y_\alpha) \le \limsup_{\alpha} h(y_\alpha) \\ &\le h(y_0) \le f(x_0, y_0) \end{aligned}$$

and hence $f(x_0, y_0) = h(y_0)$; that is, $(x_0, y_0) \in \mathrm{Gr}(T)$. Thus T has closed graph. Since X is compact, it is clear that T is upper semicontinuous (see [1]).

(4) As in the proof of (3), we can check that S has a closed graph and hence S is upper semicontinuous. This completes the proof. □

The following lemma provides a criterion for the existence of saddle points. For loose saddle points of multimaps, see [6, Lemma 2.1]. For cone saddle points of vector-valued functions, see [17, Theorem 2.4].

LEMMA 2.2. *Let Z be a real topological vector space with a partial order $\leq$ and $g_1, g_2 : Z \to \mathbb{R}$ strictly monotone functions. If $f : X \times Y \to Z$ is a vector-valued function on the Cartesian product $X \times Y$, then any semi-saddle point of $(g_1 \circ f, g_2 \circ f)$ on $X \times Y$ is also a saddle point of f on $X \times Y$.*

Proof. Let $(x_0, y_0) \in X \times Y$ be a semi-saddle point of $(g_1 \circ f, g_2 \circ f)$ on $X \times Y$. Then $g_1 \circ f(x_0, y_0) \leq g_1 \circ f(x, y_0)$ and $g_2 \circ f(x_0, y) \leq g_2 \circ f(x_0, y_0)$ for all $x \in X$ and $y \in Y$. Since g_1 and g_2 are strictly monotone, it follows that $f(x_0, y_0) \in \min f(X, y_0) \cap \max f(x_0, Y)$. In fact, if $f(x_0, y_0)$ is not a minimal point of $f(X, y_0)$, then $f(w, y_0) < f(x_0, y_0)$ for some $w \in X$ and hence by the strict monotonicity, $g_1 \circ f(w, y_0) < g_1 \circ f(x_0, y_0)$ which contradicts the above relation that $g_1 \circ f(x_0, y_0) \leq g_1 \circ f(x, y_0)$ for all $x \in X$. Similarly, we obtain that $f(x_0, y_0) \in \max f(x_0, Y)$. Therefore, (x_0, y_0) is a saddle point of f on $X \times Y$. This completes the proof. □

Our main tool is the following particular form of a fixed point theorem [11, Corollary 1.1] recently due to Park.

LEMMA 2.3. *Let X be an admissible convex subset of a Hausdorff topological vector space E and $A : X \multimap X$ a compact closed multimap with nonempty acyclic values. Then A has a fixed point.*

Now we can obtain our main result which is a generalization of [6, Theorem 3.2]. For cone saddle points on H-spaces, see [2, Theorem 2.3].

THEOREM 2.4. *Let X and Y be nonempty admissible compact convex sets in two Hausdorff topological vector spaces E and F respectively, and Z a partially ordered topological vector space. Let $f : X \times Y \to Z$ be a vector-valued function defined on the product space $X \times Y$. Suppose that there exist strictly monotone functions $g_1, g_2 : Z \to \mathbb{R}$ such that*

(1) *$g_1 \circ f$ is lower semicontinuous on $X \times Y$ and $g_1 \circ f(x, \cdot)$ is upper semicontinuous on Y for each $x \in X$;*
(2) *$g_2 \circ f$ is upper semicontinuous on $X \times Y$ and $g_2 \circ f(\cdot, y)$ is lower semicontinuous on X for each $y \in Y$;*

(3) $\{x \in X : g_1 \circ f(x,y) = \min_{x \in X} g_1 \circ f(x,y)\}$ *is acyclic for each* $y \in Y$*; and*
(4) $\{y \in Y : g_2 \circ f(x,y) = \max_{y \in Y} g_2 \circ f(x,y)\}$ *is acyclic for each* $x \in X$.

Then f *has a saddle point on* $X \times Y$.

Proof. Consider three multimaps

$$T : Y \multimap X, \quad Ty := \{x \in X : g_1 \circ f(x,y) = \min_{x \in X} g_1 \circ f(x,y)\}$$
$$S : X \multimap Y, \quad Sx := \{y \in Y : g_2 \circ f(x,y) = \max_{y \in Y} g_2 \circ f(x,y)\}$$
$$A : X \times Y \multimap X \times Y, \quad A(x,y) := (Ty, Sx).$$

For each $y \in Y$, since $g_1 \circ f(\cdot, y)$ is lower semicontinuous on the compact set X, Ty is nonempty and closed. S also has nonempty closed values. By Lemma 2.1, T and S are upper semicontinuous with nonempty closed values. Hence A is upper semicontinuous and has nonempty closed values. Therefore, A is a compact closed multimap. From (3) and (4), it follows that T and S have acyclic values. By Lemma 2.3, there is a point $(x_0, y_0) \in X \times Y$ such that $x_0 \in Ty_0$ and $y_0 \in Sx_0$. Thus, (x_0, y_0) is a semi-saddle point of $(g_1 \circ f, g_2 \circ f)$ on $X \times Y$. Since g_1 and g_2 are strictly monotone, by Lemma 2.2, (x_0, y_0) is a saddle point of f. This completes the proof. □

COROLLARY 2.5. *Let* X *and* Y *be nonempty compact convex sets in two Hausdorff locally convex topological vector spaces respectively, and* Z *a partially ordered topological vector space. Let* $f : X \times Y \to Z$ *be a continuous vector-valued function on the product space* $X \times Y$*. Suppose that there exists a continuous strictly monotone function* $g : Z \to \mathbb{R}$ *such that*

(1) *for each* $y \in Y$, $g \circ f(\cdot, y)$ *is quasiconvex on* X*; and*
(2) *for each* $x \in X$, $g \circ f(x, \cdot)$ *is quasiconcave on* Y.

Then f *has a saddle point on* $X \times Y$.

REMARK. If f is a continuous real-valued function and g is the identity map, then Corollary 2.5 reduces to [15, Theorem 4.1], where the concept of an escaping sequence is required instead of the compactness of X and Y.

Finally we show that the minimax theorem can be deduced from our saddle point theorem.

THEOREM 2.6. *Let X and Y be nonempty admissible compact convex sets in two Hausdorff topological vector spaces E and F respectively. Let $f : X \times Y \to \mathbb{R}$ be a continuous real-valued function defined on the product space $X \times Y$ such that*

(1) *$\{x \in X : f(x,y) = \min_{x\in X} f(x,y)\}$ is acyclic for each $y \in Y$; and*

(2) *$\{y \in Y : f(x,y) = \max_{y\in Y} f(x,y)\}$ is acyclic for each $x \in X$.*

Then we have the minimax theorem

$$\min_{x\in X}\max_{y\in Y} f(x,y) = \max_{y\in Y}\min_{x\in X} f(x,y).$$

Proof. Theorem 2.4 implies that there exists a point $(x_0, y_0) \in X \times Y$ such that

$$\max_{y\in Y} f(x_0,y) = f(x_0,y_0) = \min_{x\in X} f(x,y_0).$$

By Lemma 2.1, $\max_{y\in Y} f(\cdot,y)$ is lower semicontinuous on the compact set X and $\min_{x\in X} f(x,\cdot)$ is upper semicontinuous on the compact set Y. Hence we conclude that

$$\min_{x\in X}\max_{y\in Y} f(x,y) \le \max_{y\in Y} f(x_0,y) = \min_{x\in X} f(x,y_0) \le \max_{y\in Y}\min_{x\in X} f(x,y).$$

The inequality $\max_{y\in Y}\min_{x\in X} f(x,y) \le \min_{x\in X}\max_{y\in Y} f(x,y)$ is obvious. This completes the proof. □

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